Math 5421

An Introduction to Mathematical Climate Models

Spring 2025
1:25 – 3:20 Tuesdays and Thursdays
Blegen Hall 155

Richard McGehee, Instructor
458 Vincent Hall
mcgehee@umn.edu
www-users.cse.umn.edu/~mcgehee/

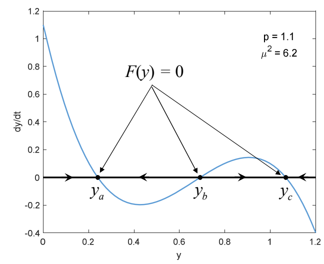
course website
https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/

2

Math 5421
Assignment 9

Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + p = F(y)$$



The rest points occur at points y where $F(y) = 0$.

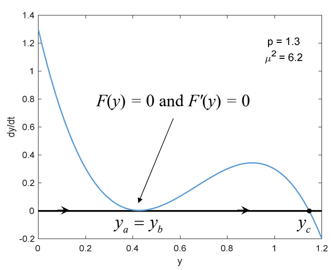
The points y_a and y_c are stable, while y_b is unstable.

3

Math 5421
Assignment 9

Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + p = F(y)$$



A saddle-node occurs where $F(y) = 0$ and $F'(y) = 0$.

4

Math 5421
Bifurcation Theory

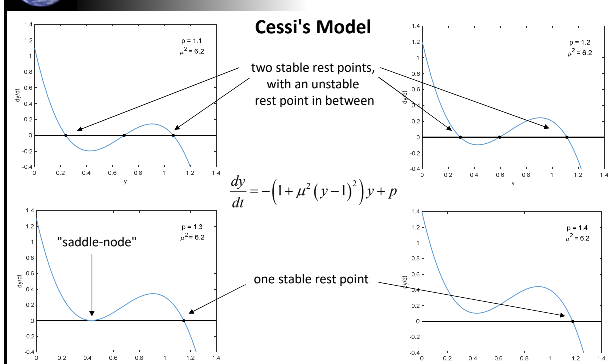
Reference

H. Kaper & H. Engler, *Mathematics & Climate*, SIAM Philadelphia 2013, Chapter 5.

5

Math 5421
Bifurcation Theory

Cessi's Model



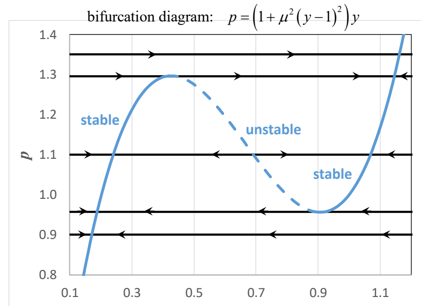
$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + p$$

6

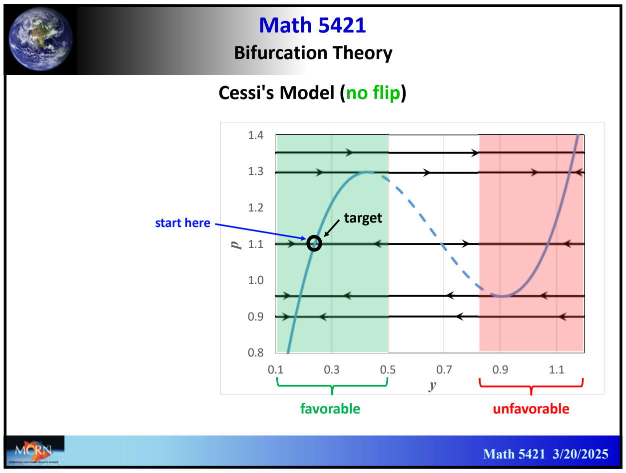
Math 5421
Bifurcation Theory

Cessi's Model

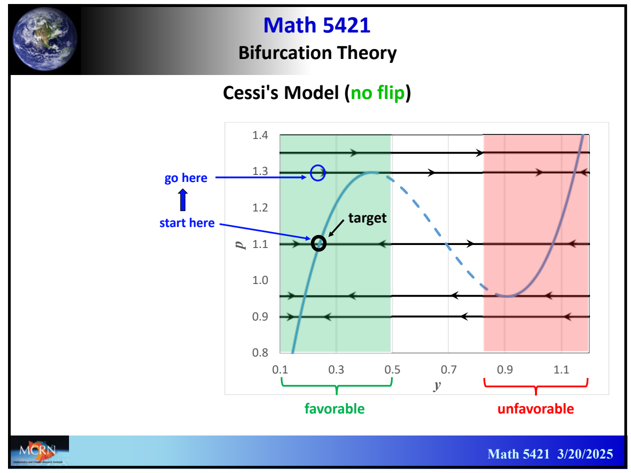
bifurcation diagram: $p = \left(1+\mu^2\left(y-1\right)^2\right)y$



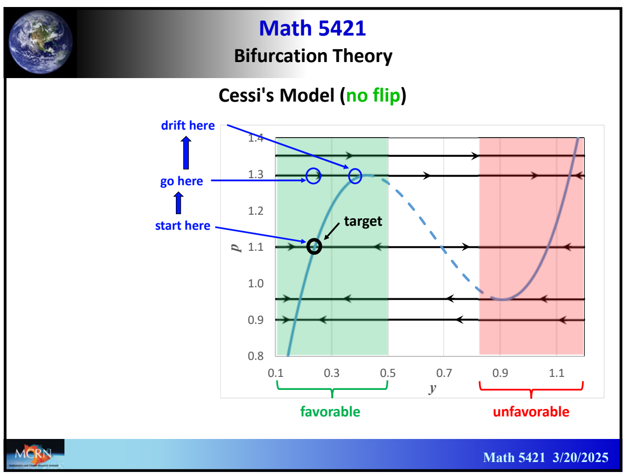
7


Math 5421
Bifurcation Theory
Cessi's Model (no flip)
start here
target
favorable
unfavorable
Math 5421 3/20/2025

