Math 5421

# An Introduction to Mathematical Climate Models

**Spring 2025**
**1:25 – 3:20 Tuesdays and Thursdays**
**Blegen Hall 155**

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course website
https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/



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## Math 5421
### Chaos

**Can We Predict the Future?**

**If we know the state of a system now, do we know its state in the future?**

For models based on differential equations, the answer is 'yes".

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n, \quad x = x_0 \text{ when } t = 0$$

If $f$ is sufficiently smooth (e.g., continuously differentiable) then there is a unique solution of the differential equations satisfying the initial condition.

**Interpretation:**

If we know the state of the system now, we can compute its state in the future.



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### Chaos

Example of a late 19th century mathematical question:

*Is the solar system stable?*

Example of a late 20th century mathematical question:

*Can we predict the weather?*

**What happened?**



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## Math 5421
### Chaos

Example of a late 19th century mathematical question:

*Is the solar system stable?*

Example of a late 20th century mathematical question:

*Can we predict the weather?*

**What happened?**

**Quantum Mechanics**
Elementary particles are probability distributions.
Heisenberg Uncertainty

**Gödel's Incompleteness Theorem**
There are mathematical statements which can be neither proved nor disproved.

**Chaos Theory**
Deterministic systems can exhibit random behavior.



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### Chaos

**Coin Flips**

***How do we model a sequence of coin flips?***

Flip a fair coin $N$ times.
Record $H$ for heads and $T$ for tails to obtain a sequences of $H$s and $T$s.

$H \quad T \quad H \quad H \quad H \quad T \quad T \quad T \quad H \quad T$

To reveal the sequence of flip results, just march down the line.

$H \quad T \quad H \quad H \quad H \quad T \quad T \quad T \quad H \quad T$
flip 1 ↑
$H \quad T \quad H \quad H \quad H \quad T \quad T \quad T \quad H \quad T$
flip 2 ↑
$H \quad T \quad H \quad H \quad H \quad T \quad T \quad T \quad H \quad T$
flip 3 ↑

etc.



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### Chaos

**Coin Flips**

*Alternatively:*

Keep the pointer fixed, and shift the sequence to the left.

$H \quad T \quad H \quad H \quad H \quad T \quad T \quad T \quad H \quad T$
flip 1 ↑
$H \quad T \quad H \quad H \quad H \quad T \quad T \quad T \quad H \quad T$
flip 2 ↑
$H \quad T \quad H \quad H \quad H \quad T \quad T \quad T \quad H \quad T$
flip 3 ↑
$H \quad T \quad H \quad H \quad H \quad T \quad T \quad T \quad H \quad T$
flip 4 ↑

etc.



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### Chaos

**Coin Flips**

Replace $H$ with 1 and $T$ with 0. Throw away stuff to the left of the arrow.

flip 1: 1 0 1 1 1 0 0 0 1 0 ↑
flip 2: 0 1 1 1 0 0 0 1 0 ↑
flip 3: 1 1 1 0 0 0 1 0 ↑
flip 4: 1 1 0 0 0 1 0 ↑

etc.



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### Chaos

**Coin Flips**

Replace $H$ with 1 and $T$ with 0. Throw away stuff to the left of the arrow.

Replace the arrow with a point.

flip 1: 1 0 1 1 1 0 0 0 1 0 ↑ .1011100010
flip 2: 0 1 1 1 0 0 0 1 0 ↑ .011100010
flip 3: 1 1 1 0 0 0 1 0 ↑ .11100010
flip 4: 1 1 0 0 0 1 0 ↑ .1100010

etc.



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### Chaos

**Coin Flips**

Replace $H$ with 1 and $T$ with 0. Throw away stuff to the left of the arrow.

Replace the arrow with a point.

Interpret as a binary number.

flip 1: 1 0 1 1 1 0 0 0 1 0 ↑ .1011100010 $\frac{1}{2}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{512} = 0.72073125$
flip 2: 0 1 1 1 0 0 0 1 0 ↑ .011100010 $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{256} = 0.44140625$
flip 3: 1 1 1 0 0 0 1 0 ↑ .11100010 $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{128} = 0.8828125$
flip 4: 1 1 0 0 0 1 0 ↑ .1100010 $\frac{1}{2}+\frac{1}{4}+\frac{1}{64} = 0.765625$

etc.



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### Chaos

**Coin Flips**

Interpret as a map.

$.1011100010 = 0.72073125 = x$

Shift the sequence to the right, leaving the binary point fixed.
Equivalent to multiplication by 2.
$1.011100010 = 1.44140625 = 2x$

Throw away the digit to the left of the binary point.
Equivalent to mod 1.
$.011100010 = .44140625 = 2x \bmod 1$

Altogether:

$$x \mapsto \varphi(x) = 2x \bmod 1$$



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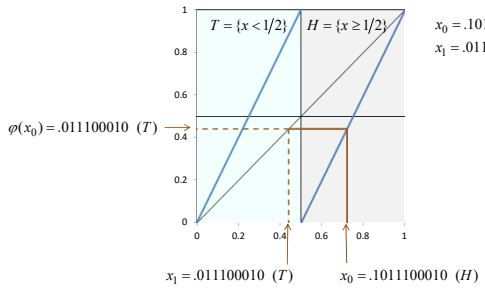
**Coin Flips**

$\varphi(x) = 2x \bmod 1$




$x_0 = .1011100010\ (H)$
$x_1 = .011100010\ (T)$



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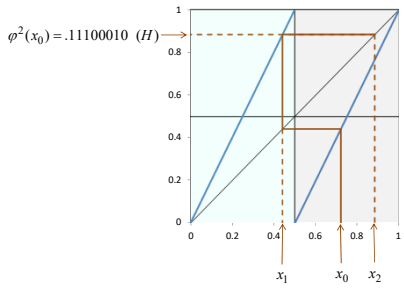
### Chaos

