Math 5421
An Introduction to Mathematical Climate Models

Spring 2025
1:25 – 3:20 Tuesdays and Thursdays
Blegen Hall 155

Richard McGehee, Instructor
458 Vincent Hall
mcgehee@umn.edu
www-users.cse.umn.edu/~mcgehee/

course website
https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/

Math 5421
Dynamical Systems

Dynamical Systems

The dependence of the solution on ***initial conditions*** is just as important as its dependence on ***time***.

$$x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n, \quad f:\mathbb{R}^n \to \mathbb{R}^n$$

intital value problem $\dot{x} = f(x)$, $x(0) = \xi$

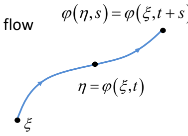
The initial value problem generates a flow

$$\varphi:\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

with properties

intital condition $\varphi(\xi,0) = \xi$

"group property" $\varphi(\varphi(\xi,t),s) = \varphi(\xi,t+s)$



If we start the system at state ξ and follow the solution for time t, then restart the system at the new state and follow the solution for time s, we end up at the same state as starting at ξ and following for time $t+s$.

Math 5421
Dynamical Systems

Classification of One Variable Linear Systems

Let's back up.
What is the classification of one variable linear systems?

$$\frac{dx}{dt} = ax$$

$a < 0$ **sink** — stable rest point

$a > 0$ **source** — unstable rest point

$a = 0$ **degenerate** — all points resting

degenerate: sink ($a < 0$), $a = 0$, source ($a > 0$)

Math 5421
Dynamical Systems

Classification of One Variable Linear Systems

Can we transform one linear equation into another with a linear change of variable?

$$\frac{dx}{dt} = ax \quad \text{to} \quad \frac{dx}{dt} = bx\,?$$

Math 5421
Dynamical Systems

Classification of One Variable Linear Systems

Can we transform one linear equation into another with a linear change of variable?

$$\frac{dx}{dt} = ax \quad \text{to} \quad \frac{dx}{dt} = bx\,?$$

Let's try.

$$x = c\xi \quad \Rightarrow \quad \frac{dx}{dt} = c\frac{d\xi}{dt} = ax = ac\xi$$

Math 5421
Dynamical Systems

Classification of One Variable Linear Systems

Can we transform one linear equation into another with a linear change of variable?

$$\frac{dx}{dt} = ax \quad \text{to} \quad \frac{dx}{dt} = bx\,?$$

Let's try.

$$x = c\xi \quad \Rightarrow \quad \frac{dx}{dt} = c\frac{d\xi}{dt} = ax = ac\xi$$

Cancel c.

$$\frac{d\xi}{dt} = a\xi$$

Math 5421
Dynamical Systems

Classification of One Variable Linear Systems

Can we transform one linear equation into another with a linear change of variable?

$$\frac{dx}{dt} = ax \quad \text{to} \quad \frac{dx}{dt} = bx\ ?$$

Let's try.

$$x = c\xi \quad \Rightarrow \quad \frac{dx}{dt} = c\frac{d\xi}{dt} = ax = ac\xi$$

no change!

No!

Cancel c.

$$\frac{d\xi}{dt} = a\xi$$

Only if $a = b$!

Math 5421 4/1/2025

8

Math 5421
Dynamical Systems

Algebraic Classification of One Variable Linear Systems

$$\frac{dx}{dt} = ax$$

They are all different.

Math 5421 4/1/2025

9

Math 5421
Dynamical Systems

Algebraic Classification of Two Variable Linear Systems

$$\frac{dx}{dt} = Ax \qquad \frac{dx}{dt} = Bx$$

Assume that there are two distinct eigenvalues for both A and B.

If the eigenvalues of A are the same as those of B, then the two systems are algebraically equivalent, i.e., one can be transformed to the other by a linear change of variables.

If the eigenvalues of A are different from those of B, then the two systems are not algebraically equivalent, i.e., one cannot be transformed to the other by a linear change of variables.

The situation is more complicated for double eigenvalues.

Math 5421 4/1/2025

10

Math 5421
Dynamical Systems

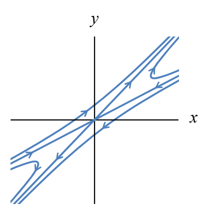
Algebraic Classification

Example

$$\frac{dx}{dt} = -3x + 4y \qquad \frac{dy}{dt} = -2x + 3y$$

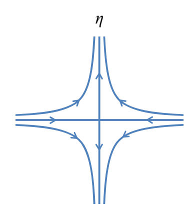
These two systems "look the same" from a linear algebra perspective, because one can be transformed to the other with a linear coordinate change.

$$\frac{d\xi}{dt} = -\xi \qquad \frac{d\eta}{dt} = \eta$$



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

Coordinate Change



Math 5421 4/1/2025

11

Math 5421
Dynamical Systems

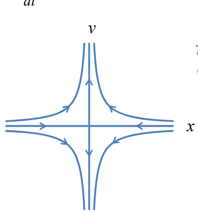
Algebraic Classification

Example

$$\frac{dx}{dt} = -2x \qquad \frac{dy}{dt} = 2y$$

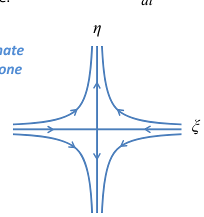
These two systems do not "look the same" from a linear algebraic perspective, because one cannot be transformed to the other with a linear coordinate change.

$$\frac{d\xi}{dt} = -\xi \qquad \frac{d\eta}{dt} = \eta$$



There is no linear coordinate change that transforms one system to the other.

Why not?



Math 5421 4/1/2025

12

Math 5421
Dynamical Systems

Algebraic Classification

Example

$$\frac{dx}{dt} = -2x \qquad \frac{dy}{dt} = 2y$$

There is no linear coordinate change that transforms one system to the other.

Why not?

$$\frac{d\xi}{dt} = -\xi \qquad \frac{d\eta}{dt} = \eta$$

Math 5421 4/1/2025

13

Math 5421
Dynamical Systems

Algebraic Classification

Example

eigenvalues -2, 2

$$\frac{dx}{dt} = -2x \qquad \frac{dy}{dt} = 2y$$

There is no linear coordinate change that transforms one system to the other.