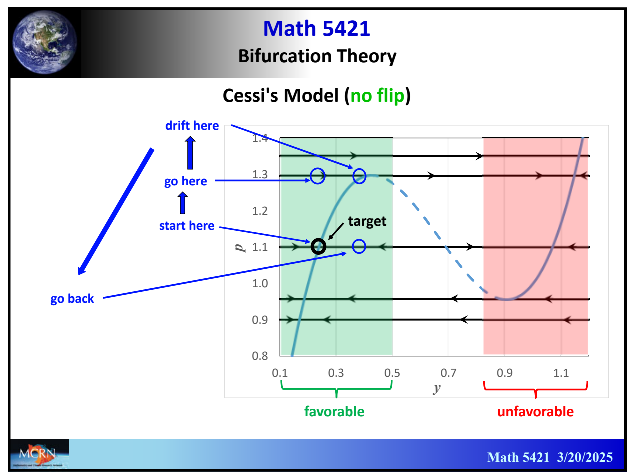
8


Math 5421
Bifurcation Theory
Cessi's Model (no flip)
go here
start here
target
favorable
unfavorable
Math 5421 3/20/2025

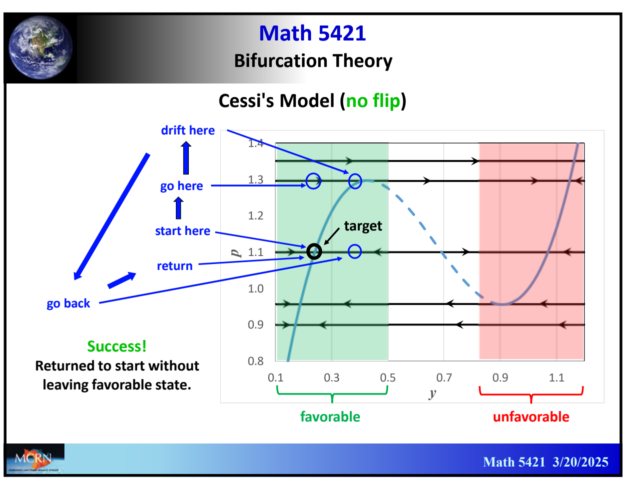
9


Math 5421
Bifurcation Theory
Cessi's Model (no flip)
drift here
go here
start here
target
favorable
unfavorable
Math 5421 3/20/2025

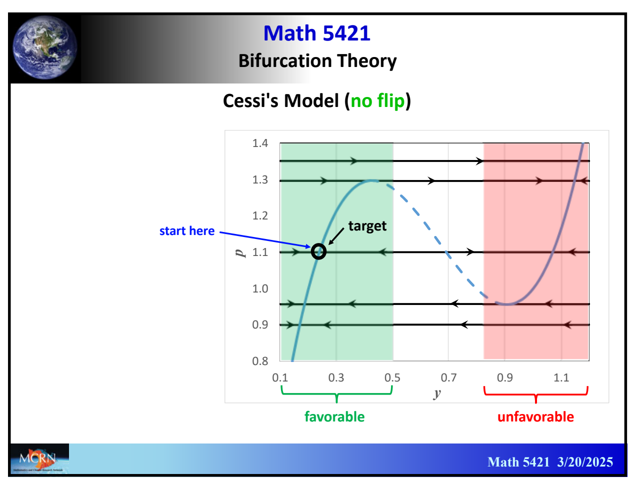
10


Math 5421
Bifurcation Theory
Cessi's Model (no flip)
drift here
go here
start here
go back
target
favorable
unfavorable
Math 5421 3/20/2025

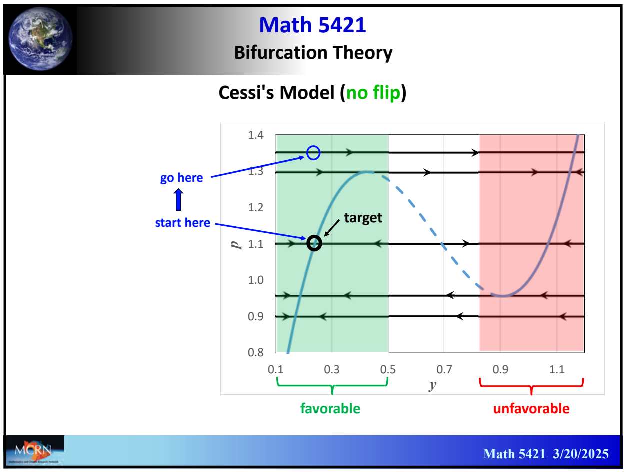
11


Math 5421
Bifurcation Theory
Cessi's Model (no flip)
drift here
go here
start here
return
go back
target
Success!
Returned to start without leaving favorable state.
favorable
unfavorable
Math 5421 3/20/2025

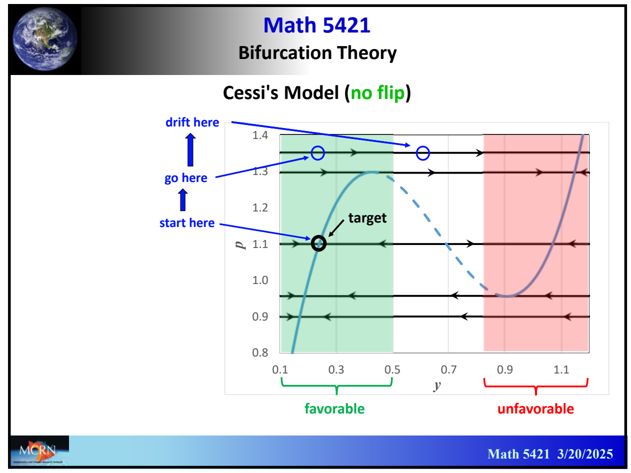
12


Math 5421
Bifurcation Theory
Cessi's Model (no flip)
start here
target
favorable
unfavorable
Math 5421 3/20/2025

