Math 5421
An Introduction to Mathematical Climate Models

Spring 2025
1:25 – 3:20 Tuesdays and Thursdays
Blegen Hall 155

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Dynamical Systems

Dynamical Systems

The dependence of the solution on ***initial conditions*** is just as important as its dependence on ***time***.

$$x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n$$

intital value problem $\quad \dot{x} = f(x), \quad x(0) = \xi$

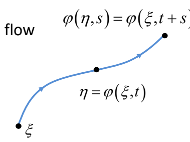
The initial value problem generates a flow

$$\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

with properties

intital condition $\quad \varphi(\xi, 0) = \xi$

"group property" $\quad \varphi(\varphi(\xi,t),s) = \varphi(\xi, t+s)$



If we start the system at state ξ and follow the solution for time t, then restart the system at the new state and follow the solution for time s, we end up at the same state as starting at ξ and following for time $t+s$.

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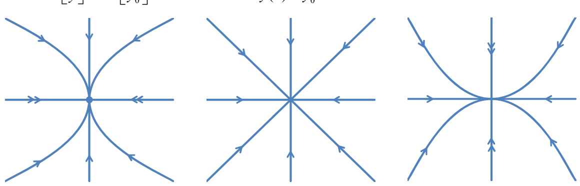
Dynamical Systems

"Phase Plane"
Example: "stable nodes"

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\dot{x} = ax, \quad \dot{y} = by, \quad x(0) = x_0, \quad y(0) = y_0$$

$$\varphi\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, t\right) = \begin{bmatrix} x_0 e^{at} \\ y_0 e^{bt} \end{bmatrix}$$



$a < b < 0$ $\qquad a = b < 0$ $\qquad b < a < 0$

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"Phase Plane"
Example: "unstable nodes"

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\dot{x} = ax, \quad \dot{y} = by, \quad x(0) = x_0, \quad y(0) = y_0$$

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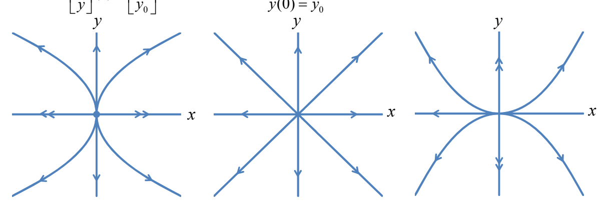

$a > b > 0$ $\qquad a = b > 0$ $\qquad 0 < a < b$

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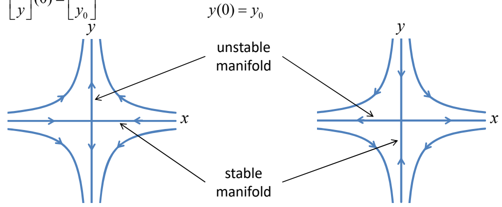
"Phase Plane"
Example: "saddles"

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\dot{x} = ax, \quad \dot{y} = by, \quad x(0) = x_0, \quad y(0) = y_0$$

$$\varphi\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, t\right) = \begin{bmatrix} x_0 e^{at} \\ y_0 e^{bt} \end{bmatrix}$$



$a < 0 < b$ $\qquad b < 0 < a$

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Dynamical Systems

Matrix Notation

$$\begin{aligned} \frac{dx}{dt} &= a_{11}x + a_{12}y \\ \frac{dy}{dt} &= a_{21}x + a_{22}y \end{aligned} \implies \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \implies \frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

Amazingly, the solution is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

Example

$$\begin{aligned} \frac{dx}{dt} &= \alpha x \\ \frac{dy}{dt} &= \beta y \end{aligned} \implies \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \implies \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} e^{\alpha t} x_0 \\ e^{\beta t} y_0 \end{bmatrix}$$

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Dynamical Systems
Eigenvalues

$$Av = \lambda v \qquad (v \neq 0)$$

The number λ is called an ***eigenvalue*** and the vector v is called an ***eigenvector***.

Example

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \beta \end{bmatrix} = \beta\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

α and β are eigenvalues with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

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Dynamical Systems
Eigenvalues

Example

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

2 and -1 are eigenvalues with corresponding eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

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Dynamical Systems
Eigenvalues

What are they good for?

To find solutions of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

If λ is an eigenvalue of A with corresponding eigenvector v, then $x(t) = e^{\lambda t}v$ is a solution.

Proof:

$$\frac{dx}{dt} = \lambda e^{\lambda t}v = e^{\lambda t}\lambda v = e^{\lambda t}Av = A\left(e^{\lambda t}v\right) = Ax$$

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Dynamical Systems
Eigenvalues

Example

$$\frac{dx}{dt} = \alpha x \qquad \frac{dy}{dt} = \beta y \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

eigenvalues: α and β

eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solutions: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{\alpha t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{\alpha t} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{\beta t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{\beta t} \end{bmatrix}$

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Dynamical Systems
Eigenvalues

Example

$$\frac{dx}{dt} = x + 2y \qquad \frac{dy}{dt} = x \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

eigenvalues: 2 and -1

eigenvectors: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Solutions: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$

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More good news:

If $x = \varphi_1(t)$ and $x = \varphi_2(t)$ are solutions of $\frac{dx}{dt} = Ax$, then $x(t) = c_1\varphi_1(t) + c_2\varphi_2(t)$ is also a solution for arbitrary constants c_1 and c_2.

