Math 5421
An Introduction to Mathematical Climate Models

Spring 2025
1:25 – 3:20 Tuesdays and Thursdays
Blegen Hall 155

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course website
https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/

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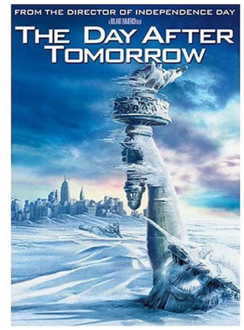
The Day After Tomorrow

What did the film get right scientifically?

What did it get wrong?

Did the portrayal of politics ring true or false?



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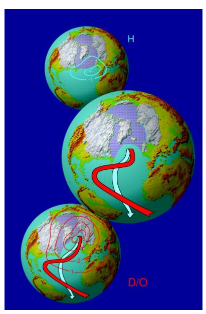
The Day After Tomorrow

Heinrich and Dansgaard-Oeschger events

What did the film get right scientifically?



http://www.pik-potsdam.de/~stefan/sampleimages.html

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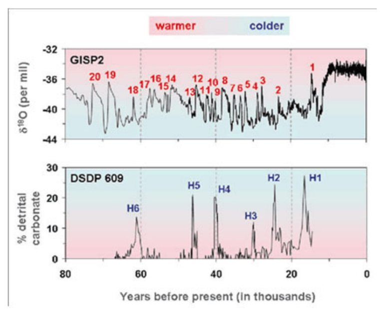
The Day After Tomorrow

Heinrich and Dansgaard-Oeschger events



http://www.ncdc.noaa.gov/paleo/abrupt/data3.html

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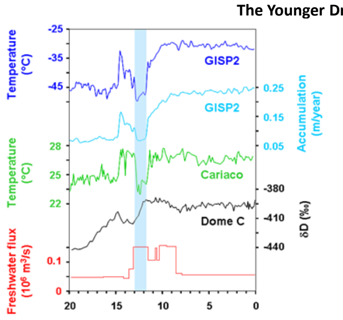
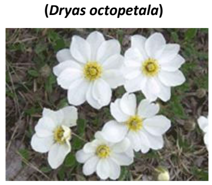
The Younger Dryas



http://www.ncdc.noaa.gov/paleo/abrupt/data4.html

Mountain Avens
(*Dryas octopetala*)



https://www.ncdc.noaa.gov/abrupt-climate-change/The%20Younger%20Dryas

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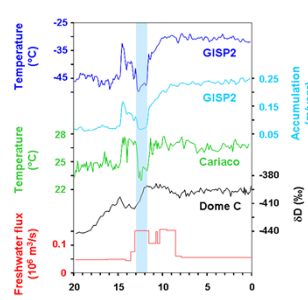
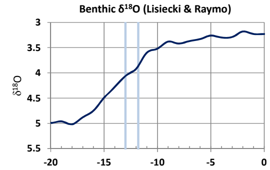
The Younger Dryas



Only a minor impact on ice volume.



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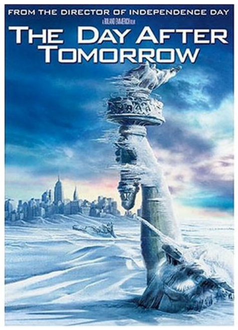
The Day After Tomorrow

What did the film get wrong scientifically?

Thermodynamics

Time Scale



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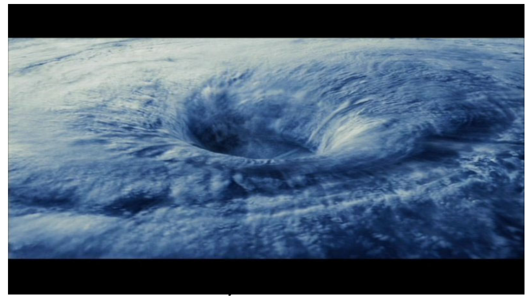
What did the film get wrong scientifically?

