Math 5421
An Introduction to Mathematical Climate Models

Spring 2025
1:25 – 3:20 Tuesdays and Thursdays
Blegen Hall 155

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course website
https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/

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Dynamical Systems

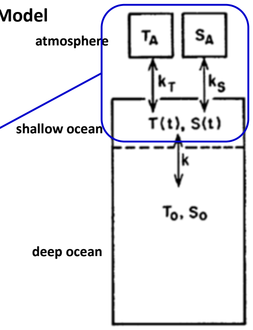
Welander's Model

The interaction between the shallow ocean and the atmosphere is modeled as a dynamic transfer of relaxation to equilibrium:

$$\frac{dT}{dt} = k_T(T_A - T),$$
$$\frac{dS}{dt} = k_S(S_A - S),$$

where k_T and k_S are positive constants. This system has a stable equilibrium point at

$$(T,S) = (T_A, S_A).$$



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Dynamical Systems

Welander's Model

Atmosphere - Ocean Surface Interaction

system of differential equations

$$\frac{dT}{dt} = k_T(T_A - T) = -k_T(T - T_A), \quad T(0) = T_0,$$
$$\frac{dS}{dt} = k_S(S_A - S) = -k_S(S - S_A), \quad S(0) = S_0.$$

initial condition

Matrix Notation

differential equation

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} -k_T & 0 \\ 0 & -k_S \end{bmatrix}\begin{bmatrix} T - T_A \\ S - S_A \end{bmatrix}, \qquad \begin{bmatrix} T \\ S \end{bmatrix}(0) = \begin{bmatrix} T_0 \\ S_0 \end{bmatrix}.$$

initial condition

singular! *ONE* vector equation

Solution

$$\begin{bmatrix} T \\ S \end{bmatrix}(t) = \begin{bmatrix} T(t) \\ S(t) \end{bmatrix} = \begin{bmatrix} T_A + (T_0 - T_A)e^{-k_T t} \\ S_A + (S_0 - S_A)e^{-k_S t} \end{bmatrix}$$

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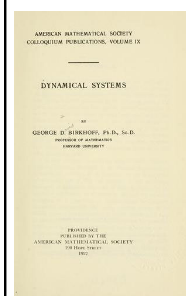
Dynamical Systems

Dynamical Systems



https://openlibrary.org/works/OL86546W/Dynamical_systems

https://en.wikipedia.org/wiki/George_David_Birkhoff

https://en.wikipedia.org/wiki/Henri_Poincar%C3%A9

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Dynamical Systems

Dynamical Systems

Basic idea:
The dependence of the solution on ***initial conditions*** is just as important as its dependence on ***time***.

Inital value problem

differential equation

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} -k_T & 0 \\ 0 & -k_S \end{bmatrix}\begin{bmatrix} T - T_A \\ S - S_A \end{bmatrix}, \qquad \begin{bmatrix} T \\ S \end{bmatrix}(0) = \begin{bmatrix} T_0 \\ S_0 \end{bmatrix}.$$

solution

$$\begin{bmatrix} T \\ S \end{bmatrix}(t) = \begin{bmatrix} T(t) \\ S(t) \end{bmatrix} = \begin{bmatrix} T_A + (T_0 - T_A)e^{-k_T t} \\ S_A + (S_0 - S_A)e^{-k_S t} \end{bmatrix}$$

flow

$$\varphi\left(\begin{bmatrix} T_0 \\ S_0 \end{bmatrix}, t\right) = \begin{bmatrix} T_A + (T_0 - T_A)e^{-k_T t} \\ S_A + (S_0 - S_A)e^{-k_S t} \end{bmatrix}$$

initial condition — time

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Dynamical Systems

Basic idea:
The dependence of the solution on ***initial conditions*** is just as important as its dependence on ***time***.

General notation:

$$x = \begin{bmatrix} T \\ S \end{bmatrix} \qquad f(x) = \begin{bmatrix} -k_T & 0 \\ 0 & -k_S \end{bmatrix}\begin{bmatrix} T - T_A \\ S - S_A \end{bmatrix} \qquad \xi = \begin{bmatrix} T_0 \\ S_0 \end{bmatrix} \qquad \dot{x} = \frac{dx}{dt} \qquad \dot{x} = f(x)$$

state variable — vector field — initial condition — dot notation — differential equation

flow

$$\varphi(\xi, t) = \varphi\left(\begin{bmatrix} T_0 \\ S_0 \end{bmatrix}, t\right) = \begin{bmatrix} T_A + (T_0 - T_A)e^{-k_T t} \\ S_A + (S_0 - S_A)e^{-k_S t} \end{bmatrix}$$

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Dynamical Systems

The dependence of the solution on ***initial conditions*** is just as important as its dependence on ***time***.

$$x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n, \quad f:\mathbb{R}^n \to \mathbb{R}^n$$

intital value problem

$$\dot{x} = f(x)$$
$$x(0) = \xi$$

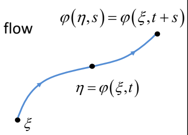
The initial value problem generates a flow

$$\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

with properties

intital condition $\varphi(\xi,0) = \xi$

"group property" $\varphi(\varphi(\xi,t),s) = \varphi(\xi,t+s)$



If we start the system at state ξ and follow the solution for time t, then restart the system at the new state and follow the solution for time s, we end up at the same state as starting at ξ and following for time $t+s$.