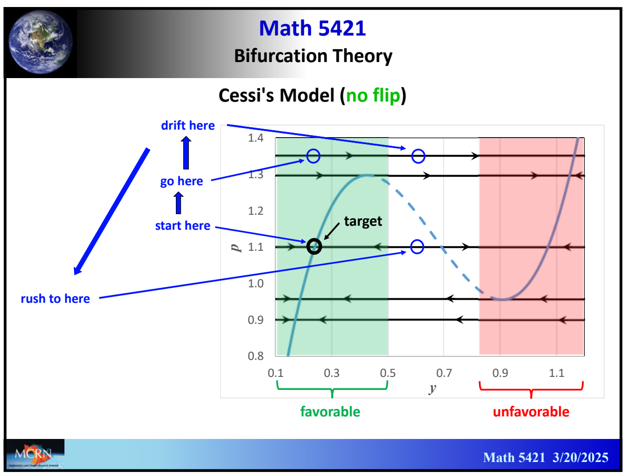
13



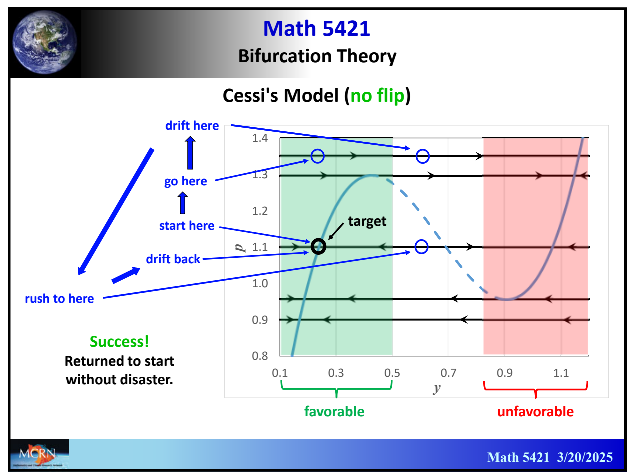
14



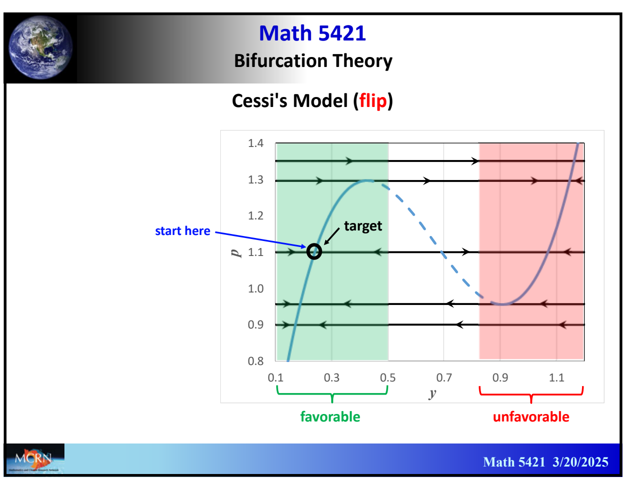
15



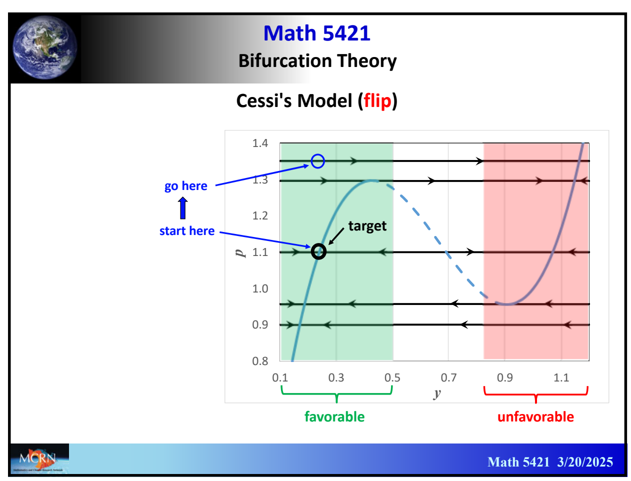
16



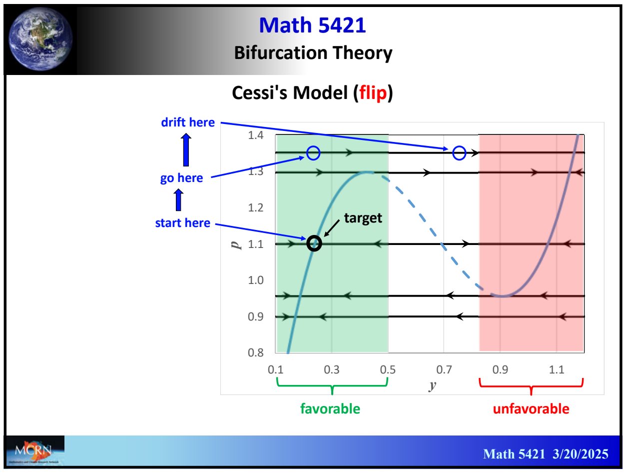
17



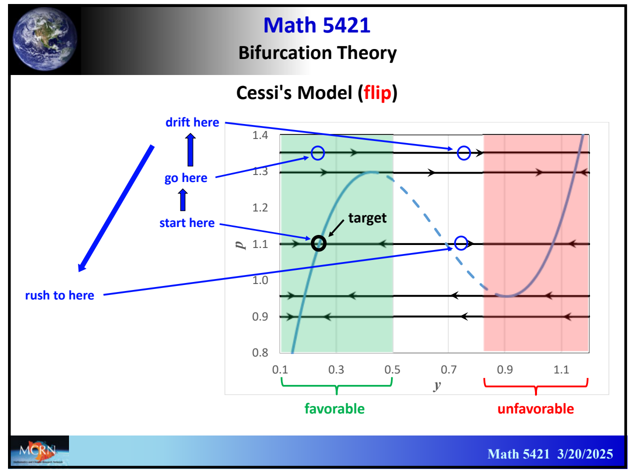
18



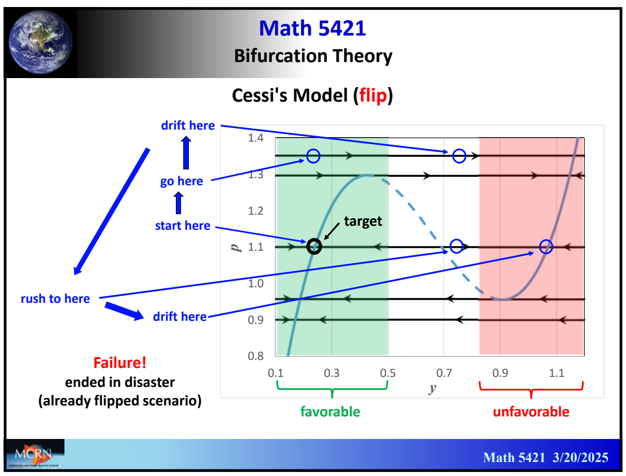