Thermodynamics Violated



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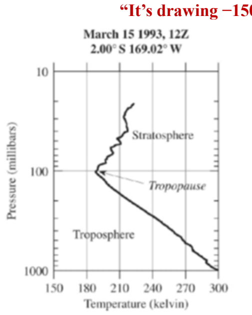
The Day After Tomorrow

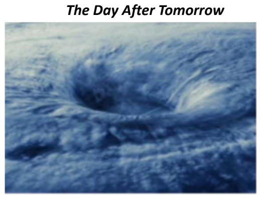
Thermodynamics Violated

"It's drawing −150º air down from the upper troposphere."

The Day After Tomorrow



Pierrehumbert, *Principles of Planetary Climate*



Upper troposphere: ~190 K

The Day After Tomorrow:

-150 °F = −101 °C = 172 K

Only a slight exaggeration.

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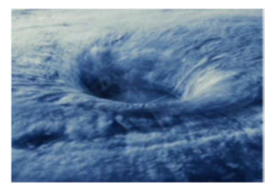
The Day After Tomorrow

Thermodynamics Violated

Professor Hall:
"It's drawing air −150º air down from the upper troposphere."

Professor Rapson:
"Wouldn't it heat up before it reached the surface?"

Professor Hall:
"No, it's descending too fast."



The Day After Tomorrow

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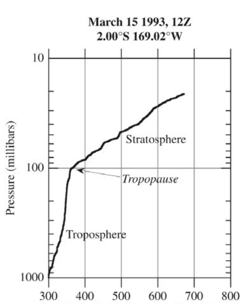
The Day After Tomorrow

Thermodynamics Violated



Pierrehumbert, *Principles of Planetary Climate*

Bringing the air down from the upper troposphere involves increasing the pressure from 0.1 atmosphere to 1 atmosphere, thereby heating it.

Potential temperature: The temperature the air would be if compressed to 1 atmosphere.

Potential temperature of the upper troposphere:

350 K = 77 °C = 171 °F

Definitely would not freeze the fuel lines of RAF helicopters.

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The Day After Tomorrow

What did the film get wrong scientifically?

Time Scale Violated

Most of the northern hemisphere land covered with 30 feet of snow, converted to about 1 meter of water.

Let's say half of all the land, or 15% of the Earth's surface.

70% of the surface is ocean, so about 15/70 or 0.2 meters of ocean evaporated and turned to snow in a few days.

Where did the energy go? How fast could that happen?

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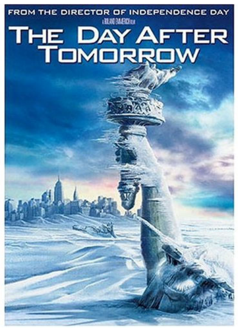
The Day After Tomorrow

What did the film get right scientifically?

What did it get wrong?

Did the portrayal of politics ring true or false?



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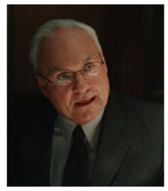
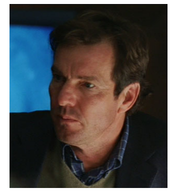
The Day After Tomorrow

Did the portrayal of politics ring true or false?



Vice President: *"Maybe you should stick to science and leave policy to us."*



Scientist: *"Well, we tried that approach. You didn't want to hear about the science when it could have made a difference."*

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Tipping Points

Tipping Points

In climate science, a tipping point is a critical threshold that, when crossed, leads to large and often irreversible changes in the climate system. If tipping points are crossed, they are likely to have severe impacts on human society. Tipping behavior is found across the climate system, in ecosystems, ice sheets, and the circulation of the ocean and atmosphere.

https://en.wikipedia.org/wiki/Tipping_points_in_the_climate_system

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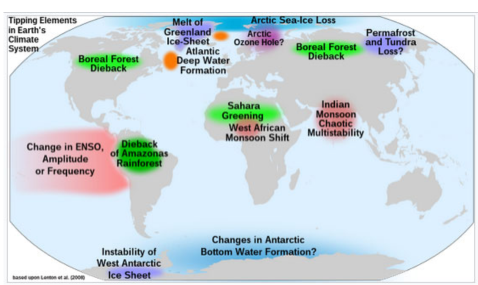
Tipping Points