Proof:

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}\left(c_1\varphi_1(t) + c_2\varphi_2(t)\right) \\ &= c_1\varphi_1'(t) + c_2\varphi_2'(t) = c_1A\varphi_1(t) + c_2A\varphi_2(t) \\ &= A\left(c_1\varphi_1(t) + c_2\varphi_2(t)\right) \\ &= Ax \end{aligned}$$

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Dynamical Systems

Eigenvalues

Example

$$\frac{dx}{dt} = \alpha x \qquad \frac{dy}{dt} = \beta y \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Solutions: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} e^{\alpha t} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} 0 \\ e^{\beta t} \end{bmatrix}$

General solution: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = c_1\begin{bmatrix} e^{\alpha t} \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{\beta t} \end{bmatrix} = \begin{bmatrix} c_1 e^{\alpha t} \\ c_2 e^{\beta t} \end{bmatrix}$

$$\boxed{\begin{aligned} x(t) &= c_1 e^{\alpha t} \\ y(t) &= c_2 e^{\beta t} \end{aligned}}$$

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Dynamical Systems

Eigenvalues

Example

$$\frac{dx}{dt} = x + 2y \qquad \frac{dy}{dt} = x \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Solutions: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$

General solution: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = c_1\begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix} + c_2\begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} 2c_1 e^{2t} - c_2 e^{-t} \\ c_1 e^{2t} + c_2 e^{-t} \end{bmatrix}$

$$\boxed{\begin{aligned} x(t) &= 2c_1 e^{2t} - c_2 e^{-t} \\ y(t) &= c_1 e^{2t} + c_2 e^{-t} \end{aligned}}$$

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Eigenvalues

$$Av = \lambda v$$

How do we find the eigenvalues?

$$Av = \lambda v \quad \Rightarrow \quad Av - \lambda v = 0 \quad \Rightarrow \quad (A - \lambda I)v = 0$$

For $(A - \lambda I)v = 0$ to have a nontrivial solution $v \neq 0$, we must have

$$\boxed{\det(A - \lambda I) = 0.}$$

Characteristic Polynomial

The roots of the characteristic polynomial are the eigenvalues of A.

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Eigenvalues

The roots of the characteristic polynomial are the eigenvalues.

Example

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

$$0 = \det(A - \lambda I) = \det\begin{bmatrix} \alpha - \lambda & 0 \\ 0 & \beta - \lambda \end{bmatrix} = (\alpha - \lambda)(\beta - \lambda)$$

The eigenvalues are the roots of $(\lambda - \alpha)(\lambda - \beta) = 0$, namely, α and β.

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Eigenvalues

Example

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$0 = \det(A - \lambda I) = \det\begin{bmatrix} 1 - \lambda & 2 \\ 1 & 0 - \lambda \end{bmatrix} = (1 - \lambda)(0 - \lambda) - 2$$

$$= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

The eigenvalues are the roots of $(\lambda - 2)(\lambda + 1) = 0$, namely, 2 and -1.

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Eigenvalues

In general

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$0 = \det(A - \lambda I) = \det\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

$$= \lambda^2 - \text{trace}(A)\lambda + \det(A)$$

$$= \lambda^2 - \tau\lambda + \delta$$

The eigenvalues are $\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$

real if $\tau^2 - 4\delta \geq 0$

complex if $\tau^2 - 4\delta < 0$

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Eigenvalues

Example

$$A=\begin{bmatrix}1&2\\1&0\end{bmatrix}$$

$$\tau=\operatorname{trace}(A)=1+0=1$$
$$\delta=\det(A)=1\cdot 0-2\cdot 1=-2$$

$$\det(A-\lambda I)=\lambda^2-\tau\lambda+\delta=\lambda^2-\lambda-2$$

The eigenvalues are $\lambda=\dfrac{\tau\pm\sqrt{\tau^2-4\delta}}{2}=\dfrac{1\pm\sqrt{1^2-4(-2)}}{2}=\dfrac{1\pm\sqrt{9}}{2}=\dfrac{1\pm3}{2}$

$\boxed{2 \text{ and } -1}$

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Eigenvalues

$$Av=\lambda v$$

How do we find the eigenvectors?

$$Av=\lambda v \;\Rightarrow\; Av-\lambda v=0 \;\Rightarrow\; (A-\lambda I)v=0$$

If we have an eigenvalue λ we can find a corresponding eigenvector by solving
$$(A-\lambda I)v=0$$
for a nontrival solution v.

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Example $A=\begin{bmatrix}1&2\\1&0\end{bmatrix}$ $\lambda_1=2,\quad \lambda_2=-1$ ← eigenvalues

$$A-\lambda_1 I=\begin{bmatrix}1-\lambda_1&2\\1&0-\lambda_1\end{bmatrix}=\begin{bmatrix}1-2&2\\1&0-2\end{bmatrix}=\begin{bmatrix}-1&2\\1&-2\end{bmatrix}$$

$$(A-\lambda_1 I)v_1=\begin{bmatrix}-1&2\\1&-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0$$

$\lambda_1=2$ $v_1=\begin{bmatrix}2\\1\end{bmatrix}$

corresponding eigenvectors

$$A-\lambda_2 I=\begin{bmatrix}1-\lambda_2&2\\1&0-\lambda_2\end{bmatrix}=\begin{bmatrix}1-(-1)&2\\1&0-(-1)\end{bmatrix}=\begin{bmatrix}2&2\\1&1\end{bmatrix}$$

$$(A-\lambda_2 I)v_2=\begin{bmatrix}2&2\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0$$

$\lambda_2=-1$ $v_2=\begin{bmatrix}-1\\1\end{bmatrix}$

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Dynamical Systems
Eigenvalues

Example

$$\frac{dx}{dt}=-2y \qquad \frac{dy}{dt}=x-3y$$

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0&-2\\1&-3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix} \qquad A=\begin{bmatrix}0&-2\\1&-3\end{bmatrix}$$

$$\tau=\operatorname{trace}(A)=0-3=-3$$
$$\delta=\det(A)=0\cdot(-3)-(-2)\cdot 1=2$$

$$\det(A-\lambda I)=\lambda^2-\tau\lambda+\delta=\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)$$

The eigenvalues are $\boxed{-1 \text{ and } -2}$

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Eigenvalues

$A=\begin{bmatrix}0&-2\\1&-3\end{bmatrix}$ $\lambda_1=-1,\quad \lambda_2=-2$ ← eigenvalues

$$A-\lambda_1 I=\begin{bmatrix}0-\lambda_1&-2\\1&-3-\lambda_1\end{bmatrix}=\begin{bmatrix}0-(-1)&-2\\1&-3-(-1)\end{bmatrix}=\begin{bmatrix}1&-2\\1&-2\end{bmatrix}$$

corresponding eigenvectors

$$(A-\lambda_1 I)v_1=\begin{bmatrix}1&-2\\1&-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0 \qquad v_1=\begin{bmatrix}2\\1\end{bmatrix}$$

$$A-\lambda_2 I=\begin{bmatrix}0-\lambda_2&-2\\1&-3-\lambda_2\end{bmatrix}=\begin{bmatrix}0-(-2)&-2\\1&-3-(-2)\end{bmatrix}=\begin{bmatrix}2&-2\\1&-1\end{bmatrix}$$

$$(A-\lambda_2 I)v_2=\begin{bmatrix}2&-2\\1&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0 \qquad v_2=\begin{bmatrix}1\\1\end{bmatrix}$$