**Coin Flips**

$\varphi(x) = 2x \bmod 1$

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$x_2 = .11100010\ (H)$



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### Chaos

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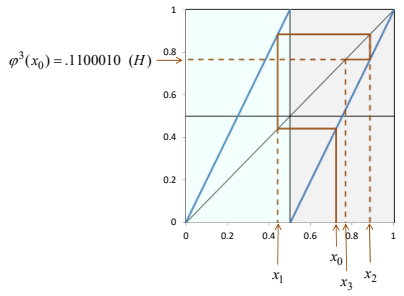
$\varphi(x) = 2x \bmod 1$




$x_0 = .1011100010 \quad (H)$
$x_1 = .011100010 \quad (T)$
$x_2 = .11100010 \quad (H)$
$x_3 = .1100010 \quad (H)$



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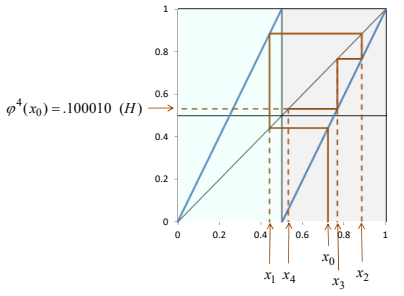
### Chaos

**Coin Flips**

$\varphi(x) = 2x \bmod 1$




$x_0 = .1011100010 \quad (H)$
$x_1 = .011100010 \quad (T)$
$x_2 = .11100010 \quad (H)$
$x_3 = .1100010 \quad (H)$
$x_4 = .100010 \quad (H)$



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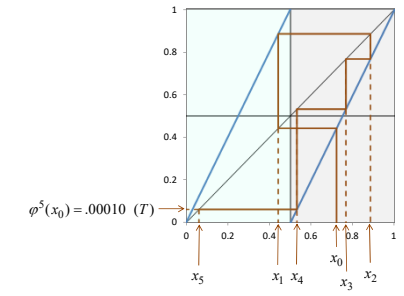
### Chaos

**Coin Flips**

$\varphi(x) = 2x \bmod 1$




$x_0 = .1011100010 \quad (H)$
$x_1 = .011100010 \quad (T)$
$x_2 = .11100010 \quad (H)$
$x_3 = .1100010 \quad (H)$
$x_4 = .100010 \quad (H)$
$x_5 = .00010 \quad (T)$



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### Chaos

**Coin Flips**

**No reason to stop.**

Flip a fair coin forever.

Record $H$ for heads and $T$ for tails to obtain a sequences of $H$s and $T$s.

$H \quad T \quad H \quad H \quad H \quad T \quad T \quad T \quad H \quad T \ldots$

Switch to 0s and 1s

$1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \ldots$

to binary

$.1011100010$

to decimal

$\frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{512} + \cdots = 0.72073125 \ldots$

same map:

$\boxed{x \mapsto \varphi(x) = 2x \bmod 1}$



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### Chaos

**Coin Flips**

**Interpretation**

For every infinite sequence of coin flips there is a binary number between 0 and 1. Numbers greater than ½ correspond to heads, while numbers less than ½ correspond to tails. Observing the events of sequential flips corresponds to shifting the binary point and discarding the integer part, which corresponds to iterating the map

$x \mapsto \varphi(x) = 2x \bmod 1$

and observing whether the iterate is to the left or to the right of ½.



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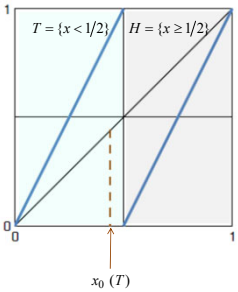
### Chaos

**Coin Flips**

$\varphi(x) = 2x \bmod 1$




$x_0 = 0.440301 \ldots \quad (T)$



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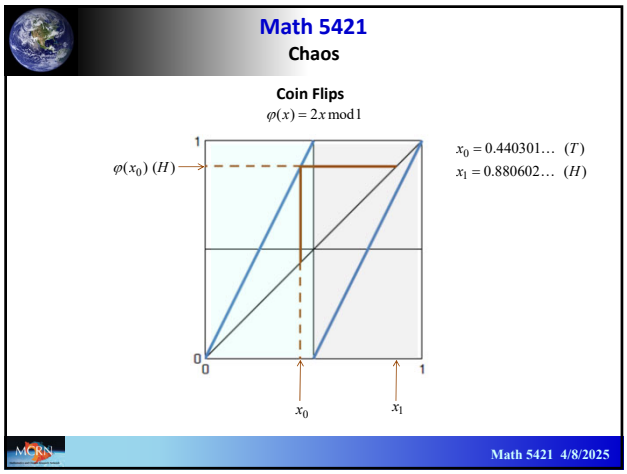


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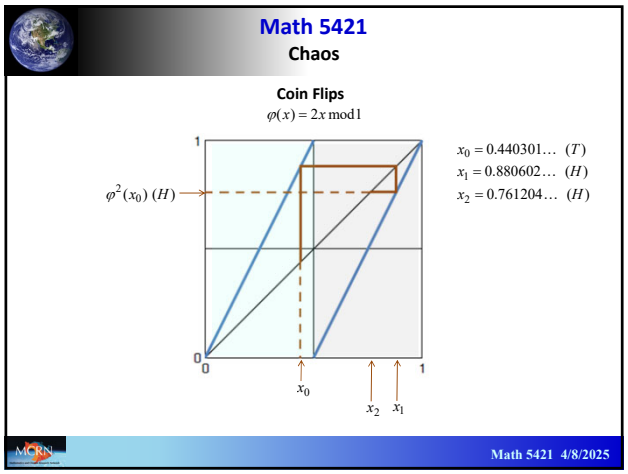