Why not?

$$\frac{d\xi}{dt} = -\xi \qquad \frac{d\eta}{dt} = \eta$$

eigenvalues -1, 1

Let S be a nonsingular matrix, and let $\begin{bmatrix} x \\ y \end{bmatrix} = S \begin{bmatrix} \xi \\ \eta \end{bmatrix}$.

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

$$S\frac{d}{dt}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{d}{dt}S\begin{bmatrix} \xi \\ \eta \end{bmatrix} = AS\begin{bmatrix} \xi \\ \eta \end{bmatrix} \implies \frac{d}{dt}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = S^{-1}AS\begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

A and $S^{-1}AS$ have the same eigenvalues.

Why?

14

Math 5421
Dynamical Systems

Algebraic Classification

A and $S^{-1}AS$ have the same eigenvalues.

Why?

Let λ be an eigenvalue of A with corresponding eigenvector v, and let S be a nonsingular matrix. Then

$$\left(S^{-1}AS\right)\left(S^{-1}v\right) = S^{-1}A\left(SS^{-1}\right)v = S^{-1}Av = S^{-1}\lambda v = \lambda S^{-1}v = \lambda\left(S^{-1}v\right),$$

so λ is an eigenvalue of $S^{-1}AS$ with corresponding eigenvector $S^{-1}v$.

Therefore, if $\frac{dx}{dt} = Ax$ can be transformed to $\frac{dx}{dt} = Bx$ by a linear coordinate change, then A and B must have the same eigenvalues.

15

Math 5421
Dynamical Systems

Algebraic Classification

Back to Example

$$\frac{dx}{dt} = -2x \qquad \frac{dy}{dt} = 2y$$

There is no linear coordinate change that transforms one system to the other.

Why not?

$$\frac{d\xi}{dt} = -\xi \qquad \frac{d\eta}{dt} = \eta$$

$\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ has eigenvalues -2 and 2, but $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ has eigenvalues -1 and 1.

There cannot be a linear coordinate change transforming one system to the other system.

16

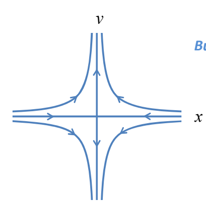
Math 5421
Dynamical Systems

Topological Classification

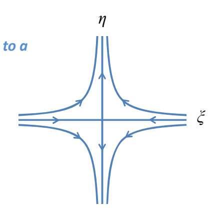
$$\frac{dx}{dt} = -2x \qquad \frac{dy}{dt} = 2y$$

These two systems do not "look the same" algebraically.

$$\frac{d\xi}{dt} = -\xi \qquad \frac{d\eta}{dt} = \eta$$



But they "look the same" to a topologist.



17

Math 5421
Dynamical Systems

Topological Classification

What do we mean when we say that two systems look the same to a topologist?

$$\frac{dx}{dt} = Ax \quad \Longleftrightarrow \quad \frac{d\xi}{dt} = B\xi$$

topologically conjugate?

These two systems are ***topologically conjugate*** if there is a <u>continuous coordinate change</u> taking one system to the other.

$x = f(\xi)$ transforms $\frac{dx}{dt} = Ax$ to $\frac{d\xi}{dt} = B\xi$.

Precisely what this means is beyond the scope of this course.

18

Math 5421
Dynamical Systems

Topological Classification

For one variable, all sources are topologically conjugate.

$$\frac{dx}{dt} = \alpha x, \quad \alpha > 0$$

$$\text{Let } x = |u|^\alpha \frac{u}{|u|} = \begin{cases} u^\alpha & u > 0 \\ -(-u)^\alpha & u < 0 \end{cases}$$

19

Math 5421
Dynamical Systems

Topological Classification

For one variable, all sources are topologically conjugate.

$$\frac{dx}{dt} = \alpha x, \quad \alpha > 0$$

$$\text{Let } x = |u|^{\alpha}\frac{u}{|u|} = \begin{cases} u^{\alpha} & u > 0 \\ -(-u)^{\alpha} & u < 0 \end{cases}$$

$$u > 0: \quad \frac{dx}{dt} = \alpha u^{\alpha-1}\frac{du}{dt} = \alpha x = \alpha u^{\alpha} = \alpha u^{\alpha-1}u \qquad \frac{du}{dt} = u$$

$$u < 0: \quad \frac{dx}{dt} = -\alpha(-u)^{\alpha-1}\left(-\frac{du}{dt}\right) = \alpha(-u)^{\alpha-1}\frac{du}{dt} = \alpha x = \alpha\left(-(-u)^{\alpha}\right) = \alpha(-u)^{\alpha-1}u \qquad \frac{du}{dt} = u$$

Math 5421 4/1/2025

20

Math 5421
Dynamical Systems

Topological Classification

For one variable, all sources are topologically conjugate.

$$\frac{dx}{dt} = \alpha x, \quad \alpha > 0$$

$$\text{Let } x = |u|^{\alpha}\frac{u}{|u|} = \begin{cases} u^{\alpha} & u > 0 \\ -(-u)^{\alpha} & u < 0 \end{cases}$$

$$u > 0: \quad \frac{dx}{dt} = \alpha u^{\alpha-1}\frac{du}{dt} = \alpha x = \alpha u^{\alpha} = \alpha u^{\alpha-1}u \qquad \frac{du}{dt} = u$$

$$u < 0: \quad \frac{dx}{dt} = -\alpha(-u)^{\alpha-1}\left(-\frac{du}{dt}\right) = \alpha(-u)^{\alpha-1}\frac{du}{dt} = \alpha x = \alpha\left(-(-u)^{\alpha}\right) = \alpha(-u)^{\alpha-1}u \qquad \frac{du}{dt} = u$$

Continuous, but not smooth at 0.

$$\frac{dx}{dt} = \alpha x \quad \Longrightarrow \quad x = |u|^{\alpha}\frac{u}{|u|} \quad \Longrightarrow \quad \frac{du}{dt} = u$$

Math 5421 4/1/2025

21

Math 5421
Dynamical Systems

Topological Classification

For one variable, all sinks are topologically conjugate.

