Math 5421
An Introduction to Mathematical Climate Models

Spring 2025
1:25 – 3:20 Tuesdays and Thursdays
Blegen Hall 155

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course website
https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/

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Can We Predict the Future?

If we know the state of a system now, do we know its state in the future?

For models based on differential equations, the answer is 'yes".

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n, \quad x = x_0 \text{ when } t = 0$$

If f is sufficiently smooth (e.g., continuously differentiable) then there is a unique solution of the differential equations satisfying the initial condition.

Interpretation:

If we know the state of the system now, we can compute its state in the future.

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Can We Predict the Future?

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n, \quad x = x_0 \text{ when } t = 0$$

If we know the state of the system now, we can compute its state in the future.

Yes, but how accurately?

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Can We Predict the Future?

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If we know the state of the system now, we can compute its state in the future.

Yes, but how accurately?

Last Time

Deterministic systems can behave randomly.

It can be impossible to tell whether a time series results from a stochastic process or a deterministic system.

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Example

$$\varphi(x) = 2x \bmod 1$$
$$x(t+1) = \varphi(x(t))$$

If all we can tell is whether $x(t)$ is less than or greater than ½, then we cannot distinguish solutions from the result of flipping a fair coin.

More precisely, let:

$$F(t) = \begin{cases} H & \text{if } x(t) > 1/2 \\ T & \text{if } x(t) < 1/2 \end{cases}$$

Then the sequence

$$F(0), F(1), F(2), \ldots$$

cannot be distinguished from a sequence produced by flipping a fair coin (for almost all starting points $x(0)$).

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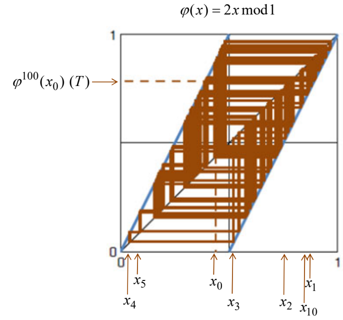
Coin Flips

$$\varphi(x) = 2x \bmod 1$$



$x_0 = 0.440301\ldots\ (T)$
$x_1 = 0.880602\ldots\ (H)$
$x_2 = 0.761204\ldots\ (H)$
$x_3 = 0.522409\ldots\ (H)$
$x_4 = 0.044818\ldots\ (T)$
$x_5 = 0.089637\ldots\ (T)$
$\vdots$
$x_{10} = 0.868400\ldots\ (H)$
$\vdots$
$x_{20} = 0.241922\ldots\ (T)$
$\vdots$
$x_{100} = 0.782505\ldots\ (H)$

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Example

$$\varphi(x) = 2x \bmod 1$$
$$x(t+1) = \varphi(x(t))$$

If all we can tell is whether $x(t)$ is less than or greater than ½, then we cannot distinguish solutions from the result of flipping a fair coin.

However …

If we know $x(0)$ precisely, then we know $x(t)$ precisely, for all time.

What can go wrong?

Suppose we know $x(0)$ only to some accuracy ε. Then, as time increases, we know $x(t)$ to less and less accuracy. Eventually, we know nothing.

For example, if $\varepsilon = ½$, we know nothing after only one time unit.

What if $\varepsilon = 10^{-6}$?

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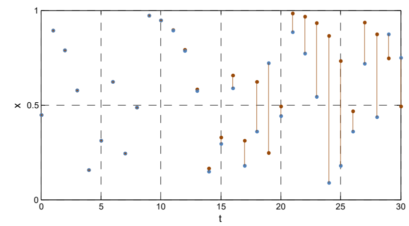
Example

$$\varphi(x) = 2x \bmod 1 \qquad x(t+1) = \varphi(x(t))$$



blue: $x_{\text{blue}}(0) = \sqrt{1/5}$

brown: $x_{\text{brown}}(0) = \sqrt{1/5} + 10^{-6}$

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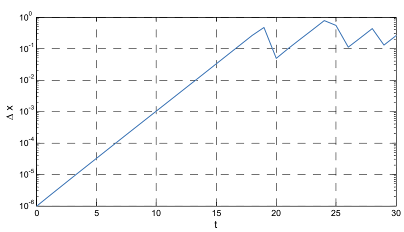
Example

$$\varphi(x) = 2x \bmod 1 \qquad x(t+1) = \varphi(x(t))$$



$$\Delta x(t) = |x_{\text{blue}}(t) - x_{\text{brown}}(t)|$$

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Lyapunov Multipliers (infinitesimal growth)

Consider: $x(t+1) = f(x(t))$

orbit: $x_{n+1} = f(x_n) = f^n(x_0), \quad x_0 = x(0)$

nearby orbit: $y_{n+1} = f(y_n) = f^n(y_0), \quad y_0 = x_0 + \xi$

$$|y_1 - x_1| = |f(y_0) - f(x_0)| = |f(x_0 + \xi) - f(x_0)| \approx |f'(x_0)\xi| = |f'(x_0)||\xi|$$
$$|y_2 - x_2| = |f(y_1) - f(x_1)| = |f(x_1 + y_1 - x_1) - f(x_1)| \approx |f'(x_1)(y_1 - x_1)| \approx |f'(x_1)||f'(x_0)||\xi|$$
$$\dots$$
$$|y_n - x_n| \approx \prod_{k=0}^{n-1} |f'(x_k)||\xi|$$

average multiplier

$$\mu(x_0, n) = \left(\prod_{k=0}^{n-1} |f'(x_k)|\right)^{1/n}$$

average exponent

$$\lambda(x_0, n) = \log(\mu(x_0, n)) = \frac{1}{n}\sum_{k=0}^{n-1} \log|f'(x_k)|$$

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Lyapunov Exponents and Multipliers

orbit: $x_{n+1} = f(x_n) = f^{n+1}(x_0), \quad x_0 = x(0)$

Lyapunov multiplier

$$\mu(x_0) = \lim_{n\to\infty} \mu(x_0, n) = \lim_{n\to\infty} \left(\prod_{k=0}^{n-1} |f'(x_k)|\right)^{1/n}$$

Lyapunov exponent

$$\lambda(x_0) = \lim_{n\to\infty} \lambda(x_0, n) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log|f'(x_k)|$$

Interpretation

If the Lyapunov exponent is greater than zero $(\lambda(x_0) > 0)$ or, equivalently, the Lyapunov multiplier is greater than one $(\mu(x_0) > 1)$, then nearby orbits diverge exponentially.