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Dynamical Systems

Example

initial value problem

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} -k_T & 0 \\ 0 & -k_S \end{bmatrix}\begin{bmatrix} T - T_A \\ S - S_A \end{bmatrix}$$
$$\begin{bmatrix} T \\ S \end{bmatrix}(0) = \begin{bmatrix} T_0 \\ S_0 \end{bmatrix}$$

flow

$$\varphi\left(\begin{bmatrix} T_0 \\ S_0 \end{bmatrix}, t\right) = \begin{bmatrix} T_A + (T_0 - T_A)e^{-k_T t} \\ S_A + (S_0 - S_A)e^{-k_S t} \end{bmatrix}$$

group property

$$\varphi\left(\begin{bmatrix} T_0 \\ S_0 \end{bmatrix}, t+s\right) = \begin{bmatrix} T_A + (T_0 - T_A)e^{-k_T(t+s)} \\ S_A + (S_0 - S_A)e^{-k_S(t+s)} \end{bmatrix} = \begin{bmatrix} T_A + (T_0 - T_A)e^{-k_T(t+s)} \\ S_A + (S_0 - S_A)e^{-k_S(t+s)} \end{bmatrix}$$

$$\varphi\left(\varphi\left(\begin{bmatrix} T_0 \\ S_0 \end{bmatrix}, t\right), s\right) = \begin{bmatrix} T_A + \left(T_A + (T_0 - T_A)e^{-k_T t} - T_A\right)e^{-k_T s} \\ S_A + \left(S_A + (S_0 - S_A)e^{-k_S t} - S_A\right)e^{-k_S s} \end{bmatrix} = \begin{bmatrix} T_A + (T_0 - T_A)e^{-k_T(t+s)} \\ S_A + (S_0 - S_A)e^{-k_S(t+s)} \end{bmatrix}$$

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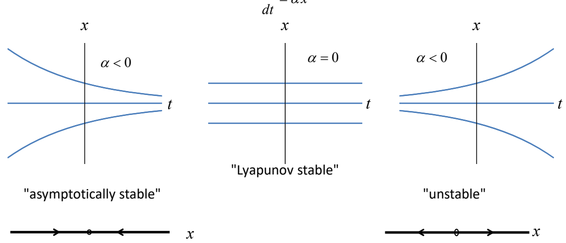
Dynamical Systems

Example

$$\frac{dx}{dt} = \alpha x$$



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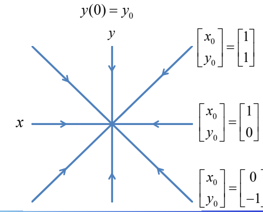
"Phase Plane"

Example

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\dot{x} = -x$$
$$\dot{y} = -y$$
$$x(0) = x_0$$
$$y(0) = y_0$$

$$\varphi\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, t\right) = \begin{bmatrix} x_0 e^{-t} \\ y_0 e^{-t} \end{bmatrix}$$



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Dynamical Systems

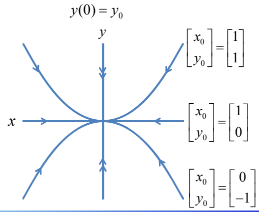
"Phase Plane"

Example

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\dot{x} = -x$$
$$\dot{y} = -2y$$
$$x(0) = x_0$$
$$y(0) = y_0$$

$$\varphi\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, t\right) = \begin{bmatrix} x_0 e^{-t} \\ y_0 e^{-2t} \end{bmatrix}$$



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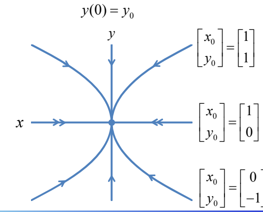
"Phase Plane"

Example

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\dot{x} = -2x$$
$$\dot{y} = -y$$
$$x(0) = x_0$$
$$y(0) = y_0$$

$$\varphi\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, t\right) = \begin{bmatrix} x_0 e^{-2t} \\ y_0 e^{-t} \end{bmatrix}$$



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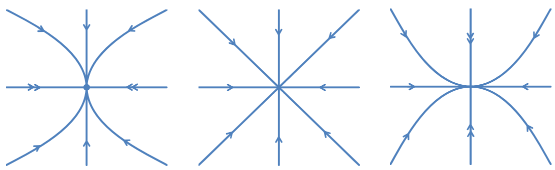
"Phase Plane"
Example: "stable nodes"

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\qquad \begin{bmatrix}x\\y\end{bmatrix}(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$$

$$\dot{x}=ax,\quad \dot{y}=by,\quad x(0)=x_0,\quad y(0)=y_0$$

$$\varphi\left(\begin{bmatrix}x_0\\y_0\end{bmatrix},t\right)=\begin{bmatrix}x_0e^{at}\\y_0e^{bt}\end{bmatrix}$$



$a<b<0$ $a=b<0$ $b<a<0$

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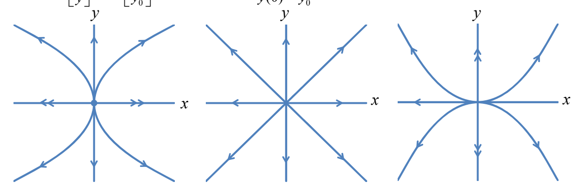
"Phase Plane"
Example: "unstable nodes"