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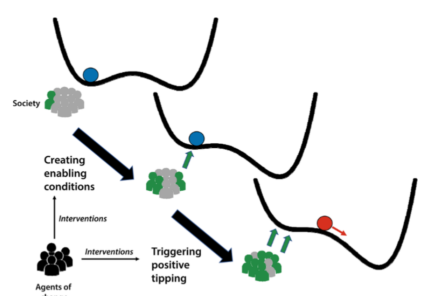
Tipping Points



https://en.wikipedia.org/wiki/Tipping_points_in_the_climate_system

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Tipping Points
Stommel's Model

References

H. Kaper & H. Engler, *Mathematics & Climate*, SIAM Philadelphia 2013, Chapter 6.

Henry Stommel, *Thermohaline Convection with Two Stable Regimes of Flow*, TELLUS **XII** (1961), 224-230.

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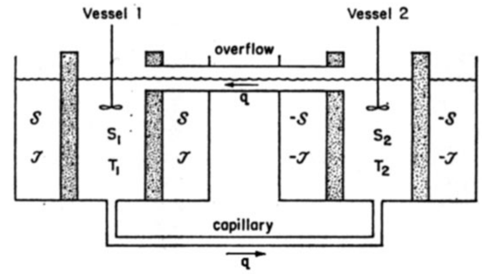
Tipping Points
Stommel's Model

Henry Stommel, *Thermohaline Convection with Two Stable Regimes of Flow*, TELLUS **XII** (1961), 224-230.



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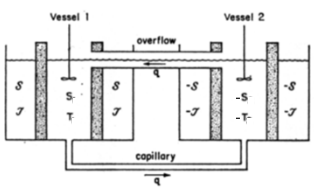
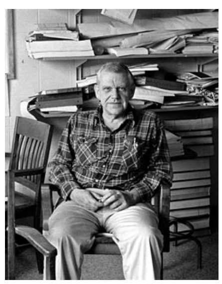
Tipping Points
Stommel's Model

$$\frac{dT}{dt} = c(T^* - T) - |2q|T$$
$$\frac{dS}{dt} = d(S^* - S) - |2q|S$$
$$kq = \rho_1 - \rho_2 = \rho_0(-2\alpha T + 2\beta S)$$





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Tipping Points
Stommel's Model

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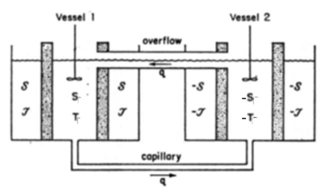

Stommel divided the ocean into two boxes, a low-latitude box, where the water is warm, and a high-latitude box, where the water is cooler. He assumed that the boxes are immersed in baths, where the temperature and salinity are constant and where each box is trying to relax to the temperature and salinity of its bath.

He reduced the system to two variables: the temperature and salinity in one of the boxes.

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Tipping Points
Stommel's Model

$$\frac{dT}{dt} = c(T^* - T) - |2q|T$$
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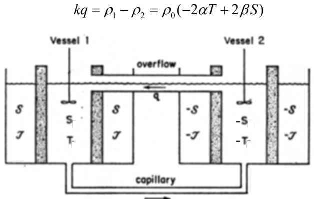

Stommel showed that this simple two box model of ocean circulation could exhibit two stable equilibrium solutions caused by interactions between temperature and salinity.

Changes in the parameters caused by events such as melting glaciers could induce the system to move from one stable state to a different stable state, perhaps an example of "tipping."