$$\frac{dx}{dt} = -\alpha x \quad \alpha > 0$$

$$\text{Let } x = |u|^{\alpha}\frac{u}{|u|} = \begin{cases} u^{\alpha} & u > 0 \\ -(-u)^{\alpha} & u < 0 \end{cases}$$

$$u > 0: \quad \frac{dx}{dt} = \alpha u^{\alpha-1}\frac{du}{dt} = -\alpha x = -\alpha u^{\alpha} = -\alpha u^{\alpha-1}u \qquad \frac{du}{dt} = -u$$

$$u < 0: \quad \frac{dx}{dt} = -\alpha(-u)^{\alpha-1}\left(-\frac{du}{dt}\right) = \alpha(-u)^{\alpha-1}\frac{du}{dt} = -\alpha x = -\alpha\left(-(-u)^{\alpha}\right) = -\alpha(-u)^{\alpha-1}u \qquad \frac{du}{dt} = -u$$

Continuous, but not smooth at 0.

$$\frac{dx}{dt} = -\alpha x \quad \alpha > 0 \quad \Longrightarrow \quad x = |u|^{\alpha}\frac{u}{|u|} \quad \Longrightarrow \quad \frac{du}{dt} = -u$$

Math 5421 4/1/2025

22

Math 5421
Dynamical Systems

Topological Classification for One Variable Linear Systems.

$$\frac{dx}{dt} = ax$$

There are exactly three classes.

1. **sources**: ($a > 0$) All sources are topologically conjugate.
2. **sinks**: ($a < 0$) All sinks are topologically conjugate.
3. **degenerate:** ($a = 0$) Only one case.

Math 5421 4/1/2025

23

Math 5421
Dynamical Systems

Topological Classification of Two Variable Linear Systems

If neither eigenvalue has zero real part, then the system is called ***hyperbolic***, in which case, there are only three classes:

1. **saddles**: One positive eigenvalue and one negative. The determinant is negative.
   Any two saddles are topologically conjugate.
2. **sources**: Both eigenvalues have positive real part. The determinant is positive, and the trace is positive.
   Any two sources are topologically conjugate.
3. **sinks**: Both eigenvalues have negative real part. The determinant is positive, and the trace is negative.
   Any two sinks are topologically conjugate.

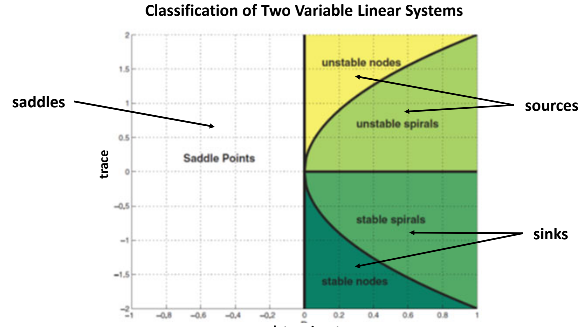
Nonhyperbolic systems are more complicated.

Math 5421 4/1/2025

24

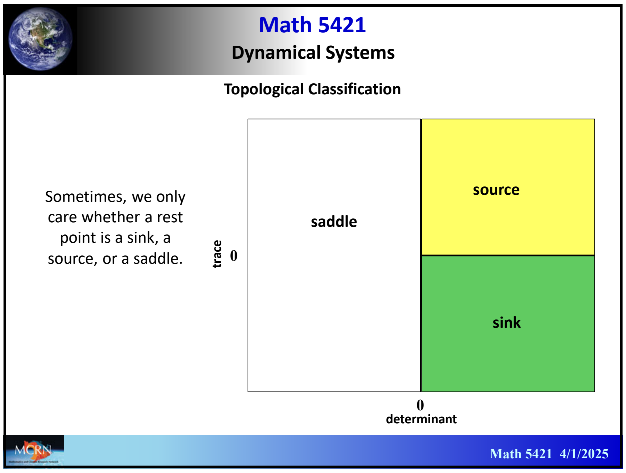
Math 5421
Dynamical Systems

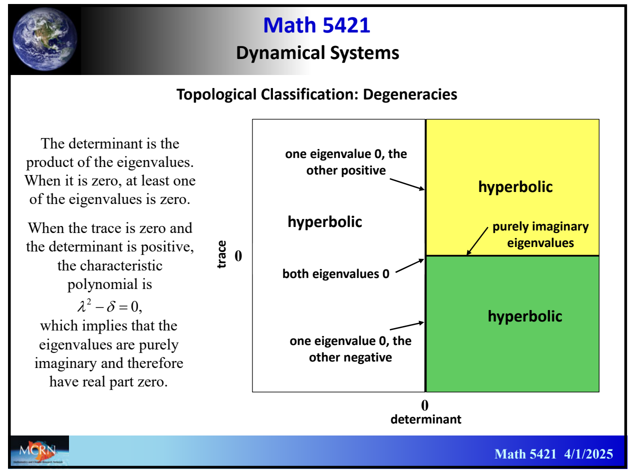
Classification of Two Variable Linear Systems