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Dynamical Systems
Eigenvalues

Example

$$\frac{dx}{dt}=-2y \qquad \frac{dy}{dt}=x-3y$$

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0&-2\\1&-3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix} \qquad A=\begin{bmatrix}0&-2\\1&-3\end{bmatrix}$$

eigenvalues: $-1 \quad -2$

eigenvectors: $\begin{bmatrix}2\\1\end{bmatrix} \quad \begin{bmatrix}1\\1\end{bmatrix}$

Solutions: $\begin{bmatrix}x\\y\end{bmatrix}(t)=e^{-t}\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}x\\y\end{bmatrix}(t)=e^{-2t}\begin{bmatrix}1\\1\end{bmatrix}$

General solution: $\begin{bmatrix}x\\y\end{bmatrix}(t)=c_1\begin{bmatrix}2e^{-t}\\e^{-t}\end{bmatrix}+c_2\begin{bmatrix}e^{-2t}\\e^{-2t}\end{bmatrix}=\begin{bmatrix}2c_1e^{-t}+c_2e^{-2t}\\c_1e^{-t}+c_2e^{-2t}\end{bmatrix}$

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Dynamical Systems

Eigenvalues

Example

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\tau = \text{trace}(A) = 0 + 0 = 0$$
$$\delta = \det(A) = 0 \cdot 0 - (-1) \cdot 1 = 1$$

$$\det(A - \lambda I) = \lambda^2 - \tau\lambda + \delta = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$$

The eigenvalues are $\boxed{i \text{ and } -i}$

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Eigenvalues

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \lambda_1 = i, \quad \lambda_2 = -i$$ ← eigenvalues

corresponding eigenvectors

$$A - \lambda_1 I = \begin{bmatrix} 0-\lambda_1 & -1 \\ 1 & 0-\lambda_1 \end{bmatrix} = \begin{bmatrix} 0-i & -1 \\ 1 & 0-i \end{bmatrix} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} 0-\lambda_2 & -1 \\ 1 & 0-\lambda_2 \end{bmatrix} = \begin{bmatrix} 0-(-i) & -1 \\ 1 & 0-(-i) \end{bmatrix} = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$$

$$(A - \lambda_2 I)v_2 = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

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Eigenvalues

Example

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

eigenvalues: $i \quad -i$

eigenvectors: $\begin{bmatrix} i \\ 1 \end{bmatrix} \quad \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Reality Check:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i\begin{bmatrix} i \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -i \end{bmatrix} = -i\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

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Eigenvalues

Example

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

eigenvalues: $i \quad -i$

eigenvectors: $\begin{bmatrix} i \\ 1 \end{bmatrix} \quad \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Solutions: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{it}\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{-it}\begin{bmatrix} -i \\ 1 \end{bmatrix}$

General solution: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = c_1\begin{bmatrix} ie^{it} \\ e^{it} \end{bmatrix} + c_2\begin{bmatrix} -ie^{-it} \\ e^{-it} \end{bmatrix} = \begin{bmatrix} ic_1e^{it} - ic_2e^{-it} \\ c_1e^{it} + c_2e^{-it} \end{bmatrix}$

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Eigenvalues

Example

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

General solution: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = c_1\begin{bmatrix} ie^{it} \\ e^{it} \end{bmatrix} + c_2\begin{bmatrix} -ie^{-it} \\ e^{-it} \end{bmatrix} = \begin{bmatrix} ic_1e^{it} - ic_2e^{-it} \\ c_1e^{it} + c_2e^{-it} \end{bmatrix}$

Let $c_1 = \frac{1}{2i}, \quad c_2 = -\frac{1}{2i}$. Then $\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} \frac{1}{2}(e^{it} + e^{-it}) \\ \frac{1}{2i}(e^{it} - e^{-it}) \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

Let $c_1 = c_2 = \frac{1}{2}$. Then $\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} \frac{i}{2}(e^{it} - e^{-it}) \\ \frac{1}{2}(e^{it} + e^{-it}) \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$

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Eigenvalues

Example

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

General solution: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = c_1\begin{bmatrix} ie^{it} \\ e^{it} \end{bmatrix} + c_2\begin{bmatrix} -ie^{-it} \\ e^{-it} \end{bmatrix} = \begin{bmatrix} ic_1e^{it} - ic_2e^{-it} \\ c_1e^{it} + c_2e^{-it} \end{bmatrix}$

Or $\begin{bmatrix} x \\ y \end{bmatrix}(t) = a\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + b\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$

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Dynamical Systems

Eigenvalues

Alternate Approach

$$\boxed{\begin{aligned}\frac{dx}{dt}&=-y\\ \frac{dy}{dt}&=x\end{aligned}}$$

Let $z=x+iy$. Then $\dfrac{dz}{dt}=\dfrac{dx}{dt}+i\dfrac{dy}{dt}=-y+ix=iz$

$$\boxed{\frac{dz}{dt}=iz}\qquad \text{Solution:}\quad z(t)=z_0e^{it}=r_0e^{i\theta_0}e^{it}=r_0e^{i(\theta_0+t)}$$

Let $z=re^{i\theta}$. Then $\dfrac{dz}{dt}=\dfrac{dr}{dt}e^{i\theta}+rie^{i\theta}\dfrac{d\theta}{dt}=iz=ire^{i\theta}$

$$\frac{dr}{dt}+ri\frac{d\theta}{dt}=0+ir\qquad \boxed{\begin{aligned}\frac{dr}{dt}&=0\\ \frac{d\theta}{dt}&=1\end{aligned}}\qquad \text{Solution:}\quad \begin{aligned}r(t)&=r_0\\ \theta(t)&=\theta_0+t\end{aligned}$$

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Summary So Far

$$\begin{aligned}\frac{dx}{dt}&=a_{11}x+a_{12}y\\ \frac{dy}{dt}&=a_{21}x+a_{22}y\end{aligned}\implies \mathbf{x}=\begin{bmatrix}x\\y\end{bmatrix}\quad A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\implies \frac{d\mathbf{x}}{dt}=A\mathbf{x}$$

Eigenvalues and eigenvectors

$$Av=\lambda v\qquad (v\neq 0)$$

If v and u are linearly independent eigenvectors with corresponding eigenvalues λ and μ, then the general solution is

$$\mathbf{x}(t)=c_1e^{\lambda t}v+c_2e^{\mu t}u$$

where c_1 and c_2 are arbitrary constants.