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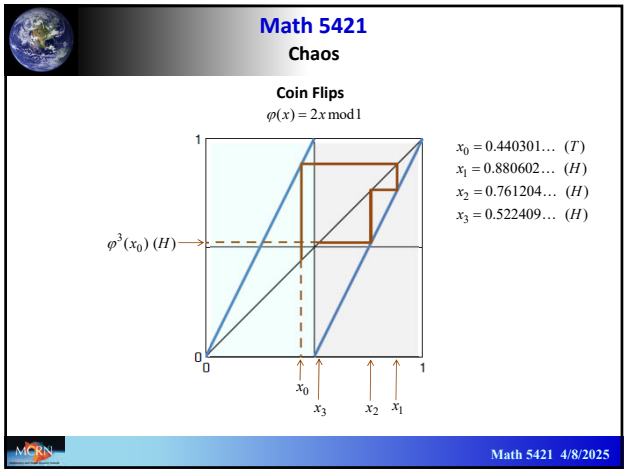


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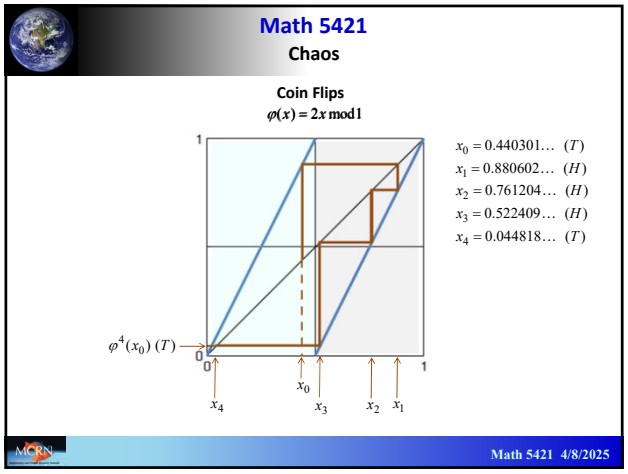


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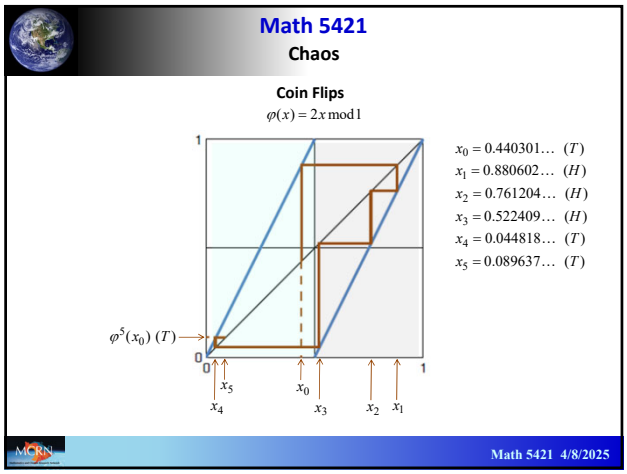


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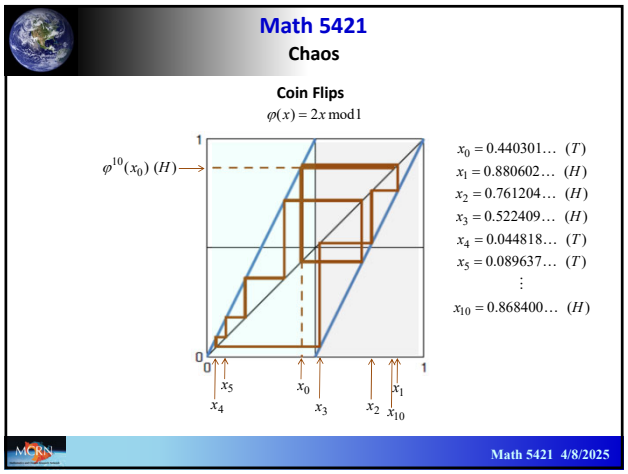


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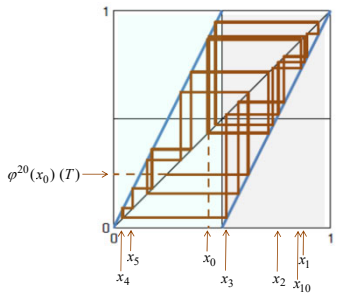
### Chaos

**Coin Flips**

$\varphi(x) = 2x \bmod 1$




$x_0 = 0.440301\ldots\ (T)$
$x_1 = 0.880602\ldots\ (H)$
$x_2 = 0.761204\ldots\ (H)$
$x_3 = 0.522409\ldots\ (H)$
$x_4 = 0.044818\ldots\ (T)$
$x_5 = 0.089637\ldots\ (T)$
$\vdots$
$x_{10} = 0.868400\ldots\ (H)$
$\vdots$
$x_{20} = 0.241922\ldots\ (T)$



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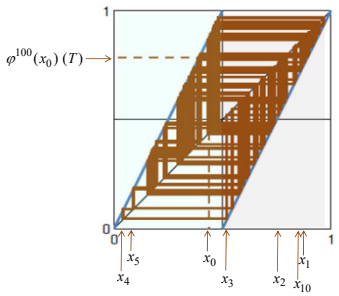
### Chaos

**Coin Flips**

$\varphi(x) = 2x \bmod 1$

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$\vdots$
$x_{10} = 0.868400\ldots\ (H)$
$\vdots$
$x_{20} = 0.241922\ldots\ (T)$
$\vdots$
$x_{100} = 0.782505\ldots\ (H)$