Math 5421 4/1/2025

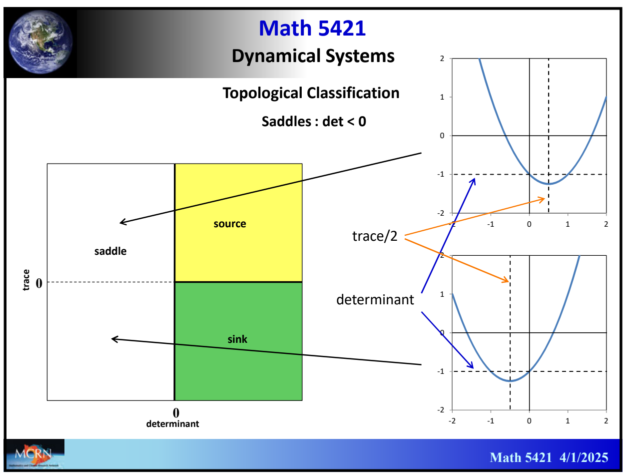
25

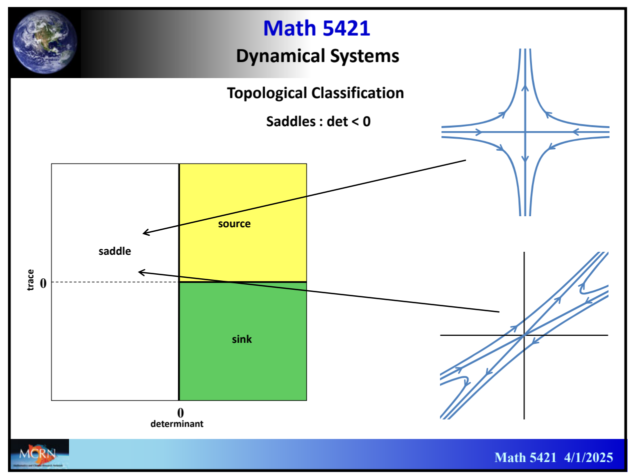




26

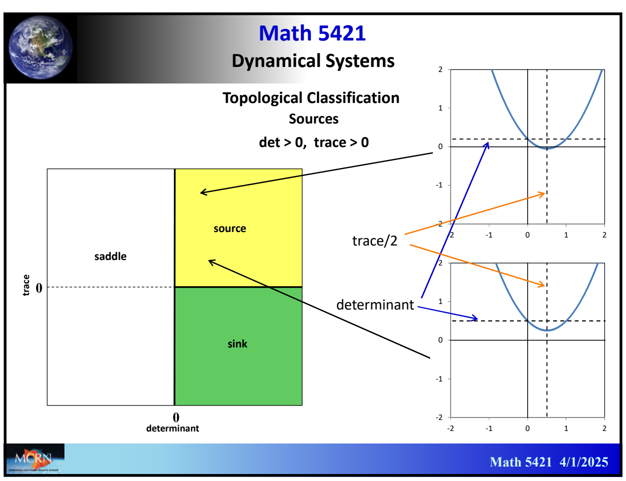
27

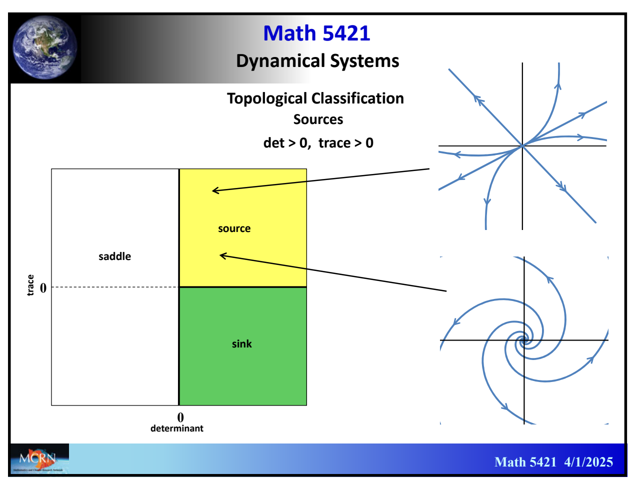




28

29



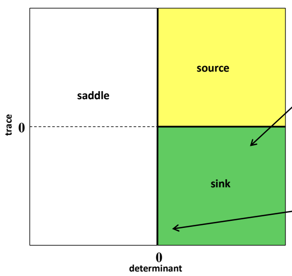


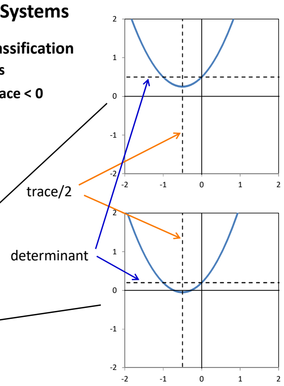
30

31

Math 5421
Dynamical Systems

Topological Classification
Sinks
det > 0, trace < 0



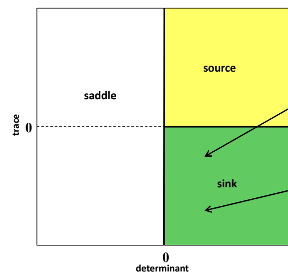


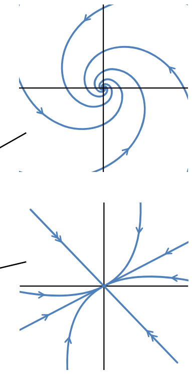
Math 5421 4/1/2025

32

Math 5421
Dynamical Systems

Topological Classification
Sinks
det > 0, trace < 0





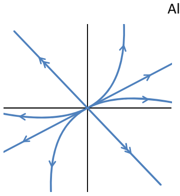
Math 5421 4/1/2025

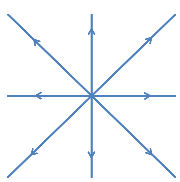
33

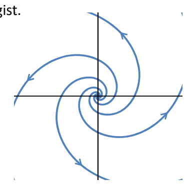
Math 5421
Dynamical Systems

Topological Classification

All sources look the same to a topologist.
Everything leaves.





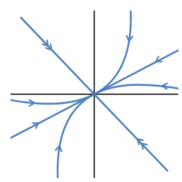


Math 5421 4/1/2025

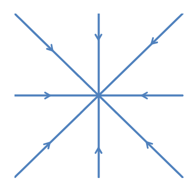
34

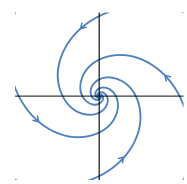
Math 5421
Dynamical Systems

Topological Classification



All sinks look the same.
Everything approaches the origin asymptotically.





Math 5421 4/1/2025

35

Math 5421
Dynamical Systems

Topological Classification
Example

$$\frac{dx}{dt} = -2x \qquad \frac{dy}{dt} = 2y$$

There is no linear coordinate change that transforms one system to the other.

$$\frac{du}{dt} = -u \qquad \frac{dv}{dt} = v$$

However, there is a continuous coordinate change that transforms one system to the other.

Apply the transformations in previous slides in the one variable case, once with $\alpha = 2$, and once with $\alpha = -2$.

$$\begin{matrix} \frac{dx}{dt} = -2x \\ \frac{dy}{dt} = 2y \end{matrix} \Longrightarrow \boxed{\begin{matrix} x = |u|u \\ y = |v|v \end{matrix}} \Longrightarrow \begin{matrix} \frac{du}{dt} = -u \\ \frac{dv}{dt} = v \end{matrix}$$

Continuous,
but not smooth at (0,0).

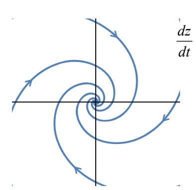
The two systems are topologically conjugate.

Math 5421 4/1/2025

36

Math 5421
Dynamical Systems

Topological Classification
What about spirals?