If the Lyapunov exponent is less than zero $(\lambda(x_0) < 0)$ or, equivalently, the Lyapunov multiplier is less than one $(\mu(x_0) < 1)$, then nearby orbits converge exponentially.

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Example

$$\varphi(x) = 2x \bmod 1$$
$$\varphi'(x) = 2$$

Lyapunov multiplier

$$\mu(x_0, n) = \left(\prod_{k=0}^{n-1} |\varphi'(x_k)|\right)^{1/n} = \left(\prod_{k=0}^{n-1} 2\right)^{1/n} = \left(2^n\right)^{1/n} = 2$$
$$\mu(x_0) = \lim_{n\to\infty} 2 = 2$$

Lyapunov exponent

$$\lambda(x_0, n) = \frac{1}{n}\sum_{k=0}^{n-1} \log|\varphi'(x_k)| = \frac{1}{n}\sum_{k=0}^{n-1} \log 2 = \frac{1}{n} n \log 2 = \log 2$$
$$\lambda(x_0) = \lim_{n\to\infty} \log 2$$

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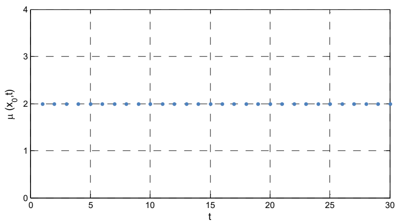
Example

$\varphi(x) = 2x \bmod 1 \qquad x(t+1) = \varphi(x(t))$



$$\mu(x_0, t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t} = 2$$

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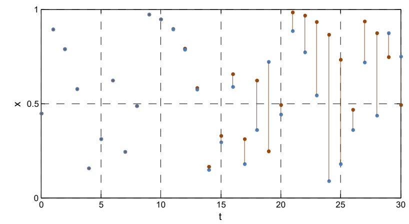
Example

$\varphi(x) = 2x \bmod 1 \qquad x(t+1) = \varphi(x(t))$



blue: $x_{\text{blue}}(0) = \sqrt{1/5}$

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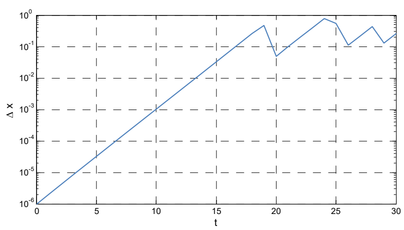
Example

$\varphi(x) = 2x \bmod 1 \qquad x(t+1) = \varphi(x(t))$



$$\Delta x(t) = |x_{\text{blue}}(t) - x_{\text{brown}}(t)|$$

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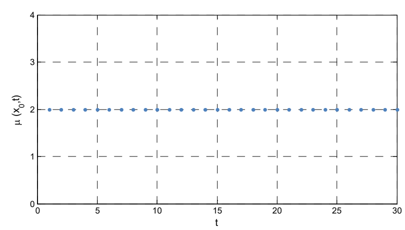
Example

$\varphi(x) = 2x \bmod 1 \qquad x(t+1) = \varphi(x(t))$



$$\mu(x_0, t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t} = 2$$

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Example

$\varphi(x) = 2x \bmod 1$

Lyapunov multiplier

$$\mu(x_0, n) = \left(\prod_{k=0}^{n-1} |\varphi'(x_k)| \right)^{1/n} = \left(\prod_{k=0}^{n-1} 2 \right)^{1/n} = \left(2^n\right)^{1/n} = 2$$

$$\mu(x_0) = \lim_{n \to \infty} 2 = 2$$

Interpretation

At each step, the error multiplies by a factor of 2. After n steps, an error of ε becomes $2^n \varepsilon$. Since the diameter of the state space is 1, to have any knowledge of the state of the system after n steps requires an initial error of $\varepsilon < 2^{-n}$.

30 steps: $2^{-30} \approx 9 \times 10^{-10}$

100 steps: $2^{-100} \approx 8 \times 10^{-31}$

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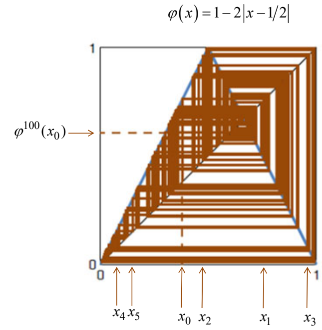
$\varphi(x) = 1 - 2|x - 1/2|$



$x_0 = 0.379750\ldots$
$x_1 = 0.759500\ldots$
$x_2 = 0.480998\ldots$
$x_3 = 0.961997\ldots$
$x_4 = 0.076004\ldots$
$x_5 = 0.152008\ldots$
$\vdots$
$x_{10} = 0.864271\ldots$
$\vdots$
$x_{20} = 0.985695\ldots$
$\vdots$
$x_{100} = 0.602054\ldots$