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\qquad \begin{bmatrix}x\\y\end{bmatrix}(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$$

$$\dot{x}=ax,\quad \dot{y}=by,\quad x(0)=x_0,\quad y(0)=y_0$$

$$\varphi\left(\begin{bmatrix}x_0\\y_0\end{bmatrix},t\right)=\begin{bmatrix}x_0e^{at}\\y_0e^{bt}\end{bmatrix}$$



$a>b>0$ $a=b>0$ $0<a<b$

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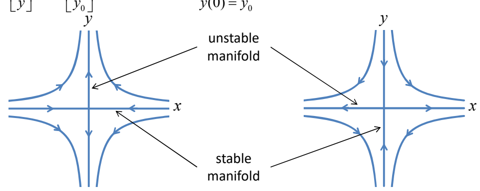
"Phase Plane"
Example: "saddles"

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\qquad \begin{bmatrix}x\\y\end{bmatrix}(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$$

$$\dot{x}=ax,\quad \dot{y}=by,\quad x(0)=x_0,\quad y(0)=y_0$$

$$\varphi\left(\begin{bmatrix}x_0\\y_0\end{bmatrix},t\right)=\begin{bmatrix}x_0e^{at}\\y_0e^{bt}\end{bmatrix}$$



$a<0<b$ $b<0<a$

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Dynamical Systems

Welander's Model

The interaction between the shallow ocean and the atmosphere is modeled as a dynamic transfer of relaxation to equilibrium:

$$\frac{dT}{dt}=k_T(T_A-T),$$
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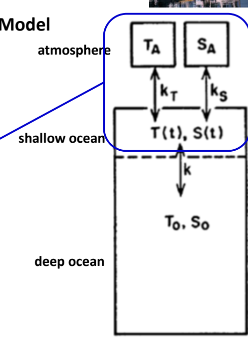

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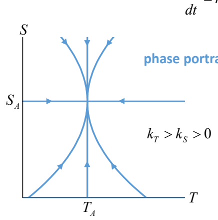
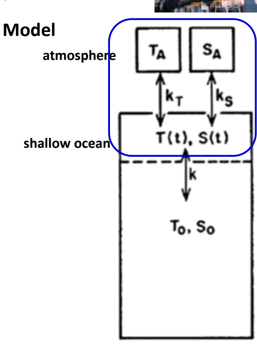
Welander's Model

$$\frac{dT}{dt}=k_T(T_A-T),$$
$$\frac{dS}{dt}=k_S(S_A-S),$$





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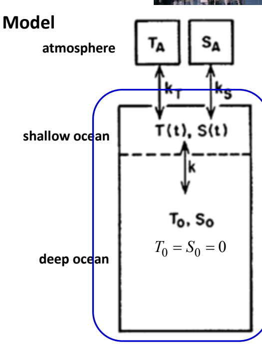
Dynamical Systems

Welander's Model

The interaction between the shallow ocean and the deep ocean is both simpler and more complex. The coordinates are chosen so that its temperature and salinity at both zero. The complication is that the mixing is determined by the density, which is a function of the temperature and salinity.

Welander simplified this complexity down to an assumption that the mixing is either zero (when the density of the shallow ocean is large) and one (when the density is low).



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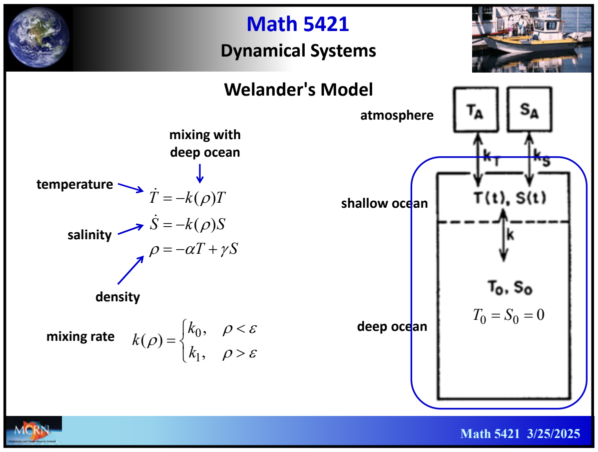

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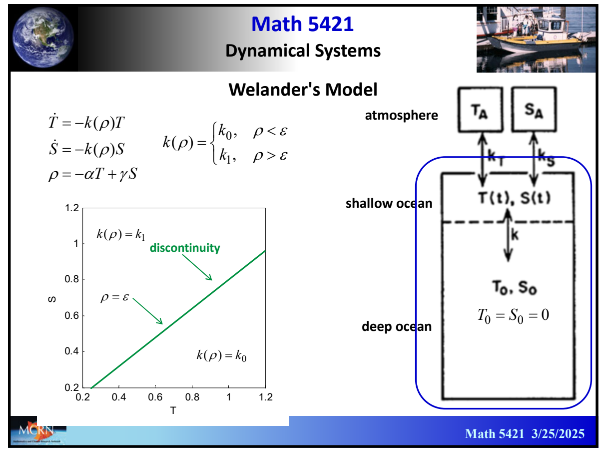