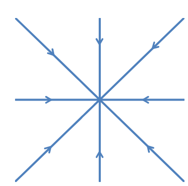


$$\frac{dz}{dt} = (-1-i)z$$

$$z = |w|^i w = e^{i\log|w|}w$$

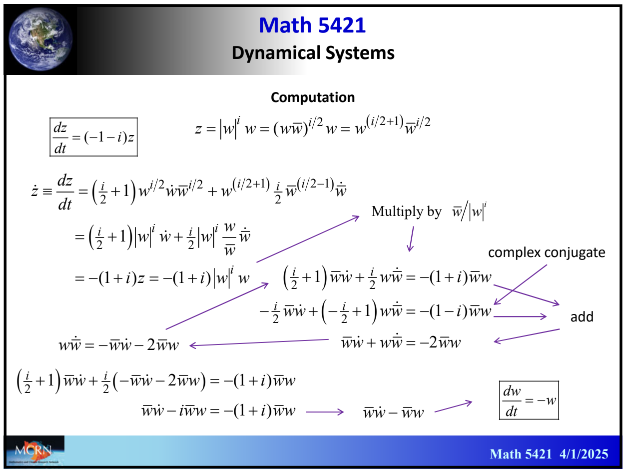
Continuous,
but not smooth at 0.

$$\frac{dw}{dt} = -w$$

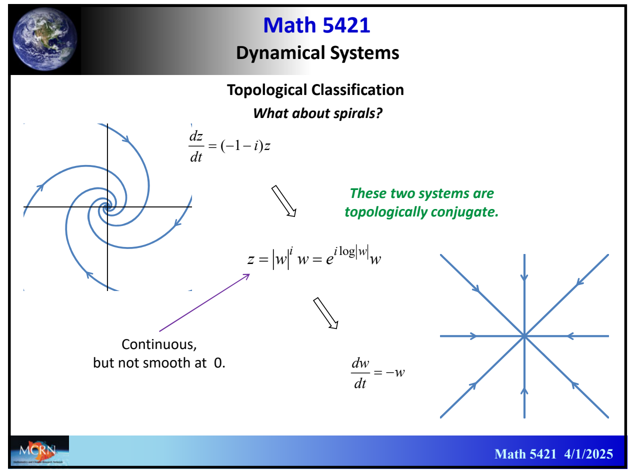


Math 5421 4/1/2025

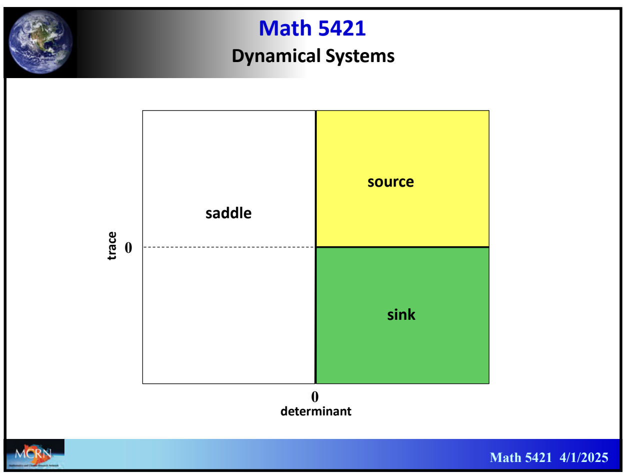
37



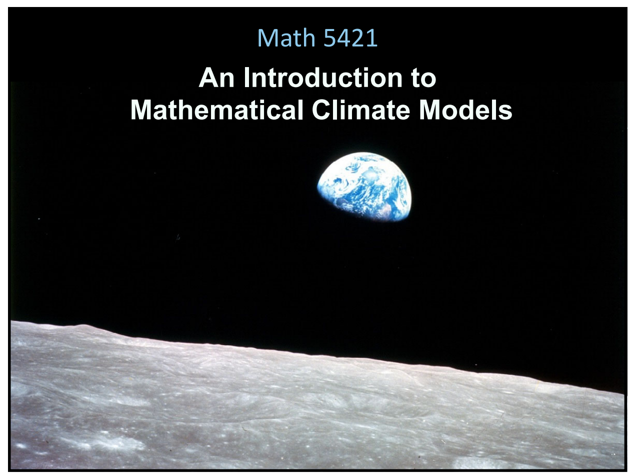
38



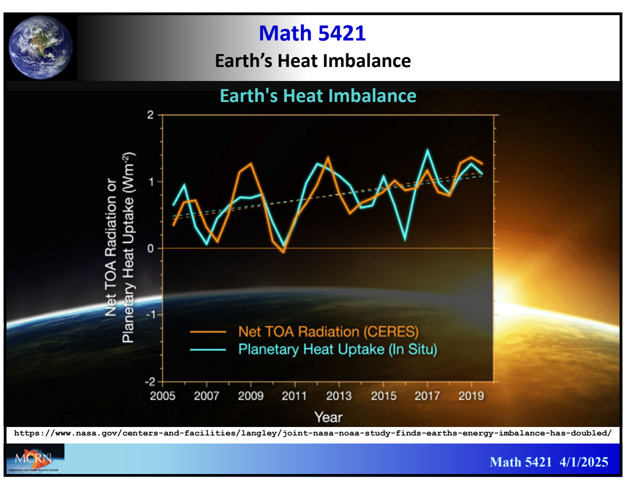
39



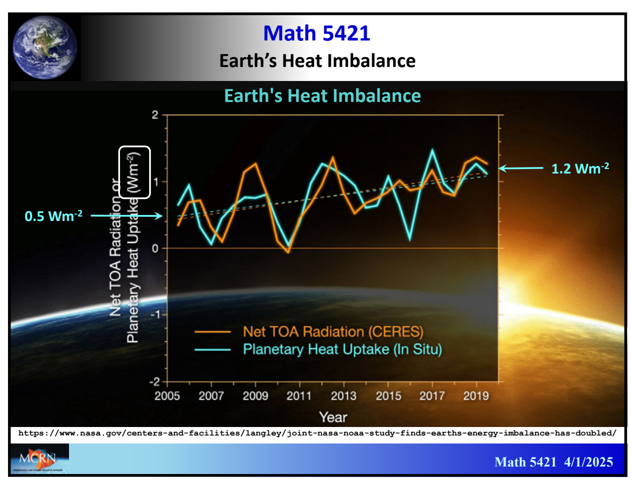
40



41



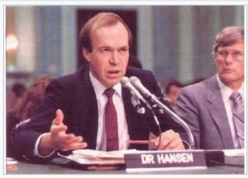
42



43

Math 5421
Earth's Heat Imbalance

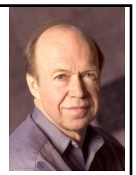
James Hansen



James Hansen arrested at a demonstration outside the White House, August 29, 2011

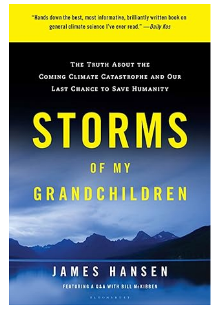
Hansen giving testimony before the United States Congress in 1988.