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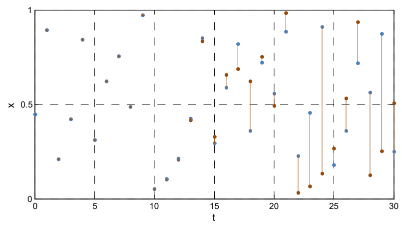
Example

$$\varphi(x) = \begin{cases} 2x, & x < 1/2 \\ 2(1-x), & x \ge 1/2 \end{cases}$$



blue: $x_{\text{blue}}(0) = \sqrt{1/5}$

brown: $x_{\text{brown}}(0) = \sqrt{1/5} + 10^{-6}$

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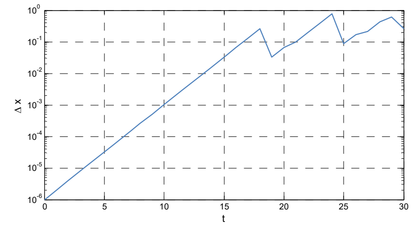
Example

$$\varphi(x) = \begin{cases} 2x, & x < 1/2 \\ 2(1-x), & x \ge 1/2 \end{cases}$$



$$\Delta x(t) = \left| x_{\text{blue}}(t) - x_{\text{brown}}(t) \right|$$

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Example

$$\varphi(x) = \begin{cases} 2x, & x < 1/2 \\ 2(1-x), & x \ge 1/2 \end{cases}$$

$$\varphi'(x) = \begin{cases} 2, & x < 1/2 \\ -2, & x \ge 1/2 \end{cases}$$

Lyapunov multiplier

$$\mu(x_0, n) = \left(\prod_{k=0}^{n-1} \left|\varphi'(x_k)\right| \right)^{1/n} = \left(\prod_{k=0}^{n-1} 2 \right)^{1/n} = \left(2^n\right)^{1/n} = 2$$

$$\mu(x_0) = \lim_{n\to\infty} 2 = 2$$

Interpretation

Same as previous example:

At each step, the error multiplies by a factor of 2.

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Example

$$\varphi(x) = \begin{cases} 2x, & x < 1/2 \\ 2(1-x), & x \ge 1/2 \end{cases}$$

$$\varphi'(x) = \begin{cases} 2, & x < 1/2 \\ -2, & x \ge 1/2 \end{cases}$$

Lyapunov multiplier

$$\mu(x_0, n) = \left(\prod_{k=0}^{n-1} \left|\varphi'(x_k)\right| \right)^{1/n} = \left(\prod_{k=0}^{n-1} 2 \right)^{1/n} = \left(2^n\right)^{1/n} = 2$$

$$\mu(x_0) = \lim_{n\to\infty} 2 = 2$$

Interpretation

Same as previous example:

At each step, the error multiplies by a factor of 2.

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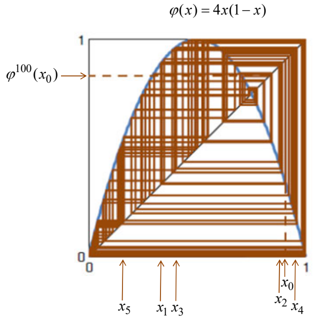
Logistic Growth

$$\varphi(x) = 4x(1-x)$$



$x_0 = 0.908778\ldots$
$x_1 = 0.331601\ldots$
$x_2 = 0.886567\ldots$
$x_3 = 0.402260\ldots$
$x_4 = 0.961788\ldots$
$x_5 = 0.147006\ldots$
$\vdots$
$x_{10} = 0.000639\ldots$
$\vdots$
$x_{20} = 0.489364\ldots$
$\vdots$
$x_{100} = 0.828471\ldots$

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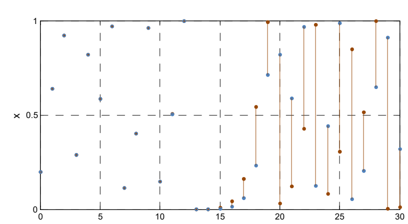
Example

$$\varphi(x) = 4x(1-x)$$



blue: $x_{\text{blue}}(0) = 1/5$

brown: $x_{\text{brown}}(0) = 1/5 + 10^{-6}$

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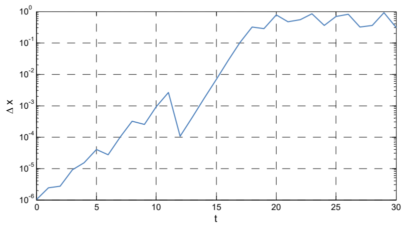
Example

$$\varphi(x)=4x(1-x)$$



$$\Delta x(t)=\left|x_{\text{blue}}(t)-x_{\text{brown}}(t)\right|$$

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Example

$$\varphi(x)=4x(1-x)$$
$$\varphi'(x)=4-8x$$

Lyapunov multiplier

$$\mu(x_0,n)=\left(\prod_{k=0}^{n-1}\left|\varphi'(x_k)\right|\right)^{1/n}=?$$
$$\mu(x_0)=\lim_{n\to\infty}\mu(x_0,n)=?$$

Let $x_0=0.2$, so $\varphi'(x_0)=4-8x_0=2.4$,

$$\mu(x_0,1)=2.4$$

$x_1=4x_0(1-x_0)=0.64$, so $\varphi'(x_1)=4-8x_1=-1.12$,

$$\mu(x_1,1)=\left(\left|\varphi'(x_0)\right|\cdot\left|\varphi'(x_1)\right|\right)^{1/2}\approx 1.64$$

$x_2=4x_1(1-x_1)=0.9216$, so $\varphi'(x_2)=4-8x_2=-3.3728$,

$$\mu(x_2,1)=\left(\left|\varphi'(x_0)\right|\cdot\left|\varphi'(x_1)\right|\cdot\left|\varphi'(x_2)\right|\right)^{1/3}\approx 2.085$$

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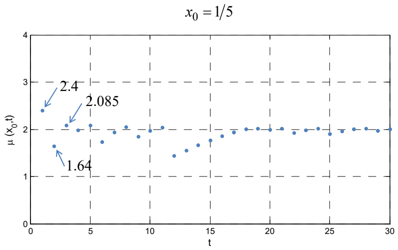
Example

$$\varphi(x)=4x(1-x)$$
$$x_0=1/5$$



$$\mu(x_0,t)=\left(\prod_{k=0}^{t-1}\left|\varphi'(x_k)\right|\right)^{1/t}$$

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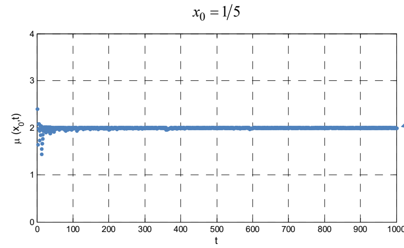
Example

$$\varphi(x)=4x(1-x)$$
$$x_0=1/5$$



Looks like $\mu(x_0)=2$

$$\mu(x_0,t)=\left(\prod_{k=0}^{t-1}\left|\varphi'(x_k)\right|\right)^{1/t}$$

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Example

$$\varphi(x)=4x(1-x)$$

Lyapunov multiplier

$$\mu(x_0,n)=\left(\prod_{k=0}^{n-1}\left|\varphi'(x_k)\right|\right)^{1/n}$$

Evidence indicates that

$$\mu(x_0)=\lim_{n\to\infty}\mu(x_0,n)=2$$

Interpretation

On average, the error multiplies by a factor of 2 at each step.