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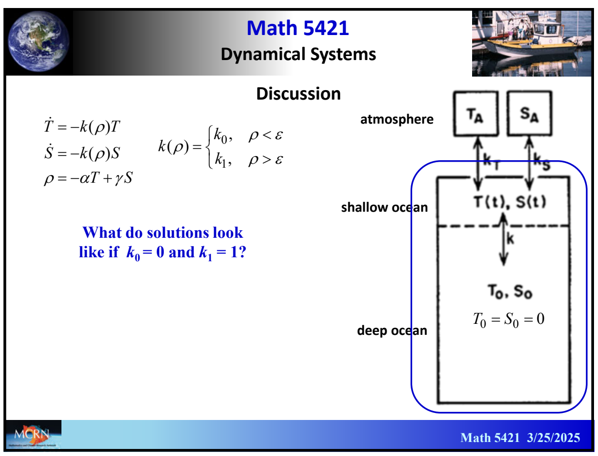

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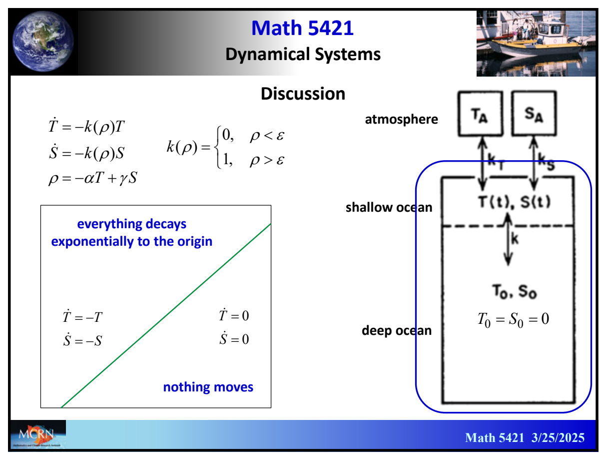

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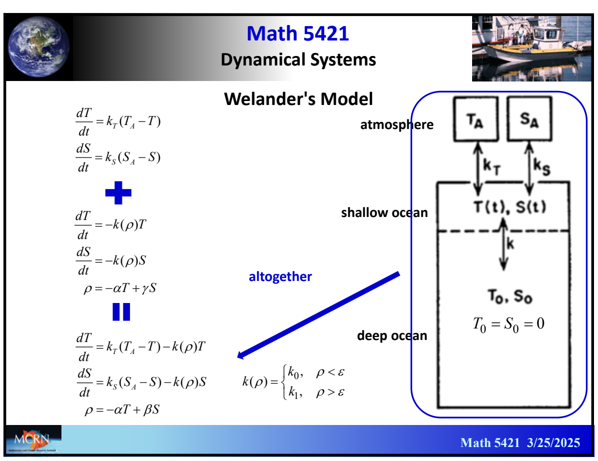

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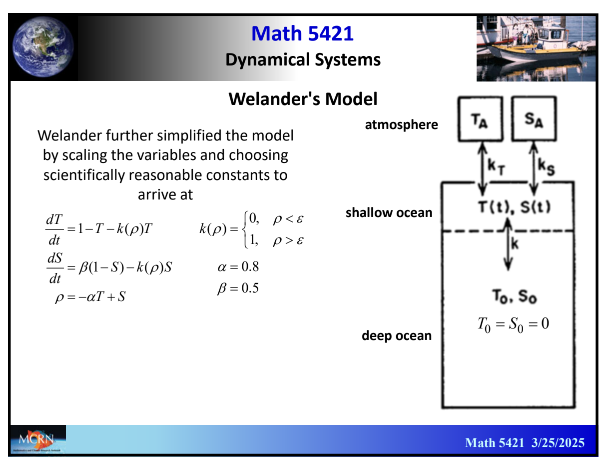

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Welander's Model

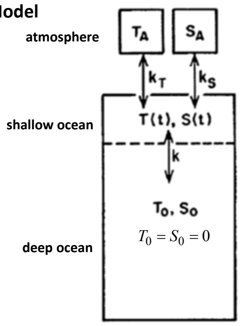
Welander further simplified the model by scaling the variables and choosing scientifically reasonable constants to arrive at

$$\frac{dT}{dt} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \boxed{\varepsilon} \\ 1, & \rho > \boxed{\varepsilon} \end{cases}$$

$$\frac{dS}{dt} = \boxed{\beta}(1-S) - k(\rho)S \qquad \alpha = 0.8$$

$$\rho = -\boxed{\alpha}T + S \qquad \beta = 0.5$$

Three essential parameters: α, β, and ε



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Scaling Idea

$$\frac{dx}{dt} = -ax + bx^2$$

two parameters

equilibria: $-ax + bx^2 = 0 \qquad x = 0$, and $x = a/b$

$0 \qquad a/b \qquad x$

scale: $x = c\xi \qquad \frac{dx}{dt} = c\frac{d\xi}{dt} = -ax + bx^2 = -ac\xi + bc^2\xi^2$

$$\frac{d\xi}{dt} = -a\xi + bc\xi^2$$

choose scaling constant: $c = a/b \qquad \frac{d\xi}{dt} = -a\left(\xi - \xi^2\right)$

only one essential parameter

$0 \qquad 1 \qquad \xi$

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Welander's Model

$$\frac{dT}{dt} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$

$$\frac{dS}{dt} = \beta(1-S) - k(\rho)S \qquad \alpha = 0.8$$

$$\rho = -\alpha T + S \qquad \beta = 0.5$$

equilibria

$$1 - T - k(\rho)T = 0$$
$$\beta(1-S) - k(\rho)S = 0$$

two cases

$\rho < \varepsilon$, so $k(\rho) = 0$ | $\rho > \varepsilon$, so $k(\rho) = 1$

$1 - T = 0 \qquad 1 - 2T = 0$

$\beta(1-S) = 0 \qquad \beta - (\beta+1)S = 0$

$$\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1/2 \\ \beta/(\beta+1) \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$$