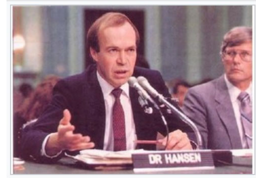
https://en.wikipedia.org/wiki/James_Hansen



44

Math 5421
Earth's Heat Imbalance
James Hansen





Hansen giving testimony before the United States Congress in 1988.

45

Math 5421
Earth's Heat Imbalance

www.sciencemag.org SCIENCE VOL 308 3 JUNE 2005

RESEARCH ARTICLES

Earth's Energy Imbalance: Confirmation and Implications

James Hansen,[1,2*] Larissa Nazarenko,[1,2] Reto Ruedy,[3] Makiko Sato,[1,2] Josh Willis,[4] Anthony Del Genio,[1,5] Dorothy Koch,[1,2] Andrew Lacis,[1,5] Ken Lo,[3] Surabi Menon,[6] Tica Novakov,[6] Judith Perlwitz,[1,2] Gary Russell,[1] Gavin A. Schmidt,[1,2] Nicholas Tausnev[3]

Our climate model, driven mainly by increasing human-made greenhouse gases and aerosols, among other forcings, calculates that Earth is now absorbing 0.85 ± 0.15 watts per square meter more energy from the Sun than it is emitting to space. This imbalance is confirmed by precise measurements of increasing ocean heat content over the past 10 years. Implications include (i) the expectation of additional global warming of about 0.6°C without further change of atmospheric composition; (ii) the confirmation of the climate system's lag in responding to forcings, implying the need for anticipatory actions to avoid any specified level of climate change; and (iii) the likelihood of acceleration of ice sheet disintegration and sea level rise.

equal forcing by CO_2 (9). F_e is an energy flux change arising in response to an imposed forcing agent. It is constant throughout the atmosphere, because it is evaluated after atmospheric temperature has been allowed to adjust to the presence of the forcing agent.

The largest forcing is due to well-mixed greenhouse gases (GHGs)—CO_2, CH_4, N_2O, CFCs (chlorofluorocarbons)—and other trace gases, totaling 2.75 W/m² in 2003 relative to the 1880 value (Table 1). Ozone (O_3) and stratospheric H_2O from oxidation of increasing CH_4 bring the total GHG forcing to 3.05 W/m² (9). Estimated uncertainty in the total GHG forcing is ~15% (11, 12).

Atmospheric aerosols cause climate forcings by reflecting and absorbing radiation, as well as through indirect effects on cloud cover and cloud albedo (11). The aerosol scenario in our model uses estimated anthropogenic emissions from fuel use statistics and includes temporal changes in fossil-fuel use technologies (13). Our parameterization of aerosol

46

Math 5421
Earth's Heat Imbalance

Table S1. Planetary Heat Storage: Ocean, Ice, Air and Land.

Energy required to melt ice and warm the air, land and ocean by specified amounts.[1]

Ocean warming by 1°C through 1 km depth of ocean. Heat storage is 1°C × 10^5 g/cm² × 1 cal/g × 4.19 joules/cal × area Earth × 0.7 ~ 15 × 10^{23} joules ~ **93 W yr/m²**.

Ice sheet melting to raise sea level 1 meter. Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C). Energy required is 100 cal/g (80 cal/g for melting). Energy for 1 meter of sea level: 100g/cm² × 100cal/g × 4.19 joules/cal × area Earth × 0.7 ~1.5 × 10^{23} joules ~ **9.3 W yr/m²**.

Sea ice melting (all sea ice on planet). Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C), and that sea ice covers 4% of the planet with mean thickness 2.5 m. Energy required is 250 g cm² × 100 cal/g (80 cal/g for melting) × 4.19 joules/cal × 0.04 × area Earth ~ 2.14 × 10^{22} joules ~ **1.3 W yr/m²**.

Air warming by 1°C. The Earth's atmospheric mass is ~ 10 m of water. Heat capacity of air ~ 0.24 cal/g/°C. Energy to raise air temperature 1°C: 1°C × 1000 g/cm² × 0.24 cal/g/°C × 4.19 joules/cal × area Earth ~ 0.26 ×10^{22} joules ~ **0.32 W yr/m².**

Land surface warming by 1°C: The depth of penetration of a thermal wave into the Earth's crust in 10 years, weighted by ΔT, is ~10 m. With density ~ 3 g/cm³, heat capacity ~ 0.2 cal/g/°C, and 0.29 fractional land coverage, land heat storage is 10^3 cm ×3 g/cm³ × 0.2 cal/g/°C × 1°C ×4.19 joules/cal × area Earth × 0.29 ~ 0.37 × 10^{22} joules ~ **0.23 W yr**. [In a century the depth of penetration is ~3 times more than in a decade, so heat storage in a century due to 1°C warming is ~ **0.7 W yr/m²**.]

[1]Note that 1 W sec = 1 joule, # sec/year ~ $\pi \times 10^7$, area Earth ~ 5.1×10^{18} cm², 1 W yr over full Earth ~ 1.61×10^{22} joules, ocean fraction of Earth ~ 0.7, 1 calorie ~ 4.19 joules.

James Hansen, et al, *Earth's Energy Imbalance: Confirmation and Implications*, SCIENCE 308 (2005), p. 1431

47

Math 5421
Earth's Heat Imbalance

Table S1. Planetary Heat Storage: Ocean, Ice, Air and Land.

Energy required to melt ice and warm the air, land and ocean by specified amounts.[1]

What's this?

Ocean warming by 1°C through 1 km depth of ocean. Heat storage is 1°C × 10^5 g/cm² × 1 cal/g × 4.19 joules/cal × area Earth × 0.7 ~ 15 × 10^{23} joules ~ 93 W yr/m².