After n steps, an error of ε becomes about $2^n\varepsilon$.

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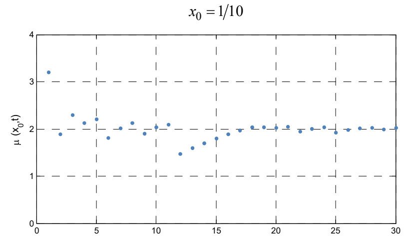
Example

$$\varphi(x)=4x(1-x)$$
$$x_0=1/10$$



$$\mu(x_0,t)=\left(\prod_{k=0}^{t-1}\left|\varphi'(x_k)\right|\right)^{1/t}$$

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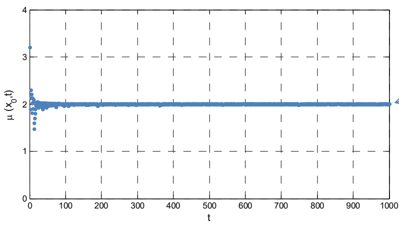
Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 1/10$$



Looks like $\mu(x_0) = 2$

$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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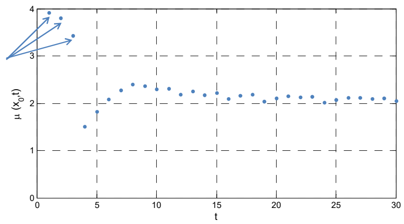
Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 1/100$$

Close to 4



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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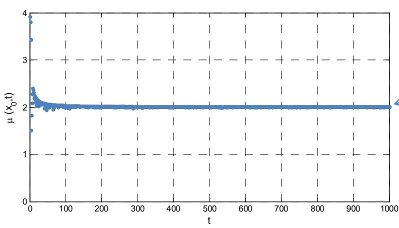
Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 1/100$$



Looks like $\mu(x_0) = 2$

$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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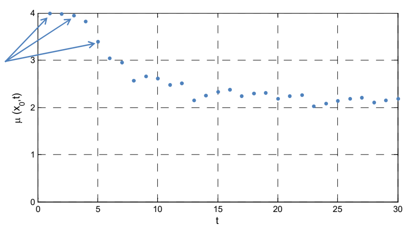
Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 1/1000$$

Close to 4



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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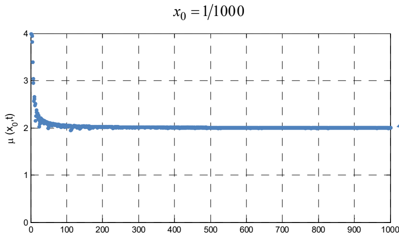
Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 1/1000$$



Looks like $\mu(x_0) = 2$

$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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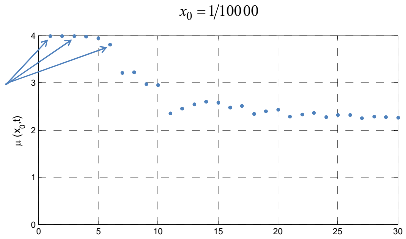
Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 1/10000$$

Close to 4



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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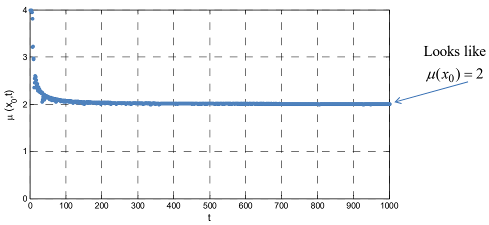
Example

$\varphi(x) = 4x(1-x)$

$x_0 = 1/10000$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

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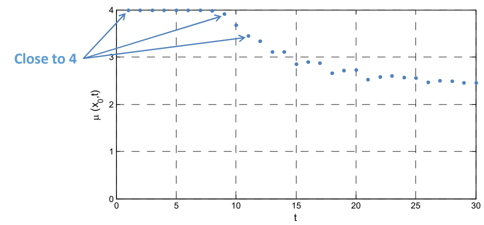
Example

$\varphi(x) = 4x(1-x)$

$x_0 = 10^{-6}$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

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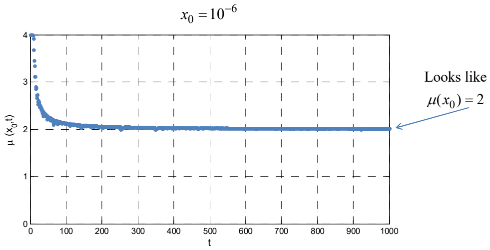
$\varphi(x) = 4x(1-x)$

$x_0 = 10^{-6}$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

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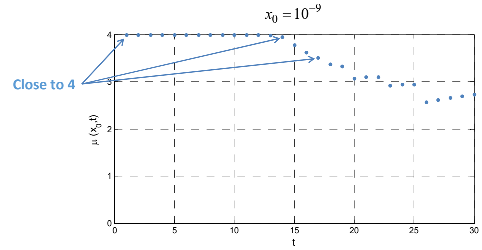
Example