19



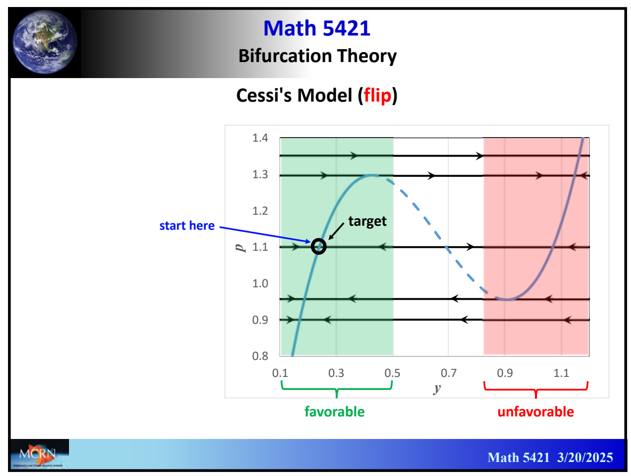
20



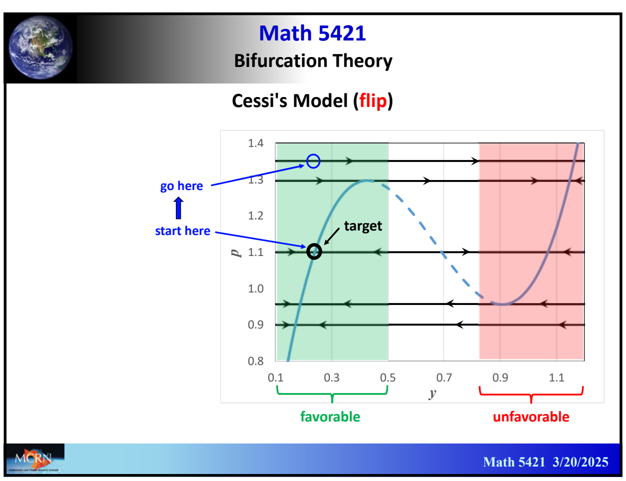
21



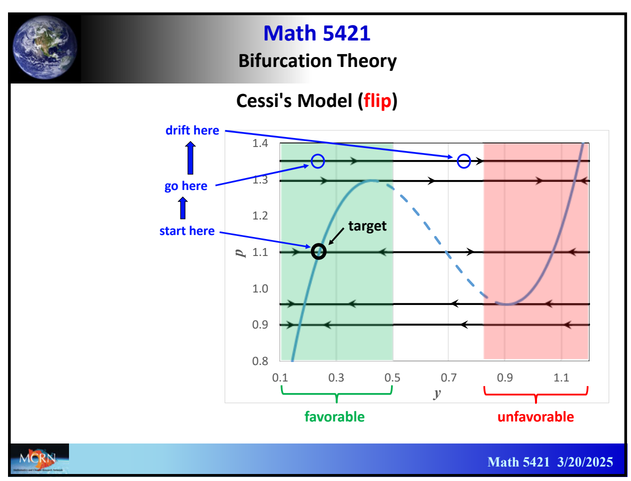
22



23

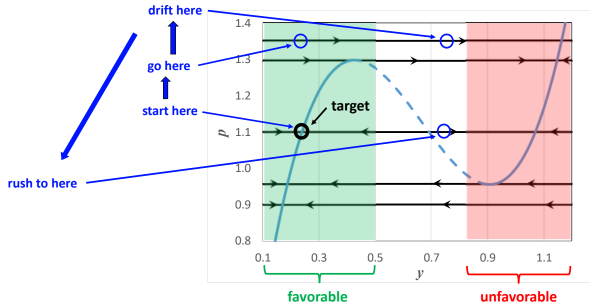


24



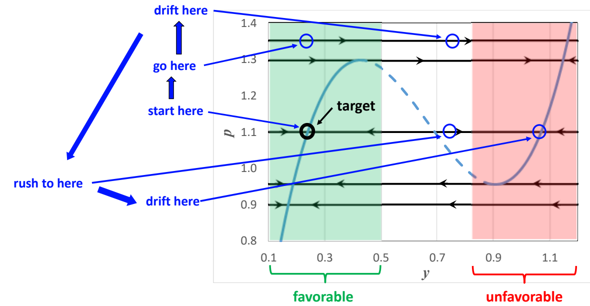
25

Math 5421
Bifurcation Theory
Cessi's Model (flip)