Ice sheet melting to raise sea level 1 meter. Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C). Energy required is 100 cal/g (80 cal/g for melting). Energy for 1 meter of sea level: 100g/cm² × 100cal/g × 4.19 joules/cal × area Earth × 0.7 ~1.5 × 10^{23} joules ~ 9.3 W yr/m².

Sea ice melting (all sea ice on planet). Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C), and that sea ice covers 4% of the planet with mean thickness 2.5 m. Energy required is 250 g cm² × 100 cal/g (80 cal/g for melting) × 4.19 joules/cal × 0.04 × area Earth ~ 2.14 × 10^{22} joules ~ 1.3 W yr/m².

Air warming by 1°C. The Earth's atmospheric mass is ~ 10 m of water. Heat capacity of air ~ 0.24 cal/g/°C. Energy to raise air temperature 1°C: 1°C × 1000 g/cm² × 0.24 cal/g/°C × 4.19 joules/cal × area Earth ~ 0.26 ×10^{22} joules ~ 0.32 W yr/m².

Land surface warming by 1°C: The depth of penetration of a thermal wave into the Earth's crust in 10 years, weighted by ΔT, is ~10 m. With density ~ 3 g/cm³, heat capacity ~ 0.2 cal/g/°C, and 0.29 fractional land coverage, land heat storage is 10^3 cm ×3 g/cm³ × 0.2 cal/g/°C × 1°C ×4.19 joules/cal × area Earth × 0.29 ~ 0.37 × 10^{22} joules ~ 0.23 W yr. [In a century the depth of penetration is ~3 times more than in a decade, so heat storage in a century due to 1°C warming is ~ 0.7 W yr/m².]

[1]Note that 1 W sec = 1 joule, # sec/year ~ $\pi \times 10^7$, area Earth ~ 5.1×10^{18} cm², 1 W yr over full Earth ~ 1.61×10^{22} joules, ocean fraction of Earth ~ 0.7, 1 calorie ~ 4.19 joules.

James Hansen, et al, Earth's Energy Imbalance: Confirmation and Implications, SCIENCE 308 (2005), p. 1431

48

Math 5421
Earth's Heat Imbalance

What is a watt-year?

A ***watt*** is a unit of power, or energy per unit time, i.e., ***one joule per second***.

A year is about 3.14×10^7 seconds, so a ***watt-year*** is about **3.14×10^7 *joules*.**

What is a watt-year per square meter?

About **3.14×10^7 *joules per square meter*.**

If the heat imbalance at the Earth's surface is one watt per square meter, then the energy absorbed over the course of a year is about **3.14×10^7 *joules per square meter*.**

Earth's surface area: about ***5.1×10^{14} square meters***.

If the heat imbalance at the Earth's surface is one watt per square meter, then the energy absorbed over the whole Earth is about ***1.61×10^{22} joules***.

49

Math 5421
Earth's Heat Imbalance

Table S1. Planetary Heat Storage: Ocean, Ice, Air and Land.

Energy required to melt ice and warm the air, land and ocean by specified amounts.[1]

What's this?

Ocean warming by 1°C through 1 km depth of ocean. Heat storage is 1°C × 10^5 g/cm^2 × 1 cal/g × 4.19 joules/cal × area Earth × 0.7 ~ 15 × 10^{23} joules ~ **93 W yr/m^2**.

Ice sheet melting to raise sea level 1 meter. Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C). Energy required is 100 cal/g (80 cal/g for melting). Energy for 1 meter of sea level: 100g/cm^2 × 100cal/g × 4.19 joules/cal × area Earth × 0.7 ~1.5 × 10^{23} joules ~ **9.3 W yr/m^2**.

Sea ice melting (all sea ice on planet). Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C), and that sea ice covers 4% of the planet with mean thickness 2.5 m. Energy required is 250 g cm^2 × 100 cal/g (80 cal/g for melting) × 4.19 joules/cal × 0.04 × area Earth ~ 2.14 × 10^{22} joules ~ **1.3 W yr/m^2**.

Air warming by 1°C. The Earth's atmospheric mass is ~ 10 m of water. Heat capacity of air ~ 0.24 cal/g/°C. Energy to raise air temperature 1°C: 1°C × 1000 g/cm^2 × 0.24 cal/g/°C × 4.19 joules/cal × area Earth ~ 0.26 ×10^{22} joules ~ **0.32 W yr/m^2.**

Land surface warming by 1°C: The depth of penetration of a thermal wave into the Earth's crust in 10 years, weighted by ΔT, is ~10 m. With density ~ 3 g/cm^3, heat capacity ~ 0.2 cal/g/°C, and 0.29 fractional land coverage, land heat storage is 10^3 cm ×3 g/cm^3 × 0.2 cal/g/°C × 1°C ×4.19 joules/cal × area Earth × 0.29 ~ 0.37 × 10^{22} joules ~ **0.23 W yr**. [In a century the depth of penetration is ~3 times more than in a decade, so heat storage in a century due to 1°C warming is ~ **0.7 W yr/m^2**.]

[1]Note that 1 W sec = 1 joule, # sec/year ~ π×10^7, area Earth ~ 5.1×10^{18} cm^2, 1 W yr over full Earth ~ 1.61×10^{22} joules, ocean fraction of Earth ~ 0.7, 1 calorie ~ 4.19 joules.

James Hansen, et al, *Earth's Energy Imbalance: Confirmation and Implications*, SCIENCE 308 (2005), p. 1431

Math 5421 4/1/2025

50

Math 5421
Earth's Heat Imbalance

How much heat does it take to melt enough of the ice sheets to raise the sea level by one meter?

Math 5421 4/1/2025

51

Math 5421
Earth's Heat Imbalance

How much heat does it take to melt enough of the ice sheets to raise the sea level by one meter?

Assumption: Ice starts at -10°C and ends up as ocean water at +15°C.
Hansen: About 1.5×10^{23} joules ≈ ***9.3 Wyr/m^2***

Math 5421 4/1/2025

52

Math 5421
Earth's Heat Imbalance

How much heat does it take to melt enough of the ice sheets to raise the sea level by one meter?

Assumption: Ice starts at -10°C and ends up as ocean water at +15°C.
Hansen: About 1.5×10^{23} joules ≈ ***9.3 Wyr/m^2***

How long does it take to melt enough of the ice sheets to raise the sea level by one meter?