$\varphi(x) = 4x(1-x)$

$x_0 = 10^{-9}$



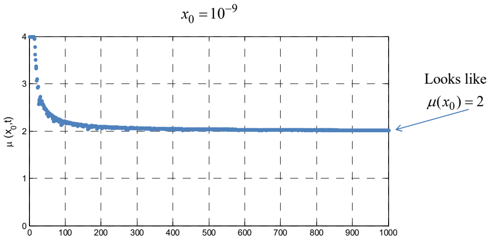
$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

Math 5421
Chaos

Example

$\varphi(x) = 4x(1-x)$

$x_0 = 10^{-9}$



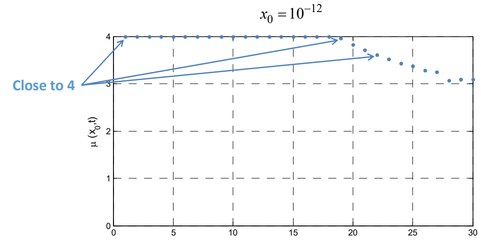
$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

Math 5421
Chaos

Example

$\varphi(x) = 4x(1-x)$

$x_0 = 10^{-12}$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

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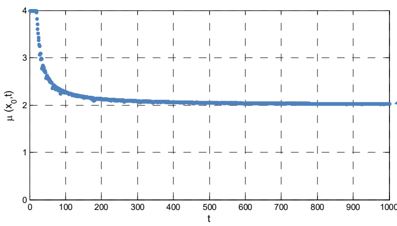
Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 10^{-12}$$



Looks like $\mu(x_0) = 2$

$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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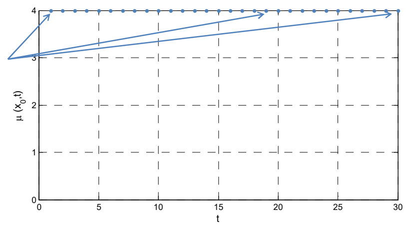
Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 0$$

Exactly 4



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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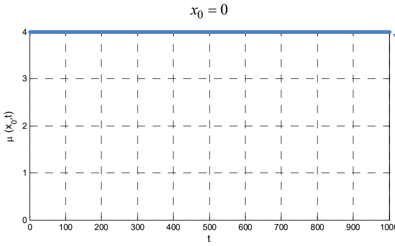
Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 0$$



Looks like $\mu(x_0) = 4$

$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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Example

$$\varphi(x) = 4x(1-x)$$

$$\varphi(0) = 0$$
$$\varphi'(0) = 4$$

Lyapunov multiplier

$$\mu(0,n) = \left(\prod_{k=0}^{n-1} |\varphi'(0)|\right)^{1/n} = \left(\prod_{k=0}^{n-1} 4\right)^{1/n} = \left(4^n\right)^{1/n} = 4$$

$$\mu(0) = \lim_{n\to\infty} \mu(0,n) = 4$$

Interpretation

The origin is a rest point with eigenvalue 4.
The Lyapunov multiplier is the eigenvalue.

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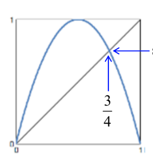
Example

$$\varphi(x) = 4x(1-x)$$

$$\varphi\left(\frac{3}{4}\right) = 4\cdot\frac{3}{4}\left(1-\frac{3}{4}\right) = \frac{3}{4}$$
$$\varphi'\left(\frac{3}{4}\right) = 4 - 8\cdot\frac{3}{4} = -2$$



slope $= -2$

Lyapunov multiplier

$$\mu\left(\frac{3}{4},n\right) = \left(\prod_{k=0}^{n-1} |\varphi'(0)|\right)^{1/n} = \left(\prod_{k=0}^{n-1} 2\right)^{1/n} = \left(2^n\right)^{1/n} = 2$$

$$\mu\left(\frac{3}{4}\right) = \lim_{n\to\infty} \mu\left(\frac{3}{4},n\right) = 2$$

Interpretation

$\frac{3}{4}$ is a rest point with eigenvalue -2.

The Lyapunov multiplier is the absolute value of the eigenvalue.

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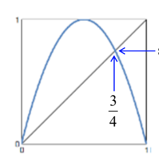
Example

$$\varphi(x) = 4x(1-x)$$

$$\varphi\left(\frac{3}{4}\right) = 4\cdot\frac{3}{4}\left(1-\frac{3}{4}\right) = \frac{3}{4}$$
$$\varphi'\left(\frac{3}{4}\right) = 4 - 8\cdot\frac{3}{4} = -2$$



slope $= -2$

Lyapunov multiplier

$$\mu\left(\frac{3}{4},n\right) = \left(\prod_{k=0}^{n-1} |\varphi'(0)|\right)^{1/n} = \left(\prod_{k=0}^{n-1} 2\right)^{1/n} = \left(2^n\right)^{1/n} = 2$$

$$\mu\left(\frac{3}{4}\right) = \lim_{n\to\infty} \mu\left(\frac{3}{4},n\right) = 2$$

Interpretation

$\frac{3}{4}$ is a rest point with eigenvalue -2.

The Lyapunov multiplier is the absolute value of the eigenvalue.

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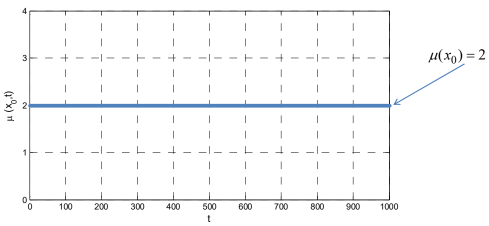
Example

$\varphi(x) = 4x(1-x)$

$x_0 = 3/4$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

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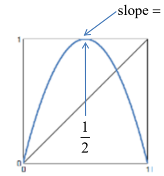
Example