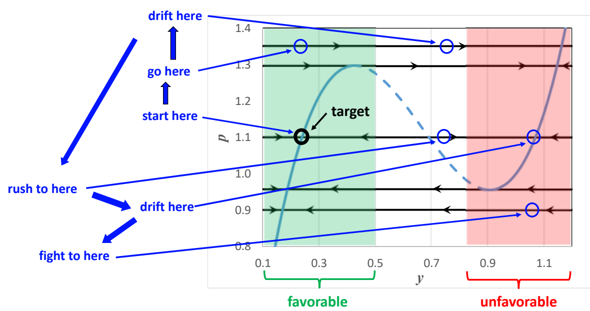
26

Math 5421
Bifurcation Theory
Cessi's Model (flip)



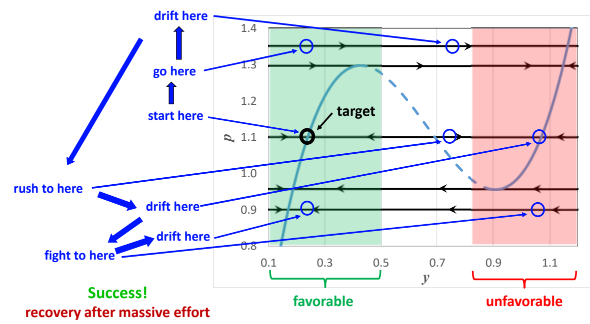
27

Math 5421
Bifurcation Theory
Cessi's Model (flip)



28

Math 5421
Bifurcation Theory
Cessi's Model (flip)

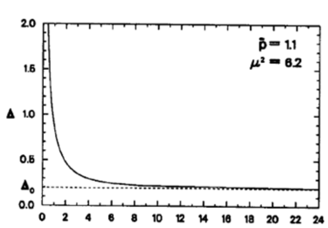


29

Math 5421
Bifurcation Theory
Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + P(t)$$

$$P(t) = p + q(t), \quad \text{where } q(t) = \begin{cases} 0, & t<0 \\ \Delta & 0 \le t \le \tau \\ 0 & t > \tau \end{cases}$$



Cessi explored the values of the parameters Δ and τ to classify where flipping occurs.

She found a critical curve in the (τ,Δ)-plane. Above the curve, flipping occurs, while below the curve, there is no flip.

30

Math 5421
An Introduction to Mathematical Climate Models

31

Math 5421
Welander's Model

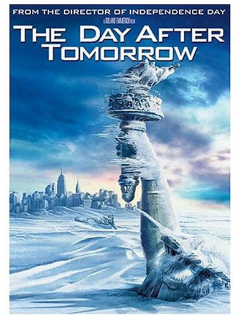
Reference

Pierre Welander, A simple heat-salt oscillator, *Dynamics of Atmospheres and Oceans* **6** (1982) 233-242.

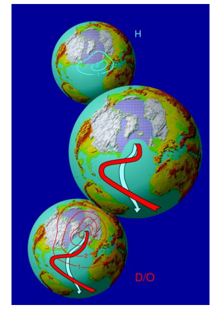
32

Math 5421
Welander's Model

Heinrich and Dansgaard-Oeschger events



What did the film get right scientifically?



http://www.pik-potsdam.de/~stefan/sampleimages.html

33

Math 5421
Welander's Model

INDEPENDENT

April 12, 2018

Gulf Stream current at 'record low' with potentially devastating consequences for weather, warn scientists

The Atlantic meridional overturning circulation (AMOC), the system of currents that transports warm water from the tropics via the Gulf Stream to the North Atlantic, plays a major role in regulating the world's climate.

A fictional depiction of AMOC's collapse was portrayed in ***The Day After Tomorrow***, and while the film's events were exaggerated, scientists say severe weather events are likely to result from the ongoing changes.

34

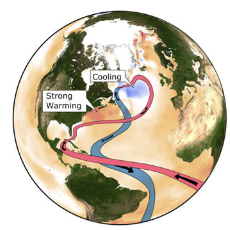
Math 5421
Welander's Model

The Washington Post

April 11, 2018

The oceans' circulation hasn't been this sluggish in 1,000 years. That's bad news.

The Atlantic Ocean circulation that carries warmth into the Northern Hemisphere's high latitudes is slowing down because of climate change, a team of scientists asserted Wednesday, suggesting one of the most feared consequences is already coming to pass.