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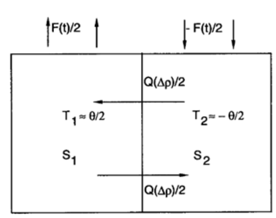
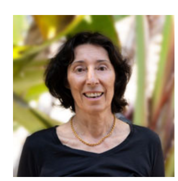
Tipping Points
Cessi's Model

Paola Cessi, A Simple Box Model of Stochastically Forced Thermohaline Flow, *Journal of Physical Oceanography* **24** (1994), 1911-1920.





https://scripps.ucsd.edu/profiles/pcessi

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Tipping Points
Cessi's Model

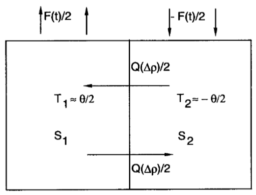

FIG. 1. The two-box model of Stommel (1961). The boxes represent two control volumes at different latitudes. Box 1 is the low-latitude box where the relaxation temperature is $\theta/2$, and box 2 is the high-latitude box where the relaxation temperature is $-\theta/2$.

Cessi constructed a variation of Stommel's model with the same two boxes, each trying to relax to its own equilibrium temperature and salinity.

Cessi was able to reduce the system to two variables by introducing the differences in temperature and salinity between the two boxes.

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Tipping Points
Cessi's Model

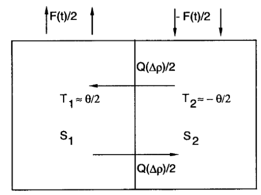

FIG. 1. The two-box model of Stommel (1961). The boxes represent two control volumes at different latitudes. Box 1 is the low-latitude box where the relaxation temperature is $\theta/2$, and box 2 is the high-latitude box where the relaxation temperature is $-\theta/2$.

Cessi deviated from Stommel by assuming that the temperature changes much faster than salinity, and she was able to reduce the system to one equation with the independent variable related to the difference in salinity between the boxes.

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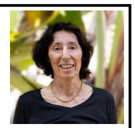
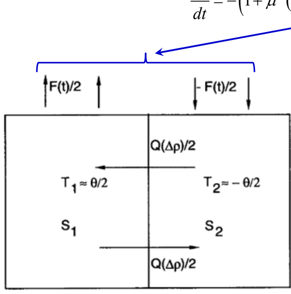
Tipping Points
Cessi's Model



$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + p$$



All variables and parameters are non-dimensional, but y is related to the salinity difference between the two boxes, μ^2 is the ratio between diffusive and advective time scales, and p is related to the infusion of fresh water.

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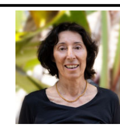
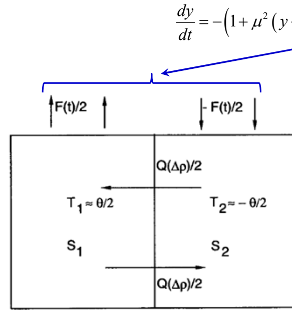
Tipping Points
Cessi's Model



$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + \boxed{p}$$



For now, we assume that the influx p of fresh water is constant, and we treat is as a parameter.

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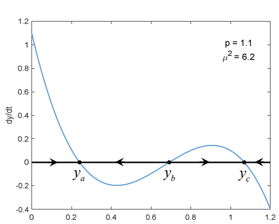
Tipping Points
Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + p$$



For $\mu^2 = 6.2$ and $p = 1.1$, there are three rest points, y_a, y_b, and y_c, as shown in the figure.

The points y_a and y_c are stable, while y_b is unstable. The existence of the two stable rest points is the main point of Stommel's model, and it is the starting point of Cessi's analysis.

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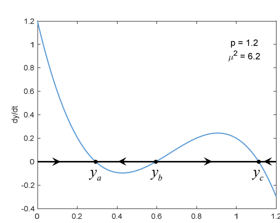
Tipping Points
Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + p$$



For $p = 1.2$, the three rest points continue. Note that y_a and y_b have moved closer together.

The points y_a and y_c continue to be stable, while y_b is unstable.

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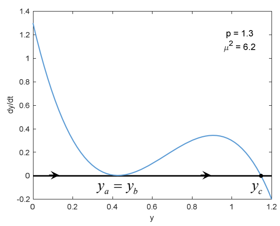
Tipping Points
Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + p$$



For $p = 1.3$, the the rest points y_a and y_b have collided.