$\varphi(x) = 4x(1-x)$



$$\varphi\left(\tfrac{1}{2}\right) = 4\tfrac{1}{2}\left(1-\tfrac{1}{2}\right) = 1, \quad \varphi^2\left(\tfrac{1}{2}\right) = \varphi(1) = 0,$$

$$\varphi^n\left(\tfrac{1}{2}\right) = 0, \quad n \ge 2$$

$$\varphi'\left(\tfrac{1}{2}\right) = 4 - 8\cdot\tfrac{1}{2} = 0$$

Lyapunov multiplier

$$\mu\left(\tfrac{1}{2},n\right) = \left(\prod_{k=0}^{n-1}\left|\varphi'\left(\varphi^k\left(\tfrac{1}{2}\right)\right)\right|\right)^{1/n} = \left(0\times\prod_{k=1}^{n-1}\left|\varphi'\left(\varphi^k\left(\tfrac{1}{2}\right)\right)\right|\right)^{1/n} = (0)^{1/n} = 0$$

$$\mu\left(\tfrac{1}{2}\right) = \lim_{n\to\infty}\mu\left(\tfrac{1}{2},n\right) = 0$$

Interpretation

If $\varphi'\left(\varphi^n(x_0)\right) = 0$ for any n, then the Lyapunov multiplier of the orbit is 0.

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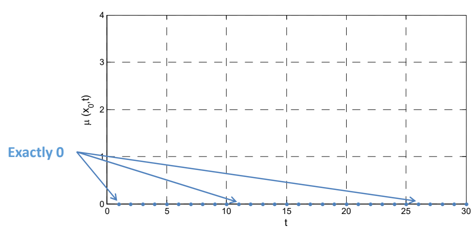
Example

$\varphi(x) = 4x(1-x)$

$x_0 = 1/2$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

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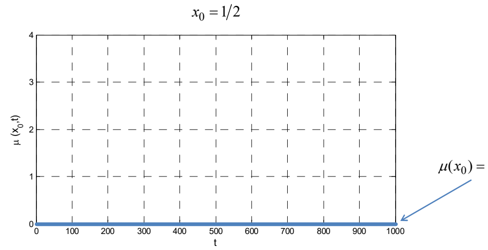
Example

$\varphi(x) = 4x(1-x)$

$x_0 = 1/2$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

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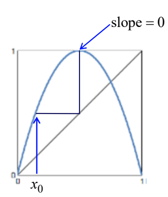
Example

$\varphi(x) = 4x(1-x)$

$x_0 = \dfrac{2-\sqrt{2}}{4}$



$$x_1 = \varphi(x_0) = 4\cdot\frac{2-\sqrt{2}}{4}\cdot\left(1-\frac{2-\sqrt{2}}{4}\right) = 4\cdot\frac{2-\sqrt{2}}{4}\cdot\left(\frac{2+\sqrt{2}}{4}\right) = 4\cdot\frac{4-2}{16} = \frac{1}{2}$$

$$\varphi'(x_1) = 0$$

Lyapunov multiplier

$$\mu(x_0,n) = \left(\prod_{k=0}^{n-1}\left|\varphi'(x_k)\right|\right)^{1/n} = \left(\varphi'(x_0)\times 0\times\prod_{k=2}^{n-1}\left|\varphi'(x_k)\right|\right)^{1/n} = (0)^{1/n} = 0$$

$$\mu(x_0) = \lim_{n\to\infty}\mu(x_0,n) = 0$$

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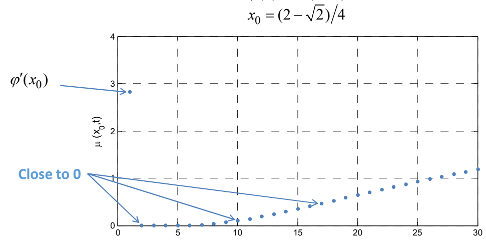
Example

$\varphi(x) = 4x(1-x)$

$x_0 = (2-\sqrt{2})/4$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

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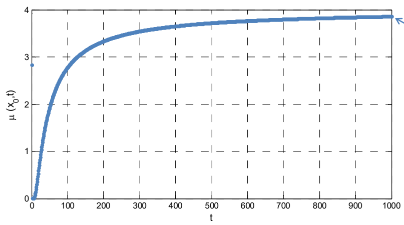
Example

$\varphi(x) = 4x(1-x)$

$x_0 = (2-\sqrt{2})/4$



Looks like $\mu(x_0) = 4$ (but it's not)

$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t}$$

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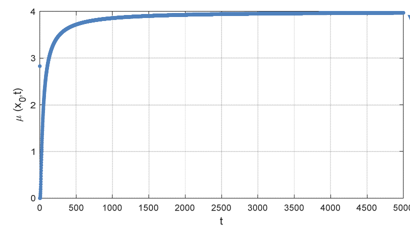
Example

$\varphi(x) = 4x(1-x)$

$x_0 = (2-\sqrt{2})/4$



Looks like $\mu(x_0) = 4$ (but it's not)

$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t}$$

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Example

$\varphi(x) = 4x(1-x)$

For almost every initial condition, the Lyapunov multiplier is 2.

$\mu(x_0) = 2$, for almost all x_0.

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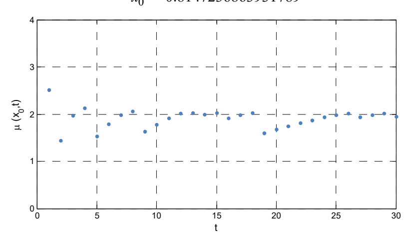
Example

$\varphi(x) = 4x(1-x)$

$x_0 = 0.8147236863931789$

random number generated by MATLAB



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t}$$

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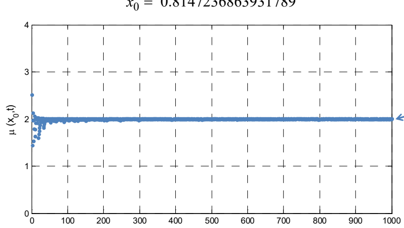
Example

$\varphi(x) = 4x(1-x)$

$x_0 = 0.8147236863931789$

random number generated by MATLAB



$\mu(x_0) = 2$

$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t}$$

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Summary

orbit: $x_{n+1} = f(x_n) = f^{n+1}(x_0), \quad x_0 = x(0)$

Lyapunov multiplier

$$\mu(x_0) = \lim_{n\to\infty} \mu(x_0,n) = \lim_{n\to\infty} \left(\prod_{k=0}^{n-1} |f'(x_k)| \right)^{1/n}$$

Lyapunov exponent

$$\lambda(x_0) = \lim_{n\to\infty} \lambda(x_0,n) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log|f'(x_k)|$$

Interpretation

If the Lyapunov exponent is greater than zero $(\lambda(x_0) > 0)$ or, equivalently, the Lyapunov multiplier is greater than one $(\mu(x_0) > 1)$, then nearby orbits diverge exponentially.