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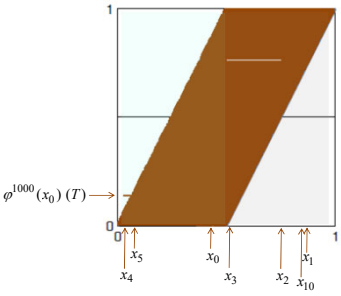
## Math 5421
### Chaos

**Coin Flips**

$\varphi(x) = 2x \bmod 1$

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$x_1 = 0.880602\ldots\ (H)$
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$x_3 = 0.522409\ldots\ (H)$
$x_4 = 0.044818\ldots\ (T)$
$x_5 = 0.089637\ldots\ (T)$
$\vdots$
$x_{10} = 0.868400\ldots\ (H)$
$\vdots$
$x_{20} = 0.241922\ldots\ (T)$
$\vdots$
$x_{1000} = 0.782505\ldots\ (H)$



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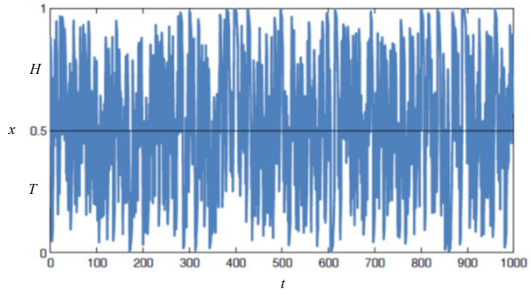
## Math 5421
### Chaos

**Coin Flips**

$\varphi(x) = 2x \bmod 1$



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### Chaos

**Coin Flips**

$\varphi(x) = 2x \bmod 1$

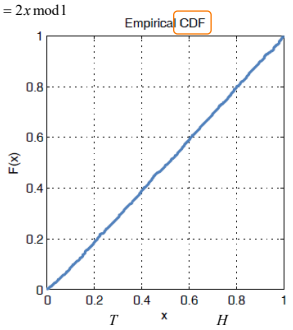
Sample = first 2000 iterates

$x$ vs the proportion of sample points whose value is below $x$.

Uniform distribution

Note that half are heads (> 1/2) and half are tails (< 1/2).

*Cumulative Distribution Funtion*



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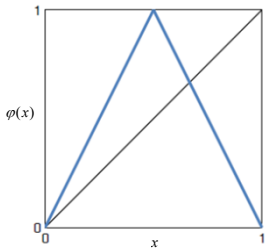
### Chaos

$$\varphi(x) = 1 - 2|x - 1/2| = \begin{cases} 2x, & x < 1/2 \\ 2(1-x), & x \geq 1/2 \end{cases}$$

continuous
piecewise linear
$|\text{slope}| = 2$




Let's try a similar map, but one that is continuous.



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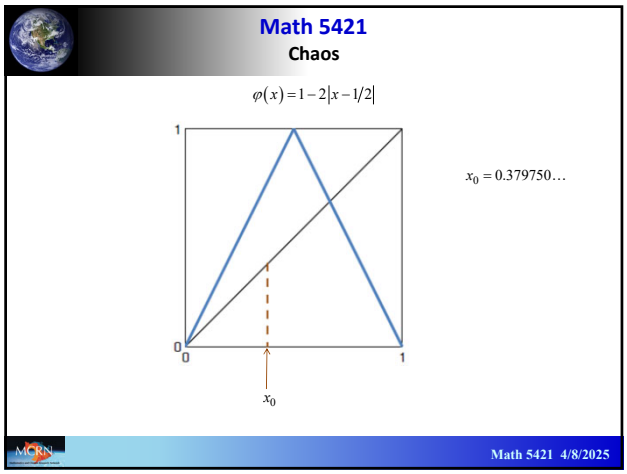


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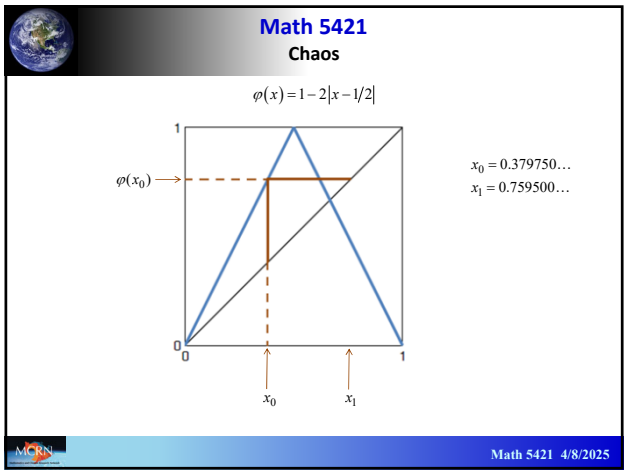


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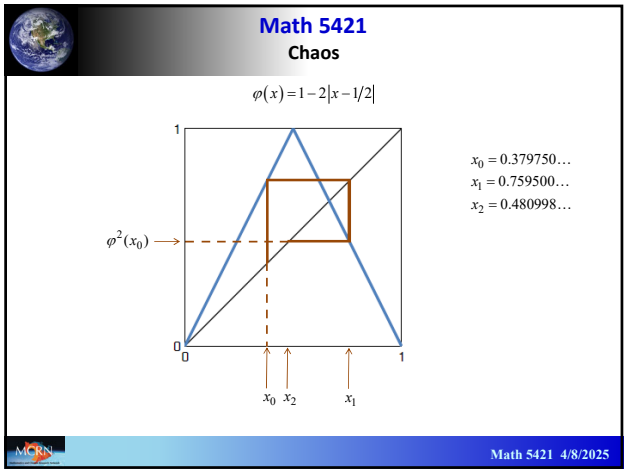