Math 5421 4/1/2025

53

Math 5421
Earth's Heat Imbalance

How much heat does it take to melt enough of the ice sheets to raise the sea level by one meter?

Assumption: Ice starts at -10°C and ends up as ocean water at +15°C.
Hansen: About 1.5×10^{23} joules ≈ ***9.3 Wyr/m^2***

How long does it take to melt enough of the ice sheets to raise the sea level by one meter?

If the heat imbalance at the Earth's surface is one watt per square meter, and if all the heat imbalance goes toward melting the ice sheets, then the time required to raise the sea level one meter is ***9.3 years***.

meters per year: 1/9.3 = ***0.108 m/yr***
= 10.8 cm/yr = 1.08 m/decade = 10.8 m/century

Math 5421 4/1/2025

54

Math 5421
Earth's Heat Imbalance

Ocean warming by 1°C through 1 km depth of ocean. Heat storage is 1°C × 10^5 g/cm^2 × 1 cal/g × 4.19 joules/cal × area Earth × 0.7 ~ 15 × 10^{23} joules ~ **93 W yr/m^2**.

Ice sheet melting to raise sea level 1 meter. Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C). Energy required is 100 cal/g (80 cal/g for melting). Energy for 1 meter of sea level: 100g/cm^2 × 100cal/g × 4.19 joules/cal × area Earth × 0.7 ~1.5 × 10^{23} joules ~ **9.3 W yr/m^2**.

Asuming an energy imbalance of 1.2 W/m^2, how long would it take to raise the temperature of the entire ocean by 5°C? (Assume that the average ocean depth is 4300 meters.) How long would it take to melt enough ice to raise the ocean depth by 70 meters?

Math 5421 4/1/2025

55

Math 5421
Earth's Heat Imbalance

Ocean warming by 1°C through 1 km depth of ocean. Heat storage is 1°C × 10^5 g/cm^2 × 1 cal/g × 4.19 joules/cal × area Earth × 0.7 ~ 15 × 10^{23} joules ~ **93 W yr/m^2**.

Ice sheet melting to raise sea level 1 meter. Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C). Energy required is 100 cal/g (80 cal/g for melting). Energy for 1 meter of sea level: 100g/cm^2 × 100cal/g × 4.19 joules/cal × area Earth × 0.7 ~1.5 × 10^{23} joules ~ **9.3 W yr/m^2**.

By some estimates, it took 20,000 years to raise the temperature of the entire ocean 5°C during the PETM (Paleoeocene-Eocene Thermal Maximum, occuring about 56 million years ago). What energy imbalance would account for that rise. Assume all other temperatures were unchanged and that the ocean depth was 4400 meters.

Math 5421 4/1/2025

56

Math 5421
Earth's Heat Imbalance

Ocean warming by 1°C through 1 km depth of ocean. Heat storage is 1°C × 10^5 g/cm^2 × 1 cal/g × 4.19 joules/cal × area Earth × 0.7 ~ 15 × 10^{23} joules ~ **93 W yr/m^2**.

Ice sheet melting to raise sea level 1 meter. Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C). Energy required is 100 cal/g (80 cal/g for melting). Energy for 1 meter of sea level: 100g/cm^2 × 100cal/g × 4.19 joules/cal × area Earth × 0.7 ~1.5 × 10^{23} joules ~ **9.3 W yr/m^2**.

What energy imbalance would be required to melt enough ice to raise the ocean depth by 70 meters in 1000 years, assuming all atmosphere, land, and ocean temperatures remained constant? What energy imbalance would accomplish the same rise in 100 years?

Math 5421 4/1/2025

57

Math 5421
Earth's Heat Imbalance

Ocean warming by 1°C through 1 km depth of ocean. Heat storage is 1°C × 10^5 g/cm^2 × 1 cal/g × 4.19 joules/cal × area Earth × 0.7 ~ 15 × 10^{23} joules ~ **93 W yr/m^2**.

Ice sheet melting to raise sea level 1 meter. Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C). Energy required is 100 cal/g (80 cal/g for melting). Energy for 1 meter of sea level: 100g/cm^2 × 100cal/g × 4.19 joules/cal × area Earth × 0.7 ~1.5 × 10^{23} joules ~ **9.3 W yr/m^2**.

Air warming by 1°C. The Earth's atmospheric mass is ~ 10 m of water. Heat capacity of air ~ 0.24 cal/g/°C. Energy to raise air temperature 1°C: 1°C × 1000 g/cm^2 × 0.24 cal/g/°C × 4.19 joules/cal × area Earth ~ 0.26 ×10^{22} joules ~ **0.32 W yr/m^2.**

Assume that, over the course of 100,000 years, the air temperature fell by 5°C, as did the top kilometer of the ocean, and ice sheets formed to lower the sea level by 125 meters. What average energy imbalance would be required?

Math 5421 4/1/2025

58

Math 5421
Earth's Heat Imbalance

Ocean warming by 1°C through 1 km depth of ocean. Heat storage is 1°C × 10^5 g/cm^2 × 1 cal/g × 4.19 joules/cal × area Earth × 0.7 ~ 15 × 10^{23} joules ~ **93 W yr/m^2**.

Ice sheet melting to raise sea level 1 meter. Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C). Energy required is 100 cal/g (80 cal/g for melting). Energy for 1 meter of sea level: 100g/cm^2 × 100cal/g × 4.19 joules/cal × area Earth × 0.7 ~1.5 × 10^{23} joules ~ **9.3 W yr/m^2**.

Air warming by 1°C. The Earth's atmospheric mass is ~ 10 m of water. Heat capacity of air ~ 0.24 cal/g/°C. Energy to raise air temperature 1°C: 1°C × 1000 g/cm^2 × 0.24 cal/g/°C × 4.19 joules/cal × area Earth ~ 0.26 ×10^{22} joules ~ **0.32 W yr/m^2.**

(*The Day After Tomorrow* scenario) Assume that, over the course of 6 weeks, the air temperature dropped by 13°C, as did the top 100 meters of the ocean, and enough snow accumulated on land to lower the sea level by 2 meters. What energy imbalance would be required? Compare your number to the current insolation and compute the heat imbalance necessary for the scenario.

Math 5421 4/1/2025

59