The point y_c continues to be stable.

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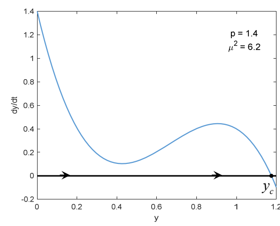
Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + p$$



For $p = 1.4$, the rest points y_a and y_b have disappeared.

The point y_c continues to be stable.

The system has "tipped" to a different state.

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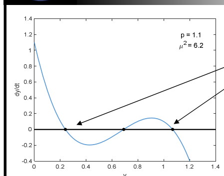
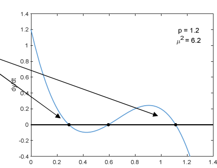
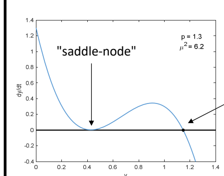
Tipping Points
Cessi's Model



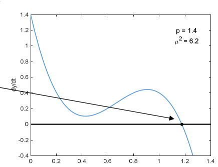
two stable rest points, with an unstable rest point in between



$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + p$$



one stable rest point



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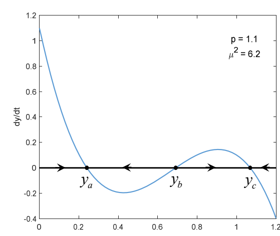
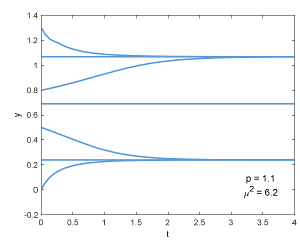
Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + p$$

differential equation



solutions



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Cessi's Model

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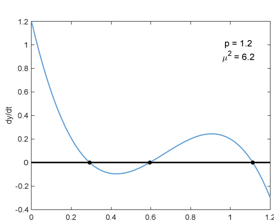
differential equation



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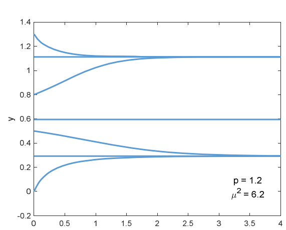

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Cessi's Model

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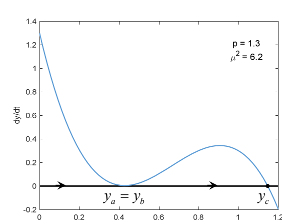
differential equation



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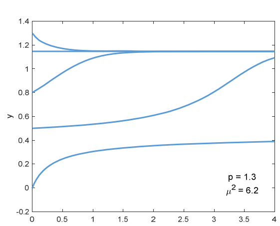

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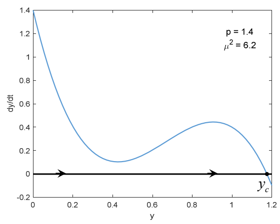
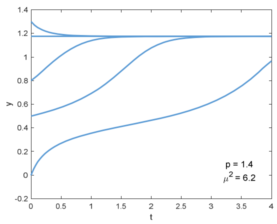
Tipping Points
Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2(y-1)^2\right)y + p$$

differential equation



solutions



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Math 5421
Tipping Points
Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2(y-1)^2\right)y + p$$

Cessi's differential equation can be written in terms of a potential function V.

$$\frac{dy}{dt} = -V'(y) = -\left(1+\mu^2(y-1)^2\right)y + p$$

We can compute V by integrating.

$$V(y) = \int V'(y)dy = \int\left(\left(1+\mu^2(y-1)^2\right)y + p\right)dy$$

$$V(y) = \mu^2\left(\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\right) + \frac{y^2}{2} - py$$

Note that the rest points of the differential equation are the maxima and minima of V.

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Math 5421
Tipping Points
Cessi's Model

$$\frac{dy}{dt} = -V'(y)$$

$$V(y) = \mu^2\left(\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\right) + \frac{y^2}{2} - py$$

Note that the rest points of the differential equation are the local maxima and minima of V.