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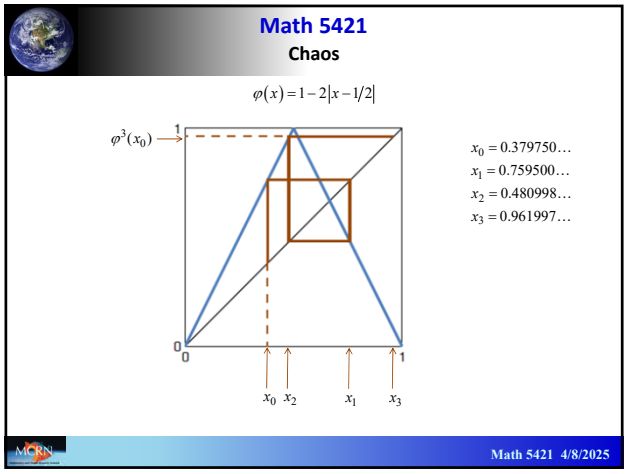


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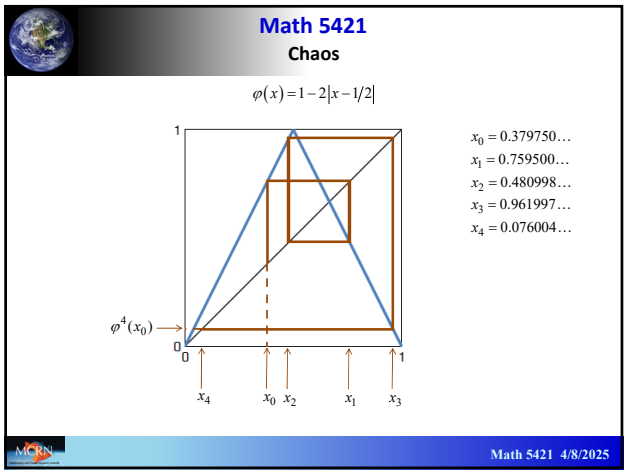


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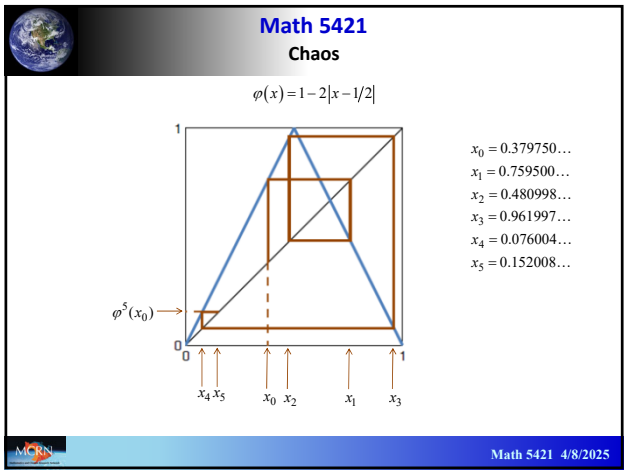


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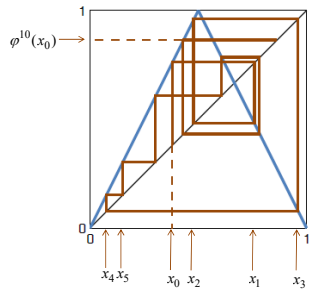
### Chaos

$\varphi(x) = 1 - 2|x - 1/2|$




$x_0 = 0.379750\ldots$
$x_1 = 0.759500\ldots$
$x_2 = 0.480998\ldots$
$x_3 = 0.961997\ldots$
$x_4 = 0.076004\ldots$
$x_5 = 0.152008\ldots$
$\vdots$
$x_{10} = 0.864271\ldots$



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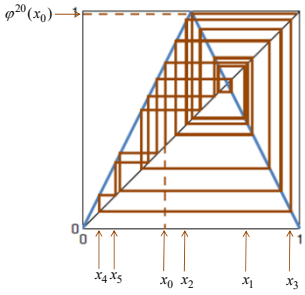
## Math 5421
### Chaos

$\varphi(x) = 1 - 2|x - 1/2|$

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$\vdots$
$x_{10} = 0.864271\ldots$
$\vdots$
$x_{20} = 0.985695\ldots$



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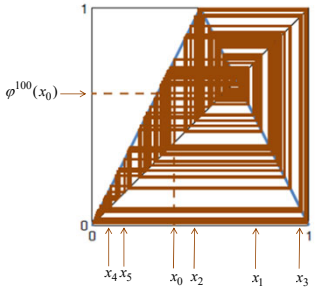
## Math 5421
### Chaos

$\varphi(x) = 1 - 2|x - 1/2|$

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$\vdots$
$x_{20} = 0.985695\ldots$
$\vdots$
$x_{100} = 0.602054\ldots$



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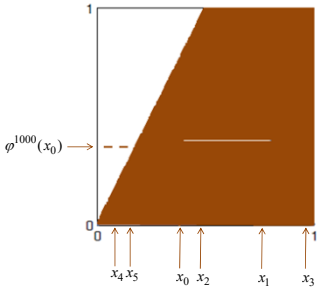
## Math 5421
### Chaos

$\varphi(x) = 1 - 2|x - 1/2|$

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$\vdots$
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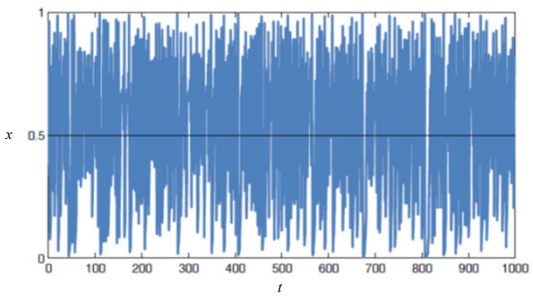
## Math 5421
### Chaos