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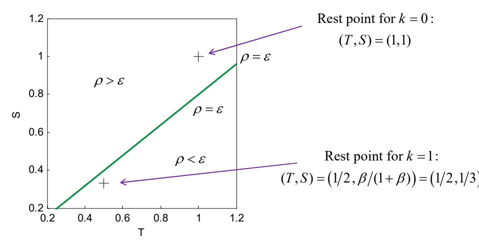
Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$
$$\rho = -\alpha T + S$$



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Dynamical Systems

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$
$$\rho = -\alpha T + S$$

At $\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\rho = -\alpha + 1 = 1/5$. At $\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$, $\rho = -\alpha/2 + 1/3 = -2/5 + 1/3 = -1/15$.

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Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$
$$\rho = -\alpha T + S$$

At $\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\rho = -\alpha + 1 = 1/5$. At $\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$, $\rho = -\alpha/2 + 1/3 = -2/5 + 1/3 = -1/15$.

Wait a minute!!

The computation yielding an equilibrium at $(T,S) = (1,1)$ is valid only if $\rho < \varepsilon$, but the density at that point is $\rho = 1/5$.

The computation yielding an equilibrium at $(T,S) = (1/2, 1/3)$ is valid only if $\rho > \varepsilon$, but the density at that point is $\rho = -1/15$.

If ε is small, neither condition holds.

No equilibrium points!

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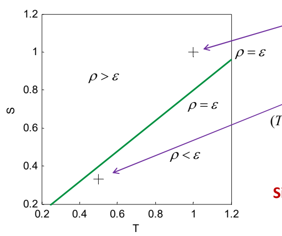
Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$
$$\rho = -\alpha T + S$$



Rest point for $k = 0$:
$(T,S) = (1,1)$

$\boxed{\text{Valid only if } \varepsilon > 1/5.}$

Rest point for $k = 1$:
$(T,S) = (1/2, \beta/(1+\beta)) = (1/2, 1/3)$

$\boxed{\text{Valid only if } \varepsilon < -1/15.}$

Since ε is small, neither rest point is located where it is valid.

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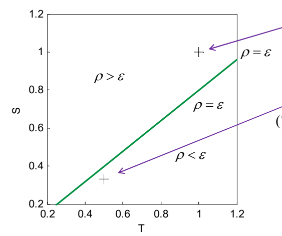
Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$
$$\rho = -\alpha T + S$$



Rest point for $k = 0$:
$(T,S) = (1,1)$

$\boxed{\text{Valid only if } \varepsilon > 1/5.}$

Rest point for $k = 1$:
$(T,S) = (1/2, \beta/(1+\beta)) = (1/2, 1/3)$

$\boxed{\text{Valid only if } \varepsilon < -1/15.}$

Let's proceed anyway.

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Math 5421
Dynamical Systems

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \begin{matrix} \alpha = 0.8 \\ \beta = 0.5 \end{matrix}$$
$$\rho = -\alpha T + S$$

First consider the case $k(\rho) = 1$, i.e., $\rho > \varepsilon$.

For $k(\rho) = 1$: $\quad \begin{matrix} \dot{T} = 1 - 2T \\ \dot{S} = \beta - (\beta+1)S \end{matrix}$

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1/2 \\ \beta/(\beta+1) \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -(\beta+1) \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -3/2 \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

What is the phase portrait for this equation?

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Math 5421
Dynamical Systems

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \begin{matrix} \alpha = 0.8 \\ \beta = 0.5 \end{matrix}$$
$$\rho = -\alpha T + S$$

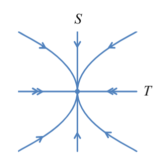
First consider the case $k(\rho) = 1$, i.e., $\rho > \varepsilon$.

For $k(\rho) = 1$: $\quad \begin{matrix} \dot{T} = 1 - 2T \\ \dot{S} = \beta - (\beta+1)S \end{matrix}$

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1/2 \\ \beta/(\beta+1) \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -(\beta+1) \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -3/2 \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

Since T decreases to equilibrium faster than S, the decay to equilibrium looks like:



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Math 5421
Dynamical Systems

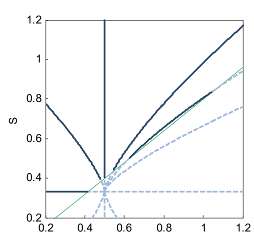
Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \begin{matrix} \alpha = 0.8 \\ \beta = 0.5 \end{matrix}$$
$$\rho = -\alpha T + S$$

First consider the case $k(\rho) = 1$, i.e., $\rho > \varepsilon$.

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -3/2 \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

Actually, it looks like:



The solid lines are above the the line $\rho = \varepsilon$, where the equation is valid, while the dotted lines show where the equation is invalid.