Linear independence: one is not a multiple of the other.

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Dynamical Systems

Coordinate Change

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix}$$

Suppose that $\boldsymbol{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}$ and $\boldsymbol{u}=\begin{bmatrix}u_1\\u_2\end{bmatrix}$ are linearly independent eigenvectors of A with corresponding eigenvalues λ and μ. Introduce new variables ξ and η:

$$\begin{bmatrix}x\\y\end{bmatrix}=\xi\boldsymbol{v}+\eta\boldsymbol{u},\quad\text{i.e.}\quad \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}v_1\xi+u_1\eta\\v_2\xi+u_2\eta\end{bmatrix}=\begin{bmatrix}v_1&u_1\\v_2&u_2\end{bmatrix}\begin{bmatrix}\xi\\\eta\end{bmatrix}=S\begin{bmatrix}\xi\\\eta\end{bmatrix}$$

where $S=\begin{bmatrix}v_1&u_1\\v_2&u_2\end{bmatrix}=\begin{bmatrix}\boldsymbol{v}\mid\boldsymbol{u}\end{bmatrix}$.

Then $S\dfrac{d}{dt}\begin{bmatrix}\xi\\\eta\end{bmatrix}=\dfrac{d}{dt}S\begin{bmatrix}\xi\\\eta\end{bmatrix}=\dfrac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix}=AS\begin{bmatrix}\xi\\\eta\end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix}\xi\\\eta\end{bmatrix}=S^{-1}AS\begin{bmatrix}\xi\\\eta\end{bmatrix}$$

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Dynamical Systems

Coordinate Change

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix}\implies \begin{bmatrix}x\\y\end{bmatrix}=S\begin{bmatrix}\xi\\\eta\end{bmatrix}\implies \frac{d}{dt}\begin{bmatrix}\xi\\\eta\end{bmatrix}=S^{-1}AS\begin{bmatrix}\xi\\\eta\end{bmatrix}$$

$$S=\begin{bmatrix}\boldsymbol{v}\mid\boldsymbol{u}\end{bmatrix}\qquad A\boldsymbol{v}=\lambda\boldsymbol{v}\qquad A\boldsymbol{u}=\mu\boldsymbol{u}$$

$$AS=A\begin{bmatrix}\boldsymbol{v}\mid\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}A\boldsymbol{v}\mid A\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\lambda\boldsymbol{v}\mid\mu\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\boldsymbol{v}\mid\boldsymbol{u}\end{bmatrix}\begin{bmatrix}\lambda&0\\0&\mu\end{bmatrix}=S\Lambda$$

$$\begin{aligned}A\begin{bmatrix}\boldsymbol{v}\mid\boldsymbol{u}\end{bmatrix}&=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}v_1&u_1\\v_2&u_2\end{bmatrix}\\&=\begin{bmatrix}a_{11}v_1+a_{12}v_2&a_{11}u_1+a_{12}u_2\\a_{21}v_1+a_{22}v_2&a_{21}u_1+a_{22}u_2\end{bmatrix}\\&=\begin{bmatrix}A\boldsymbol{v}\mid A\boldsymbol{u}\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\begin{bmatrix}\lambda\boldsymbol{v}\mid\mu\boldsymbol{u}\end{bmatrix}&=\begin{bmatrix}\lambda v_1&\mu u_1\\\lambda v_2&\mu u_2\end{bmatrix}\\&=\begin{bmatrix}v_1&u_1\\v_2&u_2\end{bmatrix}\begin{bmatrix}\lambda&0\\0&\mu\end{bmatrix}\\&=\begin{bmatrix}\boldsymbol{v}\mid\boldsymbol{u}\end{bmatrix}\begin{bmatrix}\lambda&0\\0&\mu\end{bmatrix}\end{aligned}$$

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Dynamical Systems

Coordinate Change

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix}\implies \begin{bmatrix}x\\y\end{bmatrix}=S\begin{bmatrix}\xi\\\eta\end{bmatrix}\implies \frac{d}{dt}\begin{bmatrix}\xi\\\eta\end{bmatrix}=S^{-1}AS\begin{bmatrix}\xi\\\eta\end{bmatrix}$$

$$S=\begin{bmatrix}\boldsymbol{v}\mid\boldsymbol{u}\end{bmatrix}\qquad A\boldsymbol{v}=\lambda\boldsymbol{v}\qquad A\boldsymbol{u}=\mu\boldsymbol{u}$$

$$AS=A\begin{bmatrix}\boldsymbol{v}\mid\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}A\boldsymbol{v}\mid A\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\lambda\boldsymbol{v}\mid\mu\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\boldsymbol{v}\mid\boldsymbol{u}\end{bmatrix}\begin{bmatrix}\lambda&0\\0&\mu\end{bmatrix}=S\Lambda$$

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Dynamical Systems

Coordinate Change

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix}\implies \begin{bmatrix}x\\y\end{bmatrix}=S\begin{bmatrix}\xi\\\eta\end{bmatrix}\implies \frac{d}{dt}\begin{bmatrix}\xi\\\eta\end{bmatrix}=S^{-1}AS\begin{bmatrix}\xi\\\eta\end{bmatrix}$$

$$S=\begin{bmatrix}\boldsymbol{v}\mid\boldsymbol{u}\end{bmatrix}\qquad A\boldsymbol{v}=\lambda\boldsymbol{v}\qquad A\boldsymbol{u}=\mu\boldsymbol{u}$$

$$AS=A\begin{bmatrix}\boldsymbol{v}\mid\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}A\boldsymbol{v}\mid A\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\lambda\boldsymbol{v}\mid\mu\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\boldsymbol{v}\mid\boldsymbol{u}\end{bmatrix}\begin{bmatrix}\lambda&0\\0&\mu\end{bmatrix}=S\Lambda$$

$$\Lambda=S^{-1}AS$$

$$\frac{d}{dt}\begin{bmatrix}\xi\\\eta\end{bmatrix}=S^{-1}AS\begin{bmatrix}\xi\\\eta\end{bmatrix}=\Lambda\begin{bmatrix}\xi\\\eta\end{bmatrix}$$

$$\begin{aligned}\frac{dx}{dt}&=a_{11}x+a_{12}y\\ \frac{dy}{dt}&=a_{21}x+a_{22}y\end{aligned}\implies \begin{aligned}\frac{d\xi}{dt}&=\lambda\xi\\ \frac{d\eta}{dt}&=\mu\eta\end{aligned}$$