$\varphi(x) = 1 - 2|x - 1/2|$



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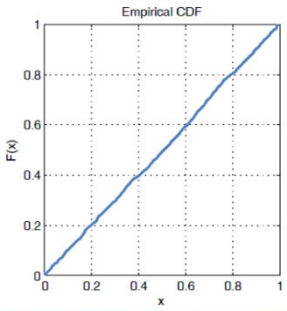
## Math 5421
### Chaos

$\varphi(x) = 1 - 2|x - 1/2|$

Sample = first 2000 iterates

$x$ vs the proportion of sample points whose value is below $x$.

Uniform distribution again.



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### Chaos

**Discrete Time Logistic Growth**

$$\varphi(x) = 4x(1-x)$$

$$N_{t+1} = rN_t\left(1-\frac{N_t}{K}\right)$$

$N_t$ : population at time $t$

$r$ : intrinsic growth rate

$K$ : carrying capacity

Small populations grow at rate $r$.

Large populations crash.

Mathematically interesting case:

$r = 4$

$K = 1$



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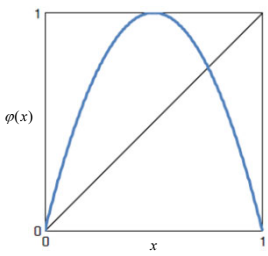
### Chaos

**Logistic Growth**

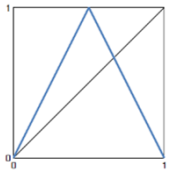
$\varphi(x) = 4x(1-x)$

continuous quadratic not 1-1




This function looks a lot like the previous function, except is is smooth instead of piecewise linear.

$\varphi(x) = 1 - 2|x - 1/2|$



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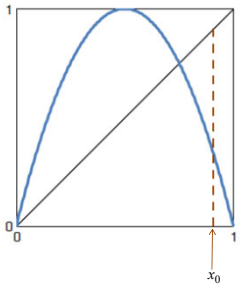
## Math 5421
### Chaos

**Logistic Growth**

$\varphi(x) = 4x(1-x)$




$x_0 = 0.908778\ldots$



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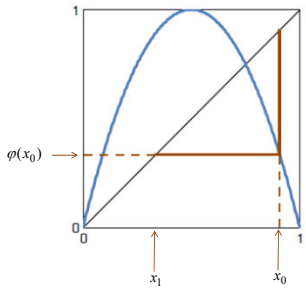
## Math 5421
### Chaos

**Logistic Growth**

$\varphi(x) = 4x(1-x)$




$x_0 = 0.908778\ldots$

$x_1 = 0.331601\ldots$



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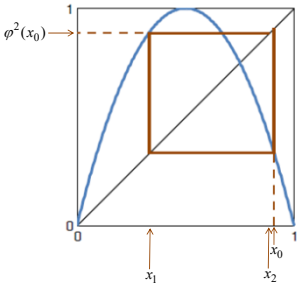
## Math 5421
### Chaos

**Logistic Growth**

$\varphi(x) = 4x(1-x)$




$x_0 = 0.908778\ldots$

$x_1 = 0.331601\ldots$

$x_2 = 0.886567\ldots$



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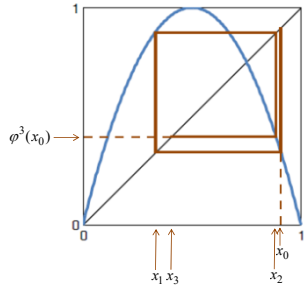
## Math 5421
### Chaos

**Logistic Growth**

$\varphi(x) = 4x(1-x)$




$x_0 = 0.908778\ldots$

$x_1 = 0.331601\ldots$

$x_2 = 0.886567\ldots$

$x_3 = 0.402260\ldots$



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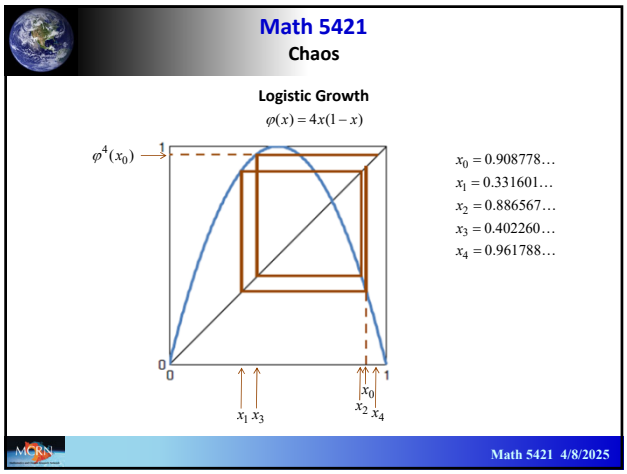
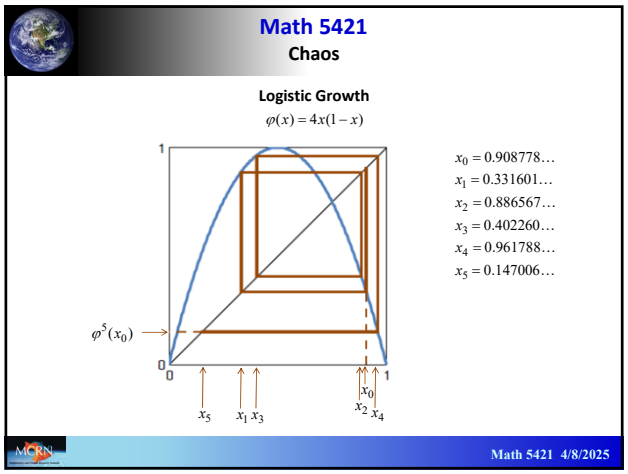
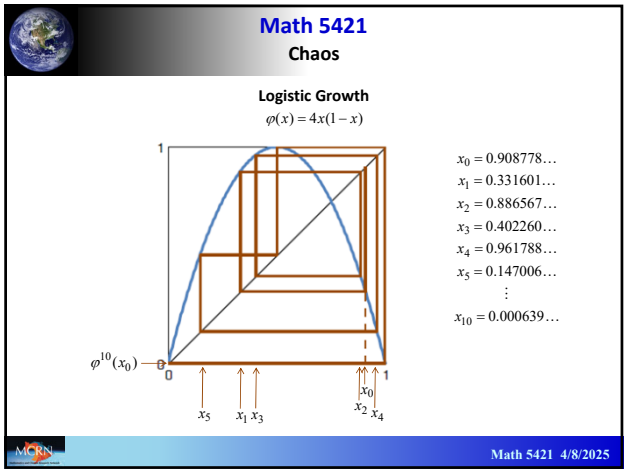