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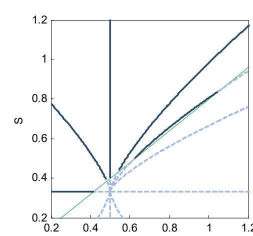
Math 5421
Dynamical Systems

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \begin{matrix} \alpha = 0.8 \\ \beta = 0.5 \end{matrix}$$
$$\rho = -\alpha T + S$$

Starting above the discontinuity line $\rho = \varepsilon$, solutions of Welander's model try to asymptotically approach the equilibrium at $(1/2, 1/3)$. Before they arrive, they hit the discontinuity line, and the system changes to the case $k(\rho) = 1$.



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Math 5421
Dynamical Systems

Welander's Model

$$\dot T = 1 - T - k(\rho)T \qquad k(\rho)=\begin{cases}0, & \rho<\varepsilon\\ 1, & \rho>\varepsilon\end{cases} \qquad \alpha = 0.8$$
$$\dot S = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

Now consider the case $k(\rho) = 0$, *i.e.*, $\rho < \varepsilon$.

For $k(\rho)=0$: $\dot T = 1 - T$, $\dot S = \beta - \beta S$

$$\frac{d}{dt}\begin{bmatrix}T\\S\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}-1 & 0\\0 & -\beta\end{bmatrix}\begin{bmatrix}T\\S\end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix}T\\S\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}-1 & 0\\0 & -1/2\end{bmatrix}\begin{bmatrix}T\\S\end{bmatrix}$$

What is the phase portrait for this equation?

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Math 5421
Dynamical Systems

Welander's Model

$$\dot T = 1 - T - k(\rho)T \qquad k(\rho)=\begin{cases}0, & \rho<\varepsilon\\ 1, & \rho>\varepsilon\end{cases} \qquad \alpha = 0.8$$
$$\dot S = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

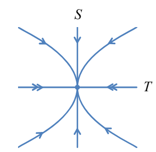
Now consider the case $k(\rho) = 0$, *i.e.*, $\rho < \varepsilon$.

For $k(\rho)=0$: $\dot T = 1 - T$, $\dot S = \beta - \beta S$

$$\frac{d}{dt}\begin{bmatrix}T\\S\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}-1 & 0\\0 & -\beta\end{bmatrix}\begin{bmatrix}T\\S\end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix}T\\S\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}-1 & 0\\0 & -1/2\end{bmatrix}\begin{bmatrix}T\\S\end{bmatrix}$$

As before, T decreases to equilibrium faster than S, so the decay to equilibrium looks like:



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Math 5421
Dynamical Systems

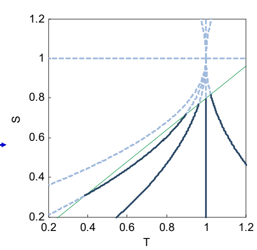
Welander's Model

$$\dot T = 1 - T - k(\rho)T \qquad k(\rho)=\begin{cases}0, & \rho<\varepsilon\\ 1, & \rho>\varepsilon\end{cases} \qquad \alpha = 0.8$$
$$\dot S = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

Now consider the case $k(\rho) = 0$, *i.e.*, $\rho < \varepsilon$.

$$\frac{d}{dt}\begin{bmatrix}T\\S\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}-1 & 0\\0 & -1/2\end{bmatrix}\begin{bmatrix}T\\S\end{bmatrix}$$

Actually, it looks like:



The solid lines are below the line $\rho = \varepsilon$, where the above equation is valid, while the dotted lines show where the equation is invalid.

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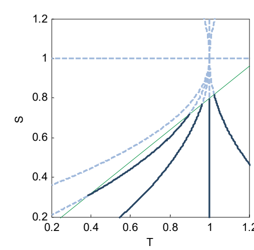
Math 5421
Dynamical Systems

Welander's Model

$$\dot T = 1 - T - k(\rho)T \qquad k(\rho)=\begin{cases}0, & \rho<\varepsilon\\ 1, & \rho>\varepsilon\end{cases} \qquad \alpha = 0.8$$
$$\dot S = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

Starting below the discontinuity line $\rho = \varepsilon$, solutions of Welander's model try to asymptotically approach the equilibrium at $(1,1)$. Before they arrive, they hit the discontinuity line, and the system changes back to the case $k(\rho) = 1$.



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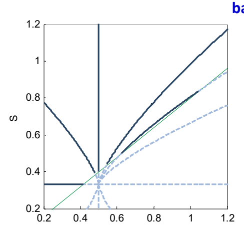
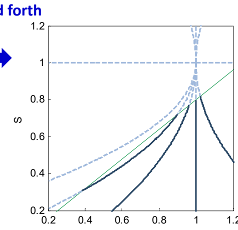
Math 5421
Dynamical Systems

Welander's Model

$$\dot T = 1 - T - k(\rho)T \qquad k(\rho)=\begin{cases}0, & \rho<\varepsilon\\ 1, & \rho>\varepsilon\end{cases} \qquad \alpha = 0.8$$
$$\dot S = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

back and forth





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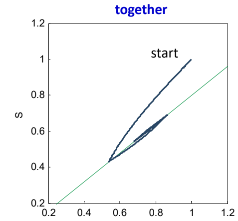
Math 5421
Dynamical Systems

Welander's Model

$$\dot T = 1 - T - k(\rho)T \qquad k(\rho)=\begin{cases}0, & \rho<\varepsilon\\ 1, & \rho>\varepsilon\end{cases} \qquad \alpha = 0.8$$
$$\dot S = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

together



narrative

The temperature and salinity both start above the line. The shallow ocean density is higher than the deep ocean density, so an overturning circulation decreases both the temperature and salinity of the shallow ocean, bringing the density down until the system hits the discontinuity line, at which point the overturning circulation stops, and the temperature and salinity then start moving toward that of the atmosphere, until they hit the discontinuity line.