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Coordinate Change

Example

$$\frac{dx}{dt} = x + 2y \qquad \frac{dy}{dt} = x \qquad A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

eigenvalues: 2 and -1

eigenvectors: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$S = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = S\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} 2\xi - \eta \\ \xi + \eta \end{bmatrix} \qquad \begin{aligned} x &= 2\xi - \eta \\ y &= \xi + \eta \end{aligned}$$

$$\begin{aligned} \frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= x \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x &= 2\xi - \eta \\ y &= \xi + \eta \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \frac{d\xi}{dt} &= 2\xi \\ \frac{d\eta}{dt} &= -\eta \end{aligned}$$

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Dynamical Systems

Coordinate Change

Example

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

eigenvalues: $i \quad -i$

eigenvectors: $\begin{bmatrix} i \\ 1 \end{bmatrix} \quad \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$$S = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = S\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} iz - iw \\ z + w \end{bmatrix} \qquad \begin{aligned} x &= iz - iw \\ y &= z + w \end{aligned}$$

$$\begin{aligned} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x &= iz - iw \\ y &= z + w \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \frac{dz}{dt} &= iz \\ \frac{dw}{dt} &= -iw \end{aligned}$$

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Dynamical Systems

Coordinate Change

Example

$$\begin{aligned} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x &= iz - iw \\ y &= z + w \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \frac{dz}{dt} &= iz \\ \frac{dw}{dt} &= -iw \end{aligned}$$

$$\begin{aligned} 2z &= y - ix \\ 2w &= y + ix \end{aligned} \quad \Rightarrow \quad w = \overline{z}$$

Note that one of these equations is redundant.

$$\begin{aligned} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x \end{aligned} \qquad \frac{dz}{dt} = iz \qquad \begin{aligned} \frac{dr}{dt} &= 0 \\ \frac{d\theta}{dt} &= 1 \end{aligned}$$

Cartesian complex polar

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Math 5421
Dynamical Systems

Coordinate Change

Example

$$\begin{aligned} \frac{dx}{dt} &= ax - \omega y \\ \frac{dy}{dt} &= \omega x + ay \end{aligned} \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & -\omega \\ \omega & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} a & -\omega \\ \omega & a \end{bmatrix}$$

$$\tau = \operatorname{trace}(A) = a + a = 2a$$
$$\delta = \det(A) = a^2 + \omega^2$$

$$\det(A - \lambda I) = \lambda^2 - \tau\lambda + \delta = \lambda^2 - 2a + a^2 + \omega^2$$

The eigenvalues are $\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2} = \dfrac{2a \pm \sqrt{4a^2 - 4a^2 - 4\omega^2}}{2} = a \pm i\omega$

$$\boxed{a + i\omega \quad \text{and} \quad a - i\omega}$$

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Dynamical Systems

Coordinate Change

$$\begin{aligned} \frac{dx}{dt} &= ax - \omega y \\ \frac{dy}{dt} &= \omega x + ay \end{aligned}$$

Let $z = x + iy$.

Then $\begin{aligned} \frac{dz}{dt} &= \frac{dx}{dt} + i\frac{dy}{dt} \\ &= ax - \omega y + i\omega x + iay \\ &= (a + i\omega)(x + iy) \end{aligned}$

$$\boxed{\frac{dz}{dt} = (a + i\omega)z}$$

Solution: $z(t) = z_0 e^{(a+i\omega)t} = r_0 e^{i\theta_0} e^{(a+i\omega)t} = r_0 e^{at} e^{i(\omega t + \theta_0)}$

Let $z = re^{i\theta}$.

Then $\dfrac{dz}{dt} = \dfrac{dr}{dt}e^{i\theta} + rie^{i\theta}\dfrac{d\theta}{dt} = (a + i\omega)z = (a + i\omega)re^{i\theta}$

$$\frac{dr}{dt} + ri\frac{d\theta}{dt} = (a + i\omega)r$$

$$\boxed{\begin{aligned} \frac{dr}{dt} &= ar \\ \frac{d\theta}{dt} &= \omega \end{aligned}}$$

Solution: $\begin{aligned} r(t) &= r_0 e^{at} \\ \theta(t) &= \theta_0 + \omega t \end{aligned}$

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Dynamical Systems

Complex Eigenvalues

If A is a matrix with real elements and if λ is an eigenvalue of A with corrresponding eigenvector v, then $\overline{\lambda}$ is an eigenvalue of A with corresponding eigenvector $\overline{v}$.

$$Av = \lambda v \quad \Rightarrow \quad \overline{A}\overline{v} = \overline{\lambda}\overline{v} \quad \Rightarrow \quad A\overline{v} = \overline{\lambda}\overline{v}$$

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Dynamical Systems

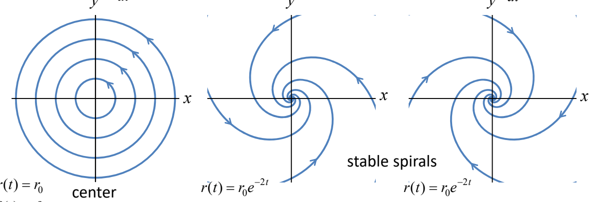
Stable Spirals

$$\frac{dx}{dt} = Ax$$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad \frac{dz}{dt} = iz$

$A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$ $\quad \frac{dz}{dt} = (-2+i)z$

$A = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$ $\quad \frac{dz}{dt} = (-2-i)z$