Nature volume 556, pages191–196 (2018)

35

Math 5421
Welander's Model

Current Atlantic Meridional Overturning Circulation weakest in last millennium

The Atlantic Meridional Overturning Circulation (AMOC)—one of Earth's major ocean circulation systems—redistributes heat on our planet and has a major impact on climate. Here, we compare a variety of published proxy records to reconstruct the evolution of the AMOC since about AD 400. A fairly consistent picture of the AMOC emerges: **after a long and relatively stable period, there was an initial weakening starting in the nineteenth century, followed by a second, more rapid, decline in the mid-twentieth century, leading to the weakest state of the AMOC occurring in recent decades.**

NATURE GEOSCIENCE | VOL 14 | MARCH 2021 | 118–120 | www.nature.com/naturegeoscience118

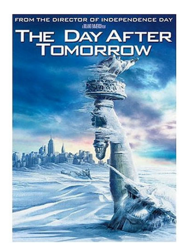
36

Math 5421
Welander's Model

The Day After Tomorrow/Film synopsis

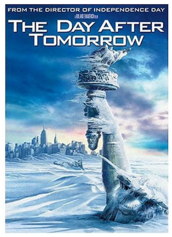
After climatologist Jack Hall (Dennis Quaid) is largely ignored by U.N. officials when presenting his environmental concerns, his research proves true when an enormous "superstorm" develops, setting off catastrophic natural disasters throughout the world. Trying to get to his son, Sam (Jake Gyllenhaal), who is trapped in New York with his friend Laura (Emmy Rossum) and others, Jack and his crew must travel by foot from Philadelphia, braving the elements, to get to Sam before it's too late.

Google Search



37

Math 5421
Welander's Model



Dansgaard-Oeschger Events

"Global warming" can cause the Northern Hemisphere to cool.

Melting ice can lower the salinity of the North Atlantic, causing a decrease in the flow of the Atlantic Meridional Overturning Circulation (AMOC), slowing the heat transfer to the Northern Hemisphere.

This phenomenon is believed to have caused the Younger Dryas.

Math 5421 3/20/2025

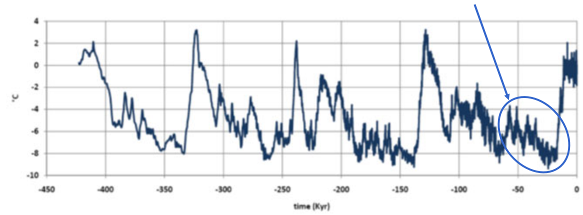
38

Math 5421
Welander's Model

Recent (last 400 Kyr) Temperature Cycles

Vostok Ice Core Data

What's with these oscillations?



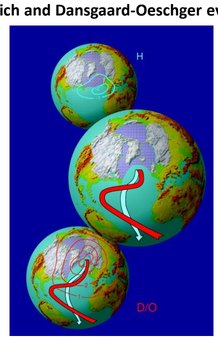
J.R. Petit, *et al* (1999) Climate and atmospheric history of the past 420,000 years from the Vostok ice core, Antarctica, *Nature* 399, 429-436.

Math 5421 3/20/2025

39

Math 5421
Welander's Model

Heinrich and Dansgaard-Oeschger events



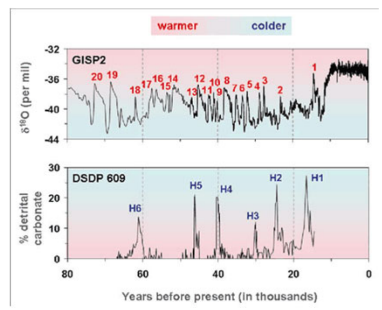
http://www.pik-potsdam.de/~stefan/sampleimages.html

Math 5421 3/20/2025

40

Math 5421
Welander's Model

Heinrich and Dansgaard-Oeschger events



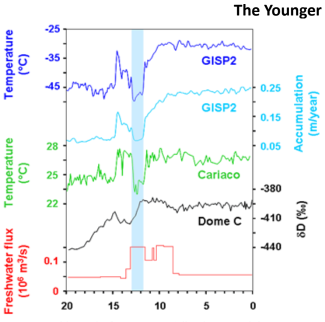
http://www.ncdc.noaa.gov/paleo/abrupt/data3.html

Math 5421 3/20/2025

41

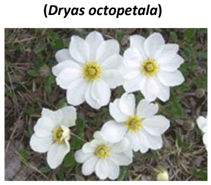
Math 5421
Welander's Model

The Younger Dryas



http://www.ncdc.noaa.gov/paleo/abrupt/data4.html

Mountain Avens
(*Dryas octopetala*)



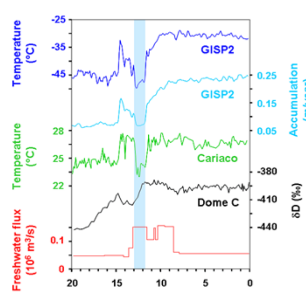
https://www.ncdc.noaa.gov/abrupt-climate-change/The%20Younger%20Dryas

Math 5421 3/20/2025

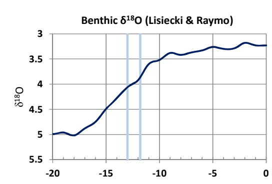
42

Math 5421
Welander's Model

The Younger Dryas



Only a minor impact on ice volume.



Math 5421 3/20/2025

43

Math 5421
Welander's Model

What caused the Dansgaard-Oeschger Oscillations?

For glacial cyles, there is overwhelming evidence that they are "paced" by the Milankovitch cycles.

None of the Milankovitch cycles has periods short enough to trigger the Dansgaard-Oeschger events, and we can't think of any other external triggers.

Could they occur spontaneously?

Math 5421 3/20/2025

44

Math 5421
Welander's Model

What caused the Dansgaard-Oeschger oscillations?

They could be self-oscillations in the natural dynamics of ocean circulation.

Pierre Welander, A simple heat-salt oscillator, *Dynamics of Atmospheres and Oceans* **6** (1982) 233-242.

R/V WeeLander is a 23-foot-long Beach Master work boat, informally named in honor of Professor Pierre Welander (1925—1996).

Math 5421 3/20/2025

45

Math 5421
Welander's Model

What caused the Dansgaard-Oeschger oscillations?