Repeat.

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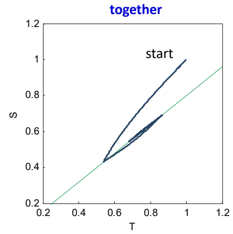
Math 5421
Dynamical Systems

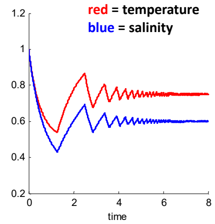
Welander's Model

$$\dot T = 1 - T - k(\rho)T \qquad k(\rho)=\begin{cases}0, & \rho<\varepsilon\\ 1, & \rho>\varepsilon\end{cases} \qquad \alpha = 0.8$$
$$\dot S = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

together



red = temperature
blue = salinity



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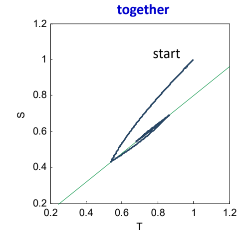
Math 5421
Dynamical Systems

Welander's Model

$$\dot T = 1 - T - k(\rho)T \qquad k(\rho)=\begin{cases}0, & \rho<\varepsilon\\ 1, & \rho>\varepsilon\end{cases} \qquad \alpha = 0.8$$
$$\dot S = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

together



narrative

The temperature and salinity both start above the line. The shallow ocean density is higher than the deep ocean density, so an overturning circulation decreases both the temperature and salinity of the shallow ocean, bringing the density down until the system hits the discontinuity line, at which point the overturning circulation stops, and the temperature and salinity then start moving toward that of the atmosphere, until they hit the discontinuity line.

Repeat.

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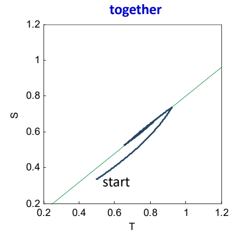
Math 5421
Dynamical Systems

Welander's Model

$$\dot T = 1 - T - k(\rho)T \qquad k(\rho)=\begin{cases}0, & \rho<\varepsilon\\ 1, & \rho>\varepsilon\end{cases} \qquad \alpha = 0.8$$
$$\dot S = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

together



narrative

The temperature and salinity both start below the line. The shallow ocean density is lower than the deep ocean density, so there is no overturning circulation. Instead, the temperature and salinity both increase as they move toward equilibrium with the atmosphere. Before the equilibrium is reached, the system hits the discontinuity line and the density become high enough so that the overturning circulation starts and continues until the density becomes so low that the circulation stops.

Repeat.

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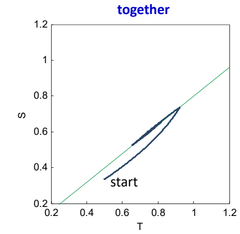
Math 5421
Dynamical Systems

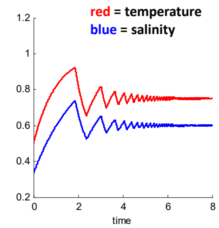
Welander's Model

$$\dot T = 1 - T - k(\rho)T \qquad k(\rho)=\begin{cases}0, & \rho<\varepsilon\\ 1, & \rho>\varepsilon\end{cases} \qquad \alpha = 0.8$$
$$\dot S = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

together



red = temperature
blue = salinity



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Math 5421
Dynamical Systems

Welander's Model

$$\dot T = 1 - T - k(\rho)T \qquad k(\rho)=\begin{cases}0, & \rho<\varepsilon\\ 1, & \rho>\varepsilon\end{cases} \qquad \alpha = 0.8$$
$$\dot S = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

The details are easier to see if we introduce a salinity anomaly measuring the deviation of the salinity from that along the discontinuity line.

$$\Delta S = S - \alpha T - \varepsilon = \rho - \varepsilon$$

Note that this anomaly also measures the density of the sytem above the critical density ε .

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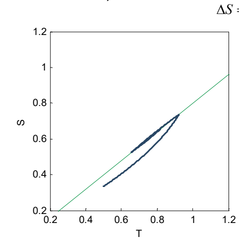
Math 5421
Dynamical Systems

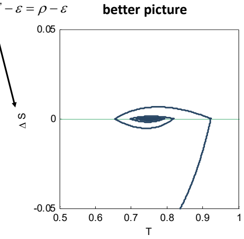
Welander's Model

$$\dot T = 1 - T - k(\rho)T \qquad k(\rho)=\begin{cases}0, & \rho<\varepsilon\\ 1, & \rho>\varepsilon\end{cases} \qquad \alpha = 0.8$$
$$\dot S = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

$$\Delta S = S - \alpha T - \varepsilon = \rho - \varepsilon$$

better picture





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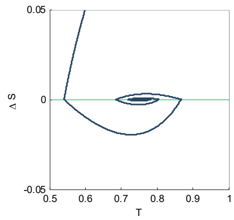
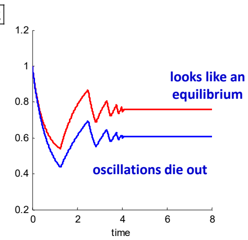
Math 5421
Dynamical Systems