If the Lyapunov exponent is less than zero $(\lambda(x_0) < 0)$ or, equivalently, the Lyapunov multiplier is less than one $(\mu(x_0) < 1)$, then nearby orbits converge exponentially.

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Positive Lyapunov exponents are associated with unpredictability.

Errors propagate exponentially.

Orbits behave randomly.

Simple deterministic systems can have random behavior.

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An Introduction to Mathematical Climate Models

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The Future

Can We Predict the Future?

More relevant question:

What Part of the Future Can't We Predict?

Solar System?

Statistical Mechanics?

Weather?

Climate?

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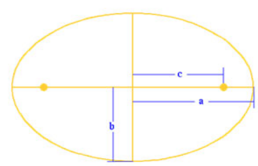
Solar System

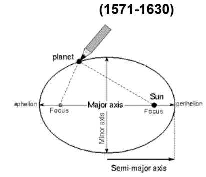
Planetary Orbits

Kepler's First Law: The orbit of every planet is an ellipse with the Sun at one of the two foci.



Eccentricity = c/a

Johannes Kepler (1571-1630)



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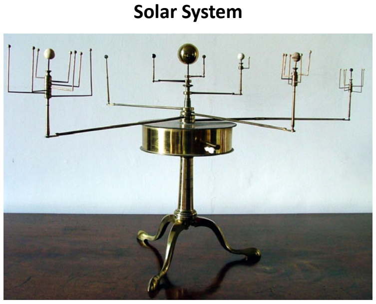

Orrery (circa 1820)

http://star.arm.ac.uk/orrery/mdpopescu_armobs_orrery.jpg

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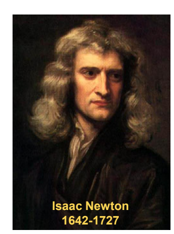
$$m_i \frac{d^2 x_i}{dt^2} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_i m_j \left(x_j - x_i\right)}{\left|x_j - x_i\right|^3}$$



Jacques Laskar (1955-)

The orbits of all the planets can be computed (both forward and backward in time) for billions of years.

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Earth's Semi-major Axis

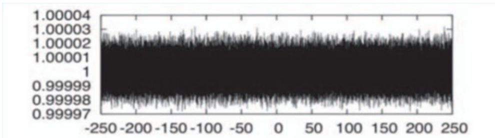

Fig. 11. Variation of the semi-major axis of the Earth–Moon barycenter (in AU) from −250 to +250 Myr.

Semi major axis does not change much and can be computed accurately for hundreds of millions of years.

J. Laskar, et al (2004) A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics* **428**, 261–285.

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Orbits of comets can be chaotic.

Jurgen Moser, *Stable and Random Motions in Dynamical Systems*, Princeton U. Press, 1973.

http://spaceplace.nasa.gov/comet-quest/en/

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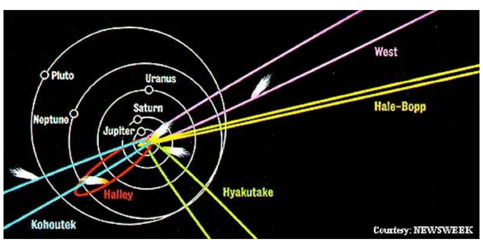
Orbits of comets can be chaotic.

timescale: millennia



http://www.enterprisemission.com/comets.html

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Orbits of the inner planets are chaotic.

timescale: gigayears

Jacques Laskar (1955-)

J. Laskar & M. Gastineau, Existence of collisional trajectories of Mercury, Mars and Venus with the Earth, *Nature* **459**, 817-819, 2009.

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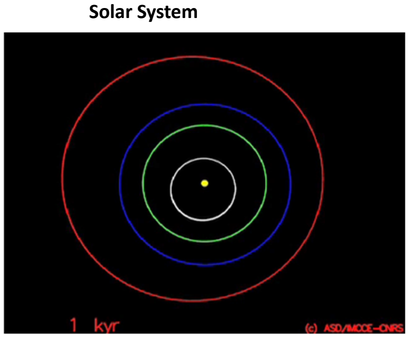
Orbits of the inner planets are chaotic.

timescale: gigayears



http://www.imcce.fr/Equipes/ASD/person/Laskar/jxl_collision.html

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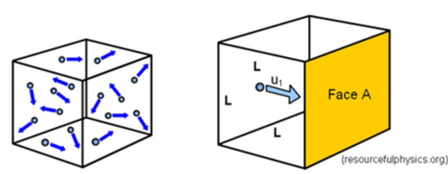
Orbits of molecules in a gas are chaotic.

timescale: microseconds

individual particles chaotic	→	statistical mechanics	→	thermodynamics predictable



http://tap.iop.org/energy/kinetic/603/page_47443.html

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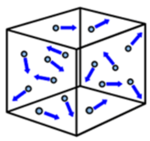
Orbits of molecules in a gas are chaotic.

timescale: microseconds

individual particles chaotic → statistical mechanics → thermodynamics predictable



http://commons.wikimedia.org/wiki/File:Balloon_free_image.jpg

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Analogy with Climate

individual particles chaotic → statistical mechanics → thermodynamics predictable

weather chaotic → ?? → climate predictable?

http://agwx.soils.wisc.edu/

http://crooksandliars.com/2014/03/out-10855-peer-reviewed-articles-climate

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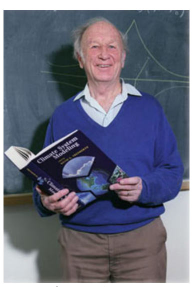
Weather is chaotic.

timescale: weeks



Ed Lorenz in 1994.
(©UCAR, Photo by Curt Zukosky.)