$r(t) = r_0$, $\theta(t) = \theta_0 + t$

$r(t) = r_0 e^{-2t}$, $\theta(t) = \theta_0 + t$

$r(t) = r_0 e^{-2t}$, $\theta(t) = \theta_0 - t$

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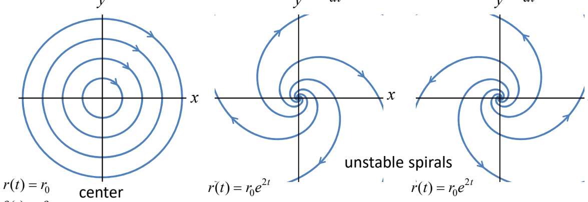
Unstable Spirals

$$\frac{dx}{dt} = Ax$$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad \frac{dz}{dt} = -iz$

$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ $\quad \frac{dz}{dt} = (2-i)z$

$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ $\quad \frac{dz}{dt} = (2+i)z$



$r(t) = r_0$, $\theta(t) = \theta_0 - t$

$r(t) = r_0 e^{2t}$, $\theta(t) = \theta_0 - t$

$r(t) = r_0 e^{2t}$, $\theta(t) = \theta_0 + t$

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Dynamical Systems

Coordinate Change

Example

$$\frac{dx}{dt} = -4x + 2y \qquad \frac{dy}{dt} = x - 5y$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} -4 & 2 \\ 1 & -5 \end{bmatrix}$$

$$\tau = \operatorname{trace}(A) = -4 - 5 = -9$$

$$\delta = \det(A) = (-4)\cdot(-5) - 2\cdot 1 = 18$$

$$\det(A - \lambda I) = \lambda^2 - \tau\lambda + \delta = \lambda^2 + 9\lambda + 18 = (\lambda+3)(\lambda+6)$$

The eigenvalues are $\lambda = -3$ and $\lambda = -6$.

$$A + 3I = \begin{bmatrix} -4+3 & 2 \\ 1 & -5+3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \qquad A + 6I = \begin{bmatrix} -4+6 & 2 \\ 1 & -5+6 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\lambda = -3 \quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \lambda = -6 \quad v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -6 \end{bmatrix} \quad S\Lambda = AS \quad \Lambda = S^{-1}AS$$

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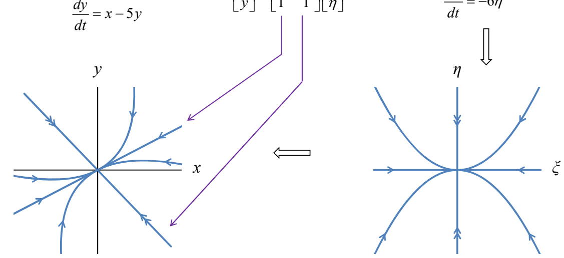
Dynamical Systems

Coordinate Change

Example

$$\frac{dx}{dt} = -4x + 2y, \quad \frac{dy}{dt} = x - 5y \quad \Longrightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} \quad \Longrightarrow \quad \frac{d\xi}{dt} = -3\xi, \quad \frac{d\eta}{dt} = -6\eta$$



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Dynamical Systems

Coordinate Change

Example

$$\frac{dx}{dt} = -3x + 4y \qquad \frac{dy}{dt} = -2x + 3y$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}$$

$$\tau = \operatorname{trace}(A) = -3 + 3 = 0$$

$$\delta = \det(A) = (-3)\cdot 3 - 4\cdot(-2) = -1$$

$$\det(A - \lambda I) = \lambda^2 - \tau\lambda + \delta = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

The eigenvalues are $\lambda = -1$ and $\lambda = 1$.

$$A + I = \begin{bmatrix} -3+1 & 4 \\ -2 & 3+1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix} \qquad A - I = \begin{bmatrix} -3-1 & 4 \\ -2 & 3-1 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix}$$

$$\lambda = -1 \quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \lambda = 1 \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad S\Lambda = AS \quad \Lambda = S^{-1}AS$$

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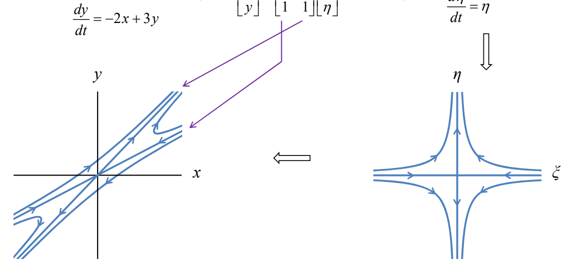
Dynamical Systems

Coordinate Change

Example

$$\frac{dx}{dt} = -3x + 4y, \quad \frac{dy}{dt} = -2x + 3y \quad \Longrightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} \quad \Longrightarrow \quad \frac{d\xi}{dt} = -\xi, \quad \frac{d\eta}{dt} = \eta$$



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Linear Systems Summary

$$\frac{dx}{dt} = Ax$$

For linear systems of ordinary differential equations, the eigenvalues and eigenvectors tell us a lot.

Reference

Kaper & Engler, *Mathematics & Climate*, SIAM 2013, Chapter 4

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Linear Systems Summary

$$\frac{dx}{dt} = Ax$$

For linear systems of ordinary differential equations, the eigenvalues and eigenvectors tell us a lot.

They tell us solutions.

If λ is an eigenvalue with corresponding eigenvector v, then $x(t) = e^{\lambda t}v$ is a solution.

They tell us simplifying transformations.

If λ_1 and λ_2 are eigenvalues of A with corresponding eigenvectors v_1 and v_2,

if $S = [v_1 \mid v_2]$, and if $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$,

then $S^{-1}AS = \Lambda$.

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Dynamical Systems

Coordinate Change

$$\frac{dx}{dt} = Ax$$

If λ_1 and λ_2 are eigenvalues of A with corresponding eigenvectors v_1 and v_2,

if $S = [v_1 \mid v_2]$, and if $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then $S^{-1}AS = \Lambda$.

$$x = Sy \;\Rightarrow\; S\frac{dy}{dt} = \frac{dx}{dt} = ASy \;\Rightarrow\; \frac{dy}{dt} = S^{-1}ASy = \Lambda y$$

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2, \quad \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 \quad \text{becomes} \quad \frac{dy_1}{dt} = \lambda_1 y_1, \quad \frac{dy_2}{dt} = \lambda_2 y_2$$

Much simpler.