50




51

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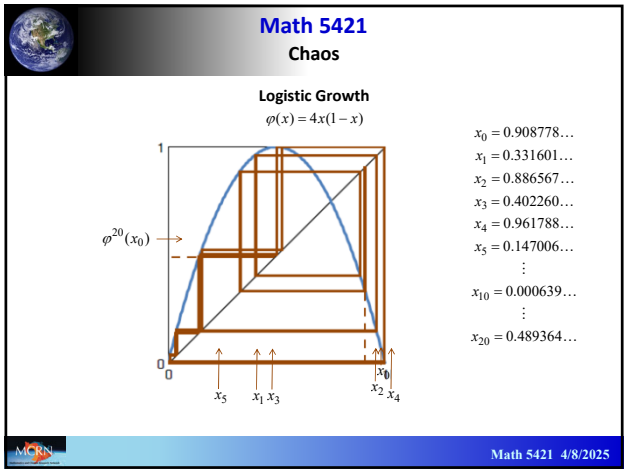
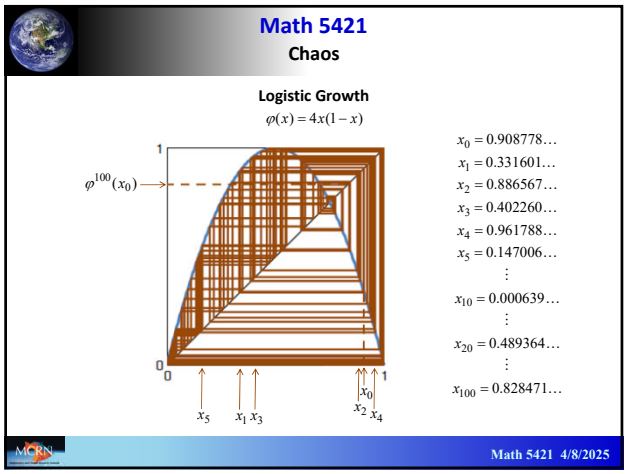



53

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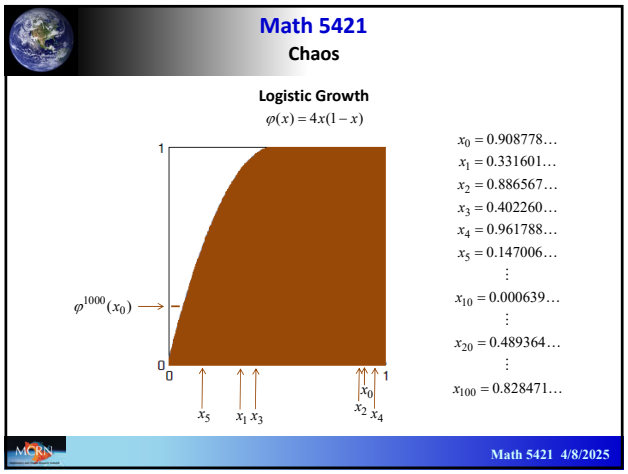


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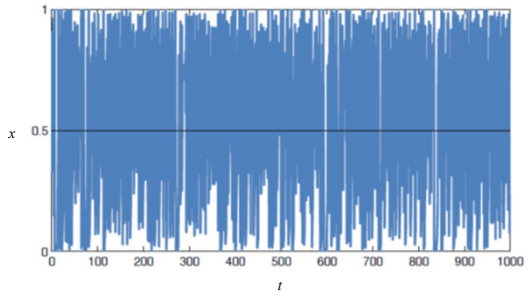
## Math 5421
### Chaos

**Logistic Growth**

$\varphi(x) = 4x(1-x)$



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## Math 5421
### Chaos

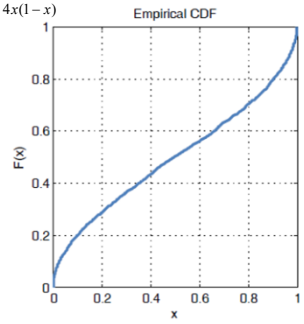
**Logistic Growth**

$\varphi(x) = 4x(1-x)$

Sample = first 2000 iterates

$x$ vs the proportion of sample points whose value is below $x$.

Continuous distribution, but not uniform.