If V is decreasing at a point y, then $V'(y)$ is negative, so $-V'(y)$ is positive, so y is increasing.

If V is increasing at a point y, then $V'(y)$ is positive, so $-V'(y)$ is negative, so y is decreasing.

Therefore, local minima of V are stable rest points, while local maxima are unstable rest points.

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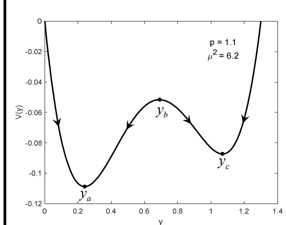
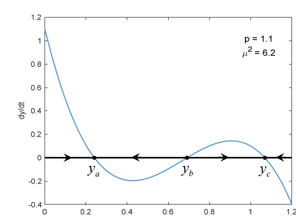
Math 5421
Tipping Points
Cessi's Model

$$V(y) = \mu^2\left(\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\right) + \frac{y^2}{2} - py$$

Think of a ball rolling downhill. The local minima of V are stable rest points, while the local maximum is an unstable rest pont.





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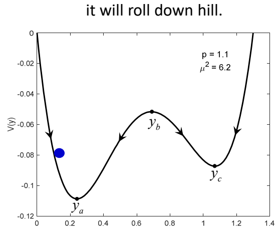
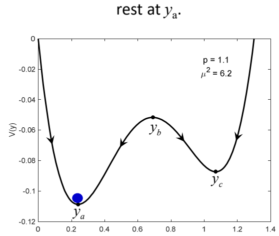
Math 5421
Tipping Points
Cessi's Model

$$V(y) = \mu^2\left(\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\right) + \frac{y^2}{2} - py$$

If a ball starts close to the local minimum y_a, it will roll down hill.



It will eventually come to rest at y_a.



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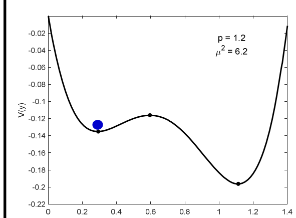
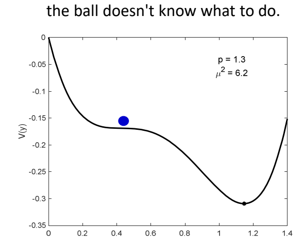
Math 5421
Tipping Points
Cessi's Model

$$V(y) = \mu^2\left(\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\right) + \frac{y^2}{2} - py$$

If the parameter p changes to 1.2, the ball will stay at rest at y_a



If the parameter p changes to 1.3, the ball doesn't know what to do.



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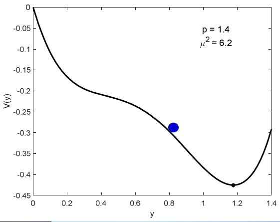
Math 5421
Tipping Points

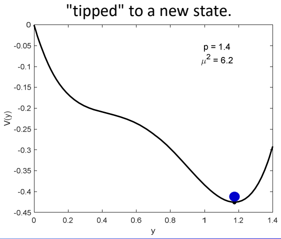
Cessi's Model

$$V(y)=\mu^2\left(\frac{y^4}{4}-\frac{2y^3}{3}+\frac{y^2}{2}\right)+\frac{y^2}{2}-py$$

If the parameter p changes to 1.4, the ball moves toward y_c



The ball eventually comes to rest at y_c. The system has "tipped" to a new state.



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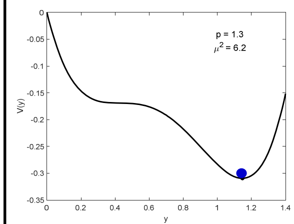
Tipping Points

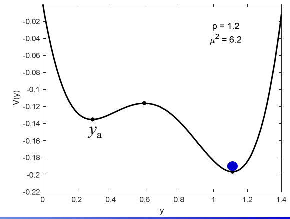
Cessi's Model

$$V(y)=\mu^2\left(\frac{y^4}{4}-\frac{2y^3}{3}+\frac{y^2}{2}\right)+\frac{y^2}{2}-py$$

If the parameter p changes back to 1.3, the ball stays at its new state.