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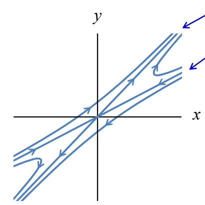
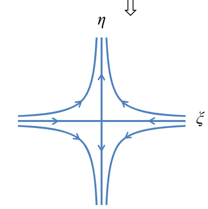
Dynamical Systems

Coordinate Change

Example

$$\frac{dx}{dt} = -3x + 4y, \quad \frac{dy}{dt} = -2x + 3y \;\Longrightarrow\; \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} \;\Longrightarrow\; \frac{d\xi}{dt} = -\xi, \quad \frac{d\eta}{dt} = \eta$$





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Dynamical Systems

Eigenvalues

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - \operatorname{trace}(A)\lambda + \det(A) = \lambda^2 - \tau\lambda + \delta = (\lambda - \lambda_1)(\lambda - \lambda_2)$$
$$= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

$$\lambda_1 + \lambda_2 = \operatorname{trace}(A)$$
$$\lambda_1\lambda_2 = \det(A)$$

The trace is the sum of the eigenvalues, while the determinant is the product of the eigenvalues.

We can characterize the dynamic behavior using the trace and the determinant.

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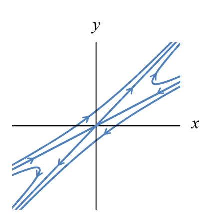
Saddles

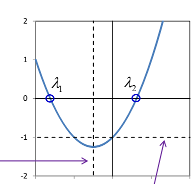
$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$$

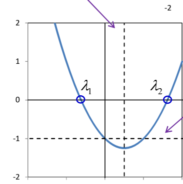
$\lambda_1\lambda_2 = \det(A) < 0$

$\lambda^2 - \operatorname{trace}\lambda + \det = 0$

One eigenvalue is positive, the other negative.







trace/2

determinant

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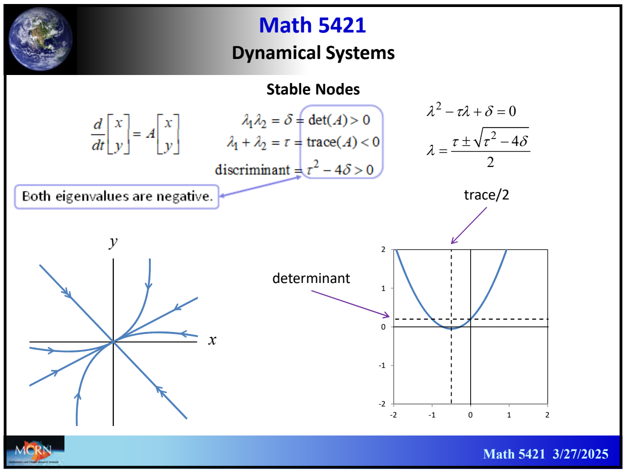

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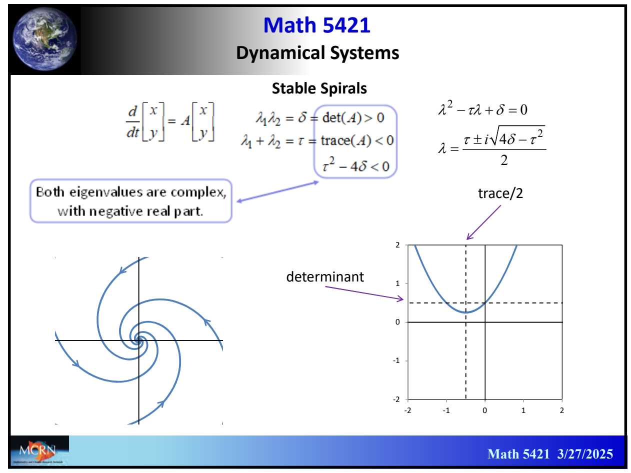

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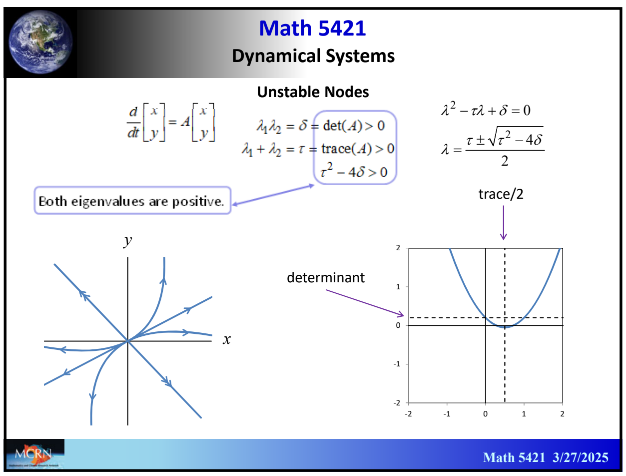

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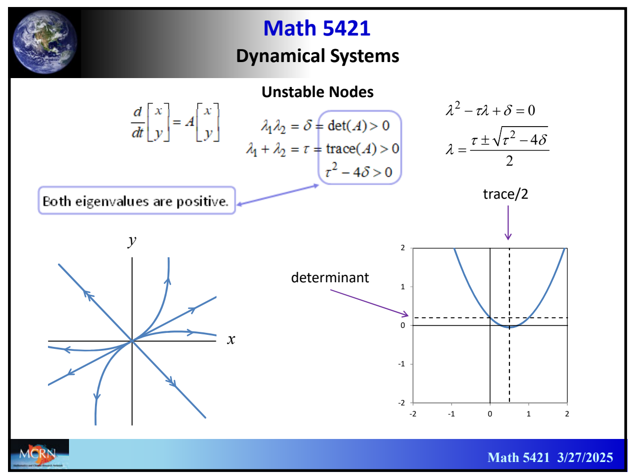