They could be self-oscillations in the natural dynamics of ocean circulation.

Welander constructed a simple (***conceptual!***) box model of ocean circulation and showed that the interactions of temperature and salinity with the atmosphere, the surface ocean, and the deep ocean could create self-oscillations.

R/V WeeLander is a 23-foot-long Beach Master work boat, informally named in honor of Professor Pierre Welander (1925—1996).

Math 5421 3/20/2025

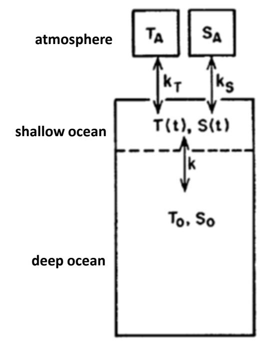
46

Math 5421
Welander's Model

Welander's model divides the Earth into three components: the "atmosphere", the "shallow ocean", and the "deep ocean".

The "shallow ocean" is represented by its temperature T and its salinity S, both of which vary with time.

The "atmosphere" is a ficticious entity encompassing and simplifying all the interactions of the shallow ocean with the rest of the planet. It is assumed to have an "effective" temperature of T_A and an "effective" salinity of S_A.



Math 5421 3/20/2025

47

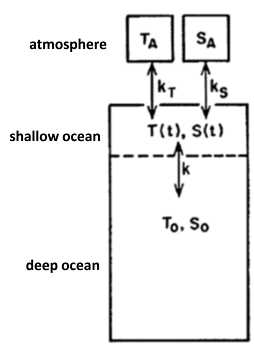
Math 5421
Welander's Model

The "deep ocean" is assumed to have a constant temperature T_0 and a constant salinity S_0.

The interaction between the shallow ocean and the atmosphere is modeled as a dynamic transfer of relaxation to equilibrium:

$$\frac{dT}{dt} = k_T(T_A - T),$$
$$\frac{dS}{dt} = k_S(S_A - S),$$

where k_T and k_S are constants.



Math 5421 3/20/2025

48

Math 5421
Dynamical Systems

How do we analyze equations like these?

$$\frac{dT}{dt} = k_T(T_A - T)$$
$$\frac{dS}{dt} = k_S(S_A - S)$$

Dynamical Systems

H. Kaper & H. Engler, *Mathematics & Climate*, SIAM Philadelphia 2013, Chapter 4.

Math 5421 3/20/2025

49

Math 5421
Dynamical Systems

Discussion

How do we solve this differential equation?

$$\frac{dT}{dt} = k_T(T_A - T)$$

50

Math 5421
Dynamical Systems

Discussion

How do we solve this differential equation?

$$\frac{dT}{dt} = k_T(T_A - T)$$

"separate variables"

$$\frac{dT}{T_A - T} = k_T dt \qquad \frac{dT}{T - T_A} = -k_T dt \qquad \int \frac{dT}{T - T_A} = \int -k_T dt$$

$$\ln(T - T_A) = -k_T t + c$$

$$T - T_A = e^{-k_T t + c} = e^c e^{-k_T t} = Ce^{-k_T t}$$

arbitrary constants

$$\boxed{T(t) = T_A + Ce^{-k_T t}}$$

general solution

51

Math 5421
Dynamical Systems

Discussion

How do we solve this differential equation?

$$\frac{dT}{dt} = k_T(T_A - T)$$

$$\boxed{T(t) = T_A + Ce^{-k_T t}}$$

general solution

check

$$\frac{dT}{dt} = \frac{d}{dt}\left(T_A + Ce^{-k_T t}\right) = 0 + Ce^{-k_T t}(-k_T) = -k_T Ce^{-k_T t}$$

$$k_T(T_A - T) = k_T\left(T_A - \left(T_A + Ce^{-k_T t}\right)\right) = -k_T Ce^{-k_T t}$$

checks!

52

Math 5421
Dynamical Systems

Discussion

$$\frac{dT}{dt} = k_T(T_A - T)$$

Find an equilibrium solution.

53

Math 5421
Dynamical Systems

Discussion

$$\frac{dT}{dt} = k_T(T_A - T)$$

Find an equilibrium solution.

Answer

$$0 = \frac{dT}{dt} = k_T(T_A - T)$$

$$\boxed{T(t) = T_A}$$

54

Math 5421
Dynamical Systems

Discussion

$$\frac{dT}{dt} = k_T(T_A - T)$$

equilibrium solution: $T(t) = T_A$

Is it stable?

55

Math 5421
Dynamical Systems

Discussion

$$\frac{dT}{dt} = k_T(T_A - T)$$

equilibrium solution: $T(t) = T_A$

Is it stable?

Yes!

general solution: $T(t) = T_A + Ce^{-k_T t}$

$$\lim_{t\to\infty} T(t) = T_A$$

Note that we are assuming that k_T is positive.

56

Math 5421
Dynamical Systems

Discussion

How do we solve this "initial value problem"?

$$\frac{dT}{dt} = k_T(T_A - T), \qquad \textit{equation}$$
$$T = T_0 \text{ when } t = 0. \qquad \textit{initial value}$$

57

Math 5421
Dynamical Systems

Discussion

How do we solve this "initial value problem"?

$$\frac{dT}{dt} = k_T(T_A - T),$$
$$T = T_0 \text{ when } t = 0.$$

general solution: $T(t) = T_A + Ce^{-k_T t}$

$$T(0) = T_A + Ce^{-k_T 0} = T_A + C = T_0$$
$$C = T_0 - T_A$$
$$T(t) = T_A + Ce^{-k_T t}$$
$$\boxed{T(t) = T_A + (T_0 - T_A)e^{-k_T t}}$$

58

Math 5421
Dynamical Systems

Discussion

How do we solve this "initial value problem"?

$$\frac{dT}{dt} = k_T(T_A - T),$$
$$T = T_0 \text{ when } t = 0.$$

solution: $T(t) = T_A + (T_0 - T_A)e^{-k_T t}$

check:

$$T'(t) = -k_T(T_0 - T_A)e^{-k_T t}$$
$$k_T(T_A - T) = k_T\left(-(T_0 - T_A)e^{-k_T t}\right)$$

equal

$$T(0) = T_A + (T_0 - T_A)e^{-k_T 0} = T_A + (T_0 - T_A) = T_0$$

checks!