If the parameter p changes back to 1.2, the ball stays at rest at y_c, even though y_a has come back into existence.



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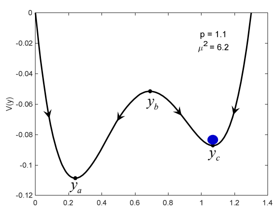
Math 5421
Tipping Points

Cessi's Model

$$V(y)=\mu^2\left(\frac{y^4}{4}-\frac{2y^3}{3}+\frac{y^2}{2}\right)+\frac{y^2}{2}-py$$



If the parameter p changes back to its original value of 1.1, the ball continues to remain in the its new state.

The system has flipped to a different state, and it has not returned to its original state, despite the parameter having returned to its original value.

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Math 5421
Tipping Points

Cessi's Model

$$\frac{dy}{dt}=-\left(1+\mu^2\left(y-1\right)^2\right)y+p$$

Cessi was interested in exploring the conditions under which the system would return to its original state instead of tipping to a new state.

She added a step function in time to the parameter p.

$$P(t)=p+q(t),\quad \text{where}\quad q(t)=\begin{cases}0, & t<0\\ \Delta & 0\le t\le\tau\\ 0 & t>\tau\end{cases}$$

The differential equation becomes

$$\frac{dy}{dt}=-\left(1+\mu^2\left(y-1\right)^2\right)y+P\left(t\right)$$

At time 0, the fresh water influx p suddenly increases by Δ, where it stays for time τ, when it returns to its original value.

Does the system flip or not?

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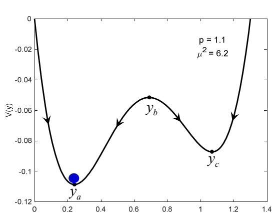
Math 5421
Tipping Points

Cessi's Model

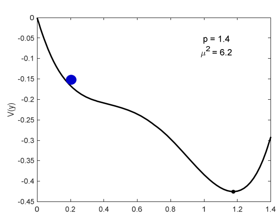
$$\frac{dy}{dt}=-\left(1+\mu^2\left(y-1\right)^2\right)y+P\left(t\right)$$

No flip scenario

The system starts at the rest point y_a.



The parameter suddenly changes.



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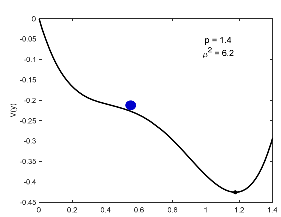
Math 5421
Tipping Points

Cessi's Model

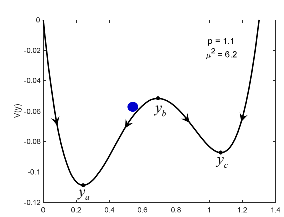
$$\frac{dy}{dt}=-\left(1+\mu^2\left(y-1\right)^2\right)y+P\left(t\right)$$

No flip scenario

The system moves toward the new stable state.



The parameter suddenly changes back to its original value.



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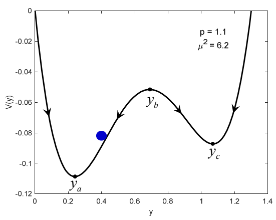
Tipping Points

Cessi's Model

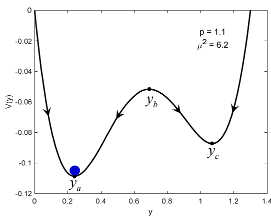
$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + P\left(t\right)$$

No flip scenario

The system moves back toward the original stable state.



The system eventually returns to its original state.