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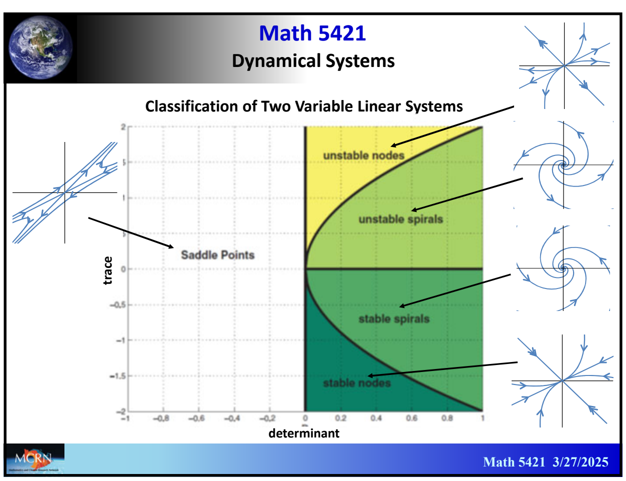

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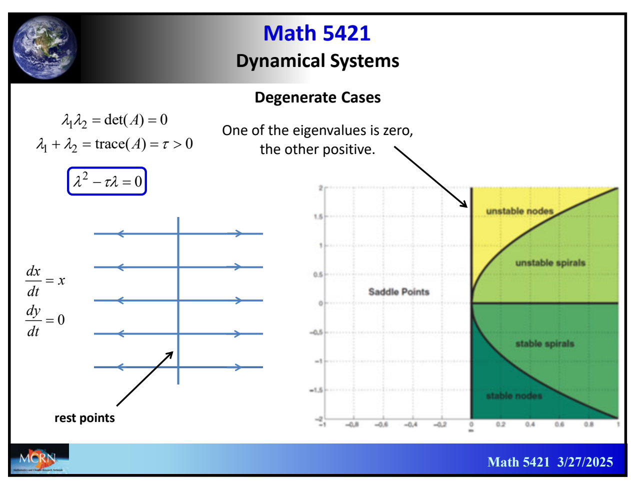

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Math 5421
Dynamical Systems

Degenerate Cases

$\lambda_1 \lambda_2 = \det(A) = 0$

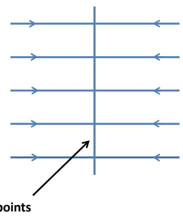
$\lambda_1 + \lambda_2 = \text{trace}(A) = \tau < 0$

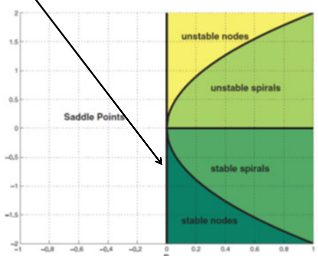
$\lambda^2 - \tau\lambda = 0$

One of the eigenvalues is zero, the other negative.

$\frac{dx}{dt} = -x$

$\frac{dy}{dt} = 0$





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Math 5421
Dynamical Systems

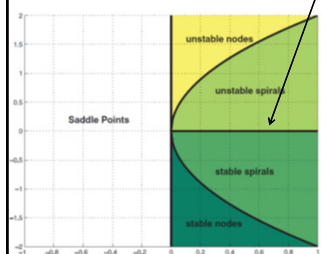
Degenerate Cases

$\lambda_1 + \lambda_2 = \text{trace}(A) = 0$

$\lambda_1 \lambda_2 = \det(A) = \delta > 0$

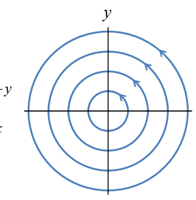
Imaginary pair of eigenvalues

$\lambda^2 + \delta = 0$



$\frac{dx}{dt} = -y$

$\frac{dy}{dt} = x$



center

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Math 5421
Dynamical Systems

Degenerate Cases

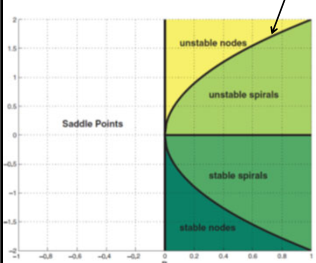
$\lambda_1 \lambda_2 = \delta = \det(A) > 0$

$\lambda_1 + \lambda_2 = \tau = \text{trace}(A) > 0$

$\tau^2 - 4\delta = 0$

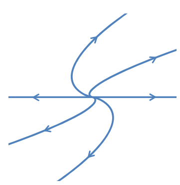
Positive double eigenvalue.
Only one eigenvector

$\lambda^2 - \tau\lambda + \frac{\tau^2}{4} = \left(\lambda - \frac{\tau}{2}\right)^2 = 0$



$\frac{dx}{dt} = x + y$

$\frac{dy}{dt} = y$



degenerate unstable node

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Math 5421
Dynamical Systems

Degenerate Cases

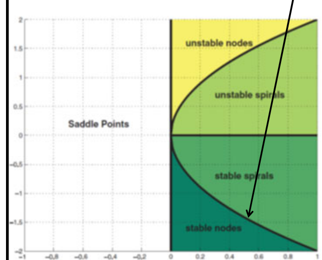
$\lambda_1 \lambda_2 = \delta = \det(A) > 0$

$\lambda_1 + \lambda_2 = \tau = \text{trace}(A) < 0$

$\tau^2 - 4\delta = 0$

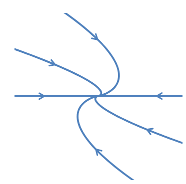
Negative double eigenvalue.
Only one eigenvector

$\lambda^2 - \tau\lambda + \frac{\tau^2}{4} = \left(\lambda - \frac{\tau}{2}\right)^2 = 0$



$\frac{dx}{dt} = -x + y$

$\frac{dy}{dt} = -y$



degenerate stable node

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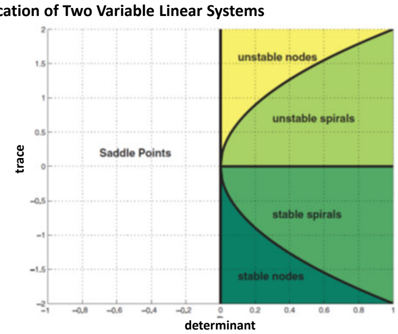
Math 5421
Dynamical Systems

Classification of Two Variable Linear Systems

$\frac{dx}{dt} = a_{11}x + a_{12}y$

$\frac{dy}{dt} = a_{21}x + a_{22}y$

We have now classified (given names) to the most common possible behaviors of two variable linear systems.



Kaper & Engler, 2013

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