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## Math 5421
### Chaos

**Discrete Time Logistic Growth**

$$N_{t+1} = rN_t\left(1-\frac{N_t}{K}\right)$$

$N_t$ : population at time $t$

$r$ : intrinsic growth rate

$K$ : carrying capacity

**Continuous Time Logistic Growth**

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$$

Measure population as fraction of carrying capacity.

$$N = Kx \quad \Rightarrow \quad K\frac{dx}{dt} = \frac{dN}{dt} = rKx(1-x)$$

$$\frac{dx}{dt} = rx(1-x)$$



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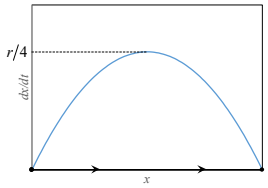
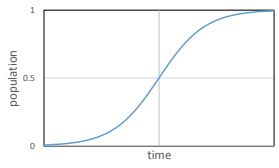
## Math 5421
### Chaos

**Continuous Time Logistic Growth**

$$\frac{dx}{dt} = rx(1-x)$$




For any positive $r$, all solutions move asymptotically from 0 to 1.



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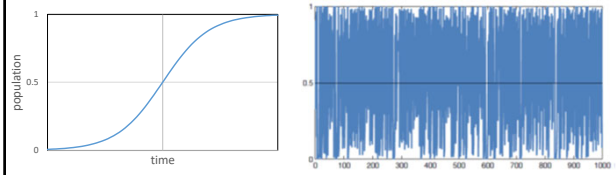
## Math 5421
### Chaos

**Continuous Time vs Discrete Time**

$r = 4$




very different



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## Math 5421
### Chaos

**Logistic Growth**

$\varphi(x) = rx(1-x)$

**Fixed Point**

$\varphi(x) = x$

anolog of rest point

$rx(1-x) = x \quad \Rightarrow \quad x = 0 \text{ or } r(1-x) = 1$

$x = 0$ is always a fixed point.

$r > 1 \quad \Rightarrow \quad x = 1 - \frac{1}{r}$ is a fixed point.



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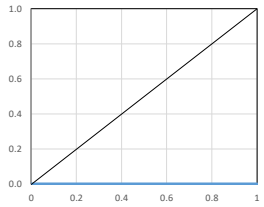
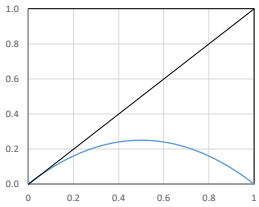
## Math 5421
### Chaos

**Logistic Growth**

$\varphi(x) = rx(1-x)$



$r = 0$: $x = 0$ is the only fixed point (super stable)



$r = 1$: $x = 0$ is still the only fixed point. becoming unstable



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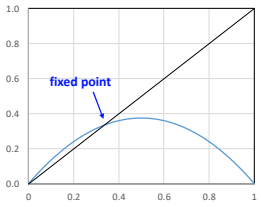
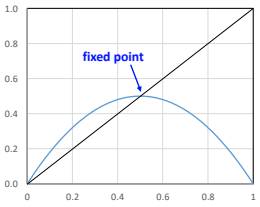
## Math 5421
### Chaos

**Logistic Growth**

$\varphi(x) = rx(1-x)$




$r = 1.5$: $x = 1/3$ is a fixed point (stable)




$r = 2$: $x = 1/2$ is a fixed point (super stable)



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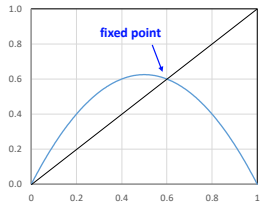
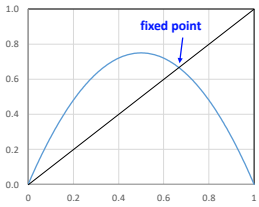
## Math 5421
### Chaos

**Logistic Growth**

$\varphi(x) = rx(1-x)$




$r = 2.5$: $x = 3/5$ is a fixed point (stable spiral)




$r = 3$: $x = 2/3$ is a fixed point (barely stable spiral)



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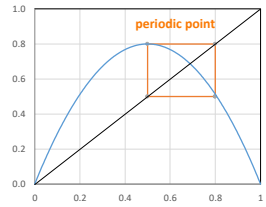
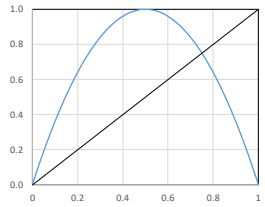
## Math 5421
### Chaos

**Logistic Growth**

$\varphi(x) = rx(1-x)$




$r = 3.2$: The fixed point has spun off a periodic point (period-doubling bifurcation)



$r = 4$: Chaos!



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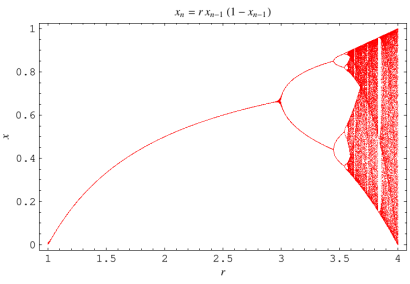
## Math 5421
### Chaos




https://mathworld.wolfram.com/LogisticMap.html



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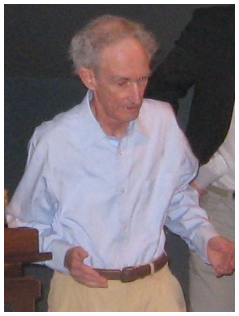
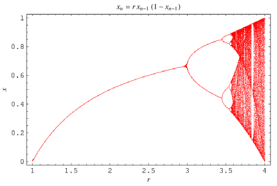
## Math 5421
### Chaos



https://en.wikipedia.org/wiki/Robert_May,_Baron_May_of_Oxford

Robert May, Simple mathematical models with very complicated dynamics, *Nature* **261**, 459-467 (1976)



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