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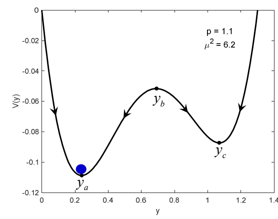
Math 5421
Tipping Points

Cessi's Model

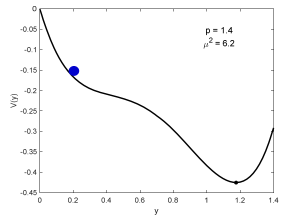
$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + P\left(t\right)$$

Flip scenario

The system starts at the rest point y_a.



The parameter suddenly changes.



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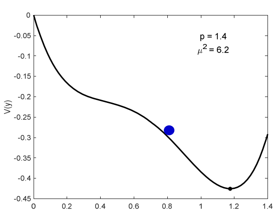
Tipping Points

Cessi's Model

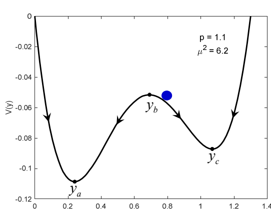
$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + P\left(t\right)$$

Flip scenario

The system moves toward the new stable state.



The parameter suddenly changes back to its original value.



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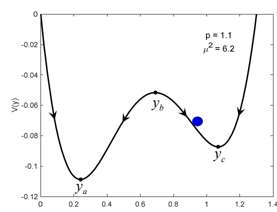
Tipping Points

Cessi's Model

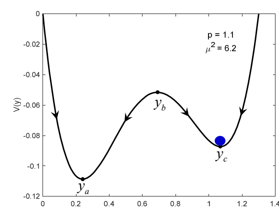
$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + P\left(t\right)$$

Flip scenario

The system continures to move toward the new stable state.



The system eventually arrives at the new state.



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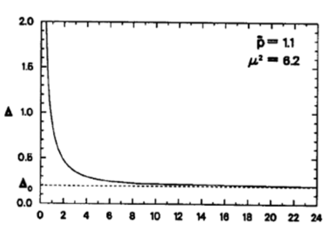
Tipping Points

Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + P\left(t\right)$$

$$P(t) = p + q(t), \quad \text{where } q(t) = \begin{cases} 0, & t<0 \\ \Delta & 0 \le t \le \tau \\ 0 & t > \tau \end{cases}$$



Cessi explored the values of the parameters Δ and τ to classify where flipping occurs.

She found a critical curve in the (τ,Δ)-plane. Above the curve, flipping occurs, while below the curve, there is no flip.

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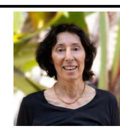
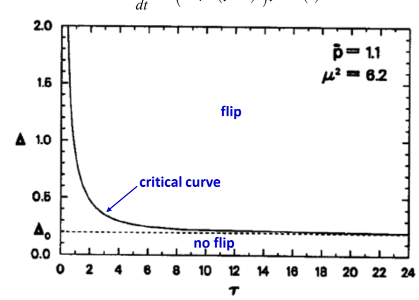
Tipping Points



Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + P\left(t\right)$$



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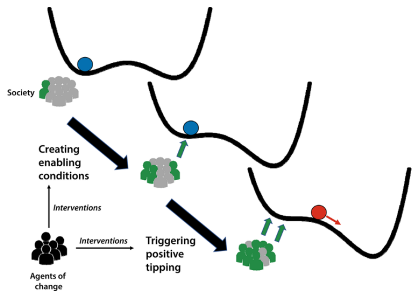

https://en.wikipedia.org/wiki/Tipping_points_in_the_climate_system

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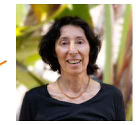
Tipping Points



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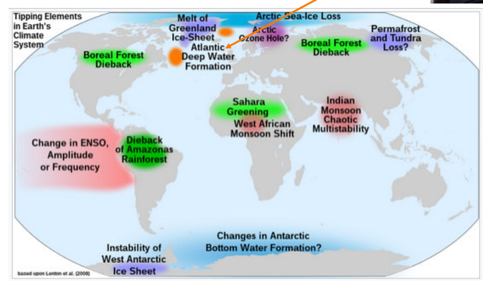

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