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

$\boxed{\varepsilon = 0.002}$





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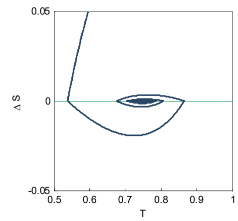
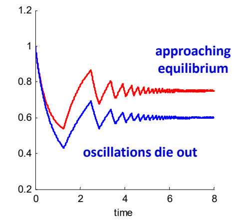
Math 5421
Dynamical Systems

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

$\boxed{\varepsilon = 0}$





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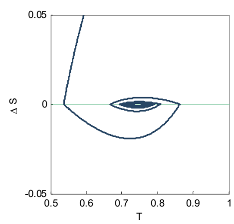
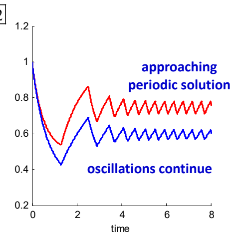
Math 5421
Dynamical Systems

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

$\boxed{\varepsilon = -0.002}$





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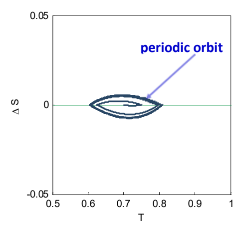
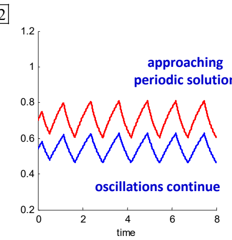
Dynamical Systems

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

$\boxed{\varepsilon = -0.02}$





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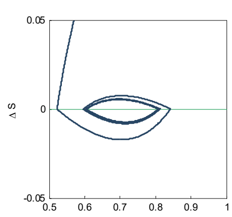
Math 5421
Dynamical Systems

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

$\boxed{\varepsilon = -0.02}$



Note that the solution started outside the periodic orbit and spiraled down to it. The amplitude of the oscillations decrease asymptotically to that of the periodic orbit.

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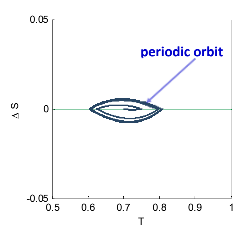
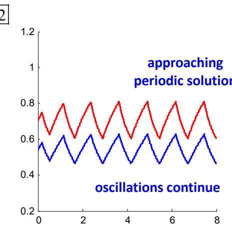
Dynamical Systems

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

$\boxed{\varepsilon = -0.02}$





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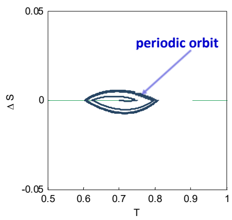
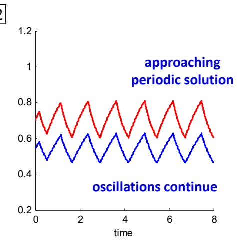
Math 5421
Dynamical Systems

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \begin{matrix} \alpha = 0.8 \\ \beta = 0.5 \end{matrix}$$
$$\rho = -\alpha T + S$$

$$\boxed{\varepsilon = -0.02}$$





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Dynamical Systems

What caused the Dansgaard-Oeschger oscillations?

They could be self-oscillations in the natural dynamics of ocean circulation.

Welander constructed a simple (*conceptual!*) box model of ocean circulation and showed that the interactions of temperature and salinity with the atmosphere, the surface ocean, and the deep ocean could create self-oscillations.

Pierre Welander, A simple heat-salt oscillator, *Dynamics of Atmospheres and Oceans* **6** (1982) 233-242.

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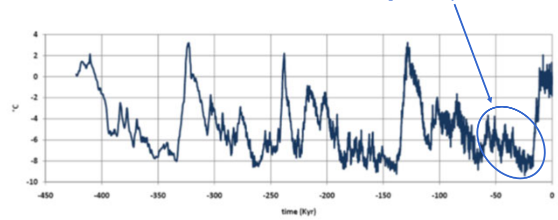
Dynamical Systems

Recent (last 400 Kyr) Temperature Cycles

Vostok Ice Core Data



J.R. Petit, *et al* (1999) Climate and atmospheric history of the past 420,000 years from the Vostok ice core, Antarctica, *Nature* 399, 429-436.

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Welander's Model

Mathematical Note

The mathematical tools to actually *prove* that Welander's model behaves in the way he described were not fully developed in 1982.

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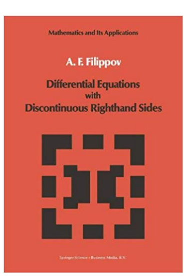
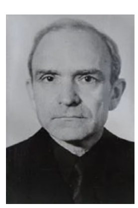


Kluwer Academic Publishers



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Mathematical Note

Welander assumed that the self-oscillations he found in his discontinuous model would hold held for a nearby smooth system.

Juliann Leifeld, PhD 2016:

Welander's assumption was correct.

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