E.N. Lorenz, Deterministic nonperiodic flow, *Journal of the Atmospheric Sciences* **20** (1963), 261-285.

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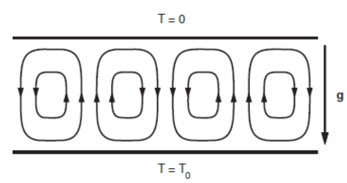
The Lorenz Equations



$$
\begin{aligned}
\dot{x} &= -\sigma x + \sigma y \\
\dot{y} &= \rho x - y - xz \\
\dot{z} &= -\beta z + xy
\end{aligned}
$$

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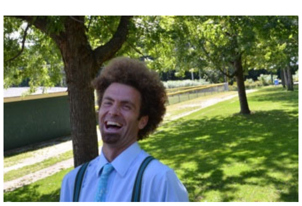
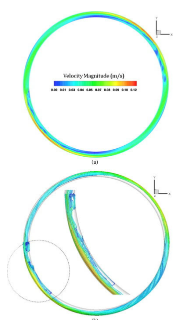
Chris Danforth's Hula Hoop

E.-H. Ridouane, C. M. Danforth, D. L. Hitt, A Numerical Investigation of 3-D Flow Regimes in a Toroidal Natural Convection Loop, *International Journal of Heat and Mass Transfer* **54** (2011), 5253-5261.





http://www.uvm.edu/~cdanfort/main/home.html

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Chris Danforth's Hula Hoop

https://www.youtube.com/watch?v=Vbni-7veJ-c#t=41

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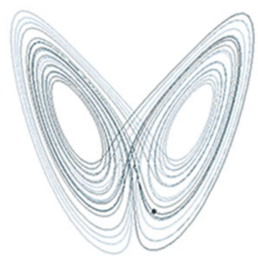
The Lorenz Attractor



$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = -\beta z + xy$$

http://en.wikipedia.org/wiki/Lorenz_system

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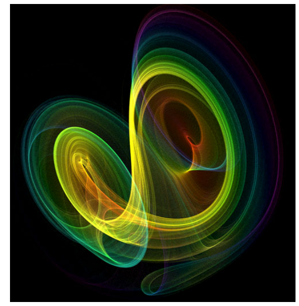
The Lorenz Attractor

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = -\beta z + xy$$



http://www.edc.ncl.ac.uk/highlight/rhnovember2006g02.php

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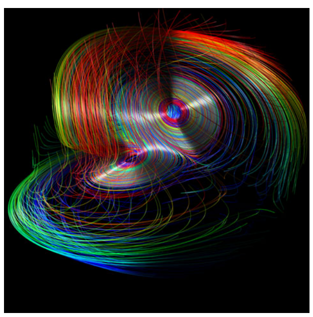
The Lorenz Attractor

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = -\beta z + xy$$



http://www.cs.swan.ac.uk/~cstony/research/star/

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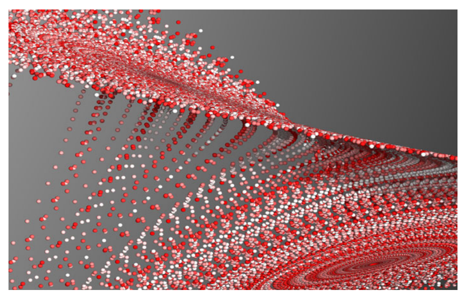
The Lorenz Attractor



$$\dot{x} = -\sigma x + \sigma y$$
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http://safepit.deviantart.com/art/Lorenz-Attractor-2-107809554

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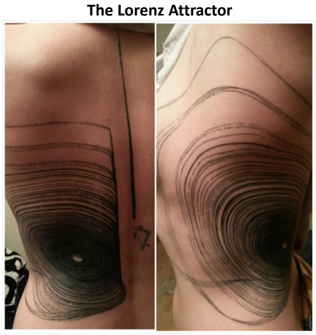

http://pikdit.com/i/lorenz-attractor-by-xoil-needleside-loic-lavenu/

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http://www.spreadshirt.com/chaotic-good-lorenz-attractor-C3376A8822413

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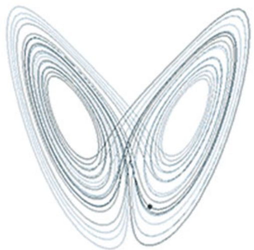

$$\dot{x} = -\sigma x + \sigma y$$
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The Lorenz attractor has a positive Lyapunov exponent. *I.e.*, nearby solutions diverge exponentially.

`http://en.wikipedia.org/wiki/Lorenz_system`

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`https://www.youtube.com/watch?v=FYE4JKAXSfY`

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$$\dot{x} = -\sigma x + \sigma y$$
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You can download a MATLAB function to make your own Lorenz attractor images:

`http://www.mathworks.com/matlabcentral/fileexchange/30066-lorenz-attaractor-plot`

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Climate vs Weather

Meteorology
The climate is the 30 year average of the weather.

Dynamical Systems
The climate is the attractor; the weather is the flow on the attractor.

Stochastic Systems
The climate is the probability distribution of the weather.

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Climate vs Weather

If we can't predict the weather, how can we predict the climate?

Perhaps statistical mechanics provides hope.

individual particles chaotic → statistical mechanics → thermodynamics predictable

weather chaotic → ?? → climate predictable ??

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Which aspects of the climate are predictable?

Perhaps Predictable	Perhaps Not
Global Mean Surface Temperature	Rainfall
Ocean Temperature	Droughts
Sea Ice Extent	Floods
Glacier and Ice Sheet Retreat	Biological Pump
Permafrost Melt	Ecosystems
	Political/Economic Response

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