59

Math 5421
Dynamical Systems

Discussion

How do we solve this "initial value problem"?

$$\frac{dT}{dt} = k_T(T_A - T), \qquad \text{equation}$$
$$T = T_0 \text{ when } t = 0. \qquad \text{initial value}$$

alternative approach:
Introduce the departure from equilibrium

$$y = T - T_A, \qquad T = T_A + y$$

$$\frac{dy}{dt} = \frac{dT}{dt} = k_T(T_A - T) = k_T(-y) = -k_T y$$

new initial value problem

$$\boxed{\frac{dy}{dt} = -k_T y, \quad y = T_0 - T_A \text{ when } t = 0.}$$

60

Math 5421
Dynamical Systems

Discussion

How do we solve this "initial value problem"?

$$\boxed{\frac{dy}{dt} = -k_T y, \quad y = T_0 - T_A \text{ when } t = 0.}$$

61

Math 5421
Dynamical Systems

Discussion

How do we solve this "initial value problem"?

$$\frac{dy}{dt} = -k_T y, \quad y = T_0 - T_A \text{ when } t = 0.$$

"separate variables"

$$\frac{dy}{y} = -k_T dt \qquad \frac{dy}{y} = -k_T dt \qquad \int_{T_0-T_A}^{y} \frac{dy}{y} = \int_0^t -k_T dt$$

$$\ln y\Big|_{T_0-T_A}^{y} = -k_T t\Big|_0^t \qquad \ln y - \ln(T_0 - T_A) = -k_T t \qquad \frac{y}{T_0 - T_A} = e^{-k_T t}$$

$$y(t) = (T_0 - T_A)e^{-k_T t}$$

stable

$$y \to 0 \text{ as } t \to 0$$

Math 5421 3/20/2025

62

Math 5421
Dynamical Systems

Summary

$$\frac{dT}{dt} = k_T(T_A - T)$$

This differential equation has a stable equilibrium solution $T = T_A$.

Remember Welander?

T is the Welander temperature.

Salinity:

$$\frac{dS}{dt} = k_S(S_A - S)$$

This differential equation also has a stable equilibrium solution $S = S_A$.

Math 5421 3/20/2025

63

Math 5421
Dynamical Systems

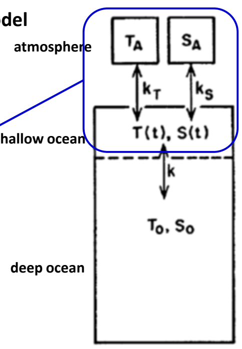
Welander's Model

The interaction between the shallow ocean and the atmosphere is modeled as a dynamic transfer of relaxation to equilibrium:

$$\frac{dT}{dt} = k_T(T_A - T),$$
$$\frac{dS}{dt} = k_S(S_A - S),$$

where k_T and k_S are positive constants. This system has a stable equilibrium point at

$$(T, S) = (T_A, S_A).$$



Math 5421 3/20/2025

64

Math 5421
Dynamical Systems

Welander's Model

Atmosphere - Ocean Surface Interaction

system of differential equations

$$\begin{cases} \dfrac{dT}{dt} = k_T(T_A - T) = -k_T(T - T_A), & T(0) = T_0, \\ \dfrac{dS}{dt} = k_S(S_A - S) = -k_S(S - S_A), & S(0) = S_0. \end{cases}$$

initial condition

Matrix Notation

differential equation

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} -k_T & 0 \\ 0 & -k_S \end{bmatrix}\begin{bmatrix} T - T_A \\ S - S_A \end{bmatrix}, \qquad \begin{bmatrix} T \\ S \end{bmatrix}(0) = \begin{bmatrix} T_0 \\ S_0 \end{bmatrix}.$$

initial condition

singular! *ONE* vector equation

Solution

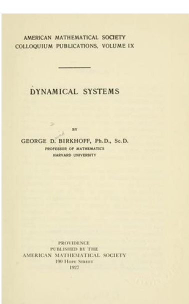
$$\begin{bmatrix} T \\ S \end{bmatrix}(t) = \begin{bmatrix} T(t) \\ S(t) \end{bmatrix} = \begin{bmatrix} T_A + (T_0 - T_A)e^{-k_T t} \\ S_A + (S_0 - S_A)e^{-k_S t} \end{bmatrix}$$

Math 5421 3/20/2025

65

Math 5421
Dynamical Systems

Dynamical Systems



AMERICAN MATHEMATICAL SOCIETY
COLLOQUIUM PUBLICATIONS, VOLUME IX

DYNAMICAL SYSTEMS

BY
GEORGE D. BIRKHOFF, Ph.D., Sc.D.
PROFESSOR OF MATHEMATICS
HARVARD UNIVERSITY

PROVIDENCE
PUBLISHED BY THE
AMERICAN MATHEMATICAL SOCIETY
1927

https://openlibrary.org/works/OL86546W/Dynamical_systems

https://en.wikipedia.org/wiki/George_David_Birkhoff

https://en.wikipedia.org/wiki/Henri_Poincar%C3%A9

Math 5421 3/20/2025

66