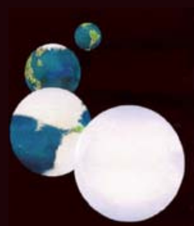
Budyko's Ice Cover Model

Richard McGehee



Seminar on the Mathematics of Climate Change
School of Mathematics
February 5, 2008

Budyko's Model

References

Classic Paper:

M. I. Budyko, The effect of solar radiation variation on the climate of the Earth, Tellus 21 (1969), 611-619.

Modern Interpretation:

K.K. Tung, Topics in Mathematical Modeling, Princeton University Press, 2007. (Chapter 8)

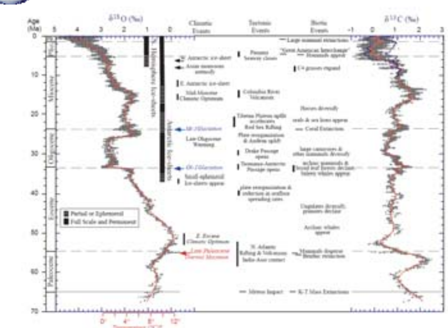
Budyko's Model

Story

http://www.eps.harvard.edu/people/faculty/hoffman/Snowball-fig11.jpg

Budyko's Model

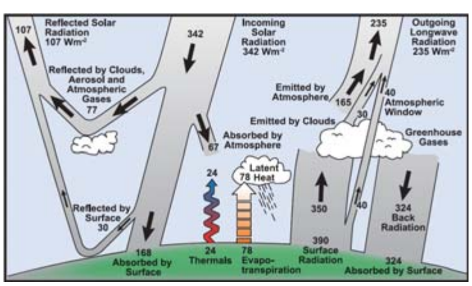
Story



Zachos, et al, *Science* **292** April 27 2001, pp.686-693

Budyko's Model

Heat Balance



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

Budyko's Model

Insolation (Incoming Solar Radiation)

Intensity of solar radiation at Earth's orbital radius:

$1372\ Wm^{-2}$

Cross section intercepted by the Earth:

πr^2

Surface area of the Earth:

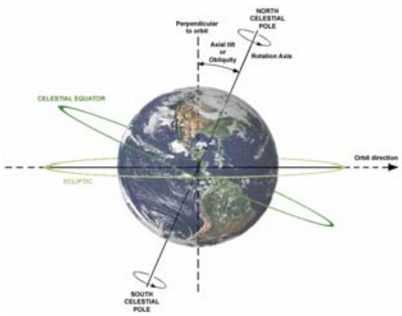
$4\pi r^2$

Average surface insolation = 1372/4 = 343 Wm^{-2}

$$Q = \frac{1372}{4} = 343\ Wm^{-2}$$

Budyko's Model

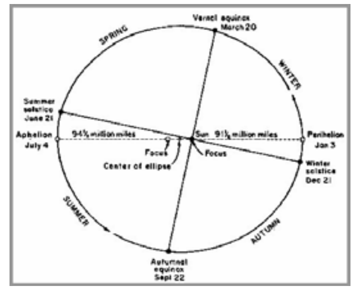
Obliquity



http://upload.wikimedia.org/wikipedia/commons/6/61/AxialTiltObliquity.png

Budyko's Model

Eccentricity



http://www.crrel.usace.army.mil/permafrosttunnel/Ice_Age_Earth_Orbit.jpg

Budyko's Model

Average Insolation

Annual average insolation as a function of latitude θ , where $y = \sin\theta$

$$\boxed{Qs(y)}$$

The function s is normalized so that

$$\int_0^1 s(y)\,dy = 1$$

If eccentricity = obliquity = 0, then

$$s(y) = \frac{4}{\pi}\sqrt{1-s^2}$$

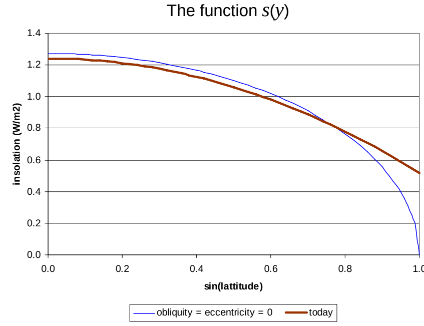
Under today's orbital elements, $s(y)$ can be approximated by a quadratic:

$$s(y) \approx 1 - 0.241\left(3y^2 - 1\right)$$

Budyko's Model

Annual Average Insolation

The function $s(y)$



Budyko's Model

Temperature

Surface temperature in degrees Centigrade is taken to be a decreasing function of latitude.

$$T \in \mathcal{F} = \{T : [0,1] \to \mathbb{R} : T \text{ is decreasing}\}$$

We consider equilibria, where surface temperature is independent of time. Sometimes we will think dynamically, in which case we will write

$$T(t) \in \mathcal{F}, \qquad T(t)(y) \in \mathbb{R}.$$

Budyko's Model

Global MeanTemperature

Compute the average temperature over the Earth's surface.

$$\begin{aligned}\bar{T} &= \frac{1}{4\pi r^2}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} T(\sin\varphi) r^2 \cos\varphi\, d\varphi\, d\theta \\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} T(\sin\varphi)\cos\varphi\, d\varphi = \frac{1}{2}\int_{-1}^{1} T(y)\,dy \\ &= \int_0^1 T(y)\,dy\end{aligned}$$

Note that we have extended T as an even function of latitude.

Budyko's Model
Ice Cover

The ice cover is assumed to be a function of surface temperature. There is a critical temperature, T_c=-10ºC, at which ice can accumulate.

$$T > T_c \Rightarrow \text{no ice}$$
$$T < T_c \Rightarrow \text{ice}$$

Since temperature decreases with latitude, there is a latitude dividing ice from no ice.

$$T(y_s) = T_c$$

Note that y_s is a function of the function T.

$$y_s(T) = T^{-1}(T_c) \in [0,1], \qquad y_s : \mathcal{F} \to [0,1]$$

Budyko's Model
Albedo

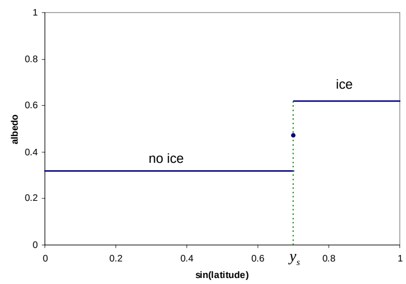
Albedo α measures reflectivity. α=0 corresponds to complete absorption. α=1 corresponds to complete reflection. The albedo of the Earth's surface depends on the ice cover.

$$\text{no ice: } \alpha_1 = 0.32$$
$$\text{ice: } \alpha_2 = 0.62$$

Since the ice cover is a function of surface temperature, which is a function of latitude, we write

$$\alpha(T)(y) = \begin{cases} \alpha_1 & y < y_s(T) \\ \alpha_2 & y > y_s(T) \\ \alpha_0 & y = y_s(T) \end{cases} \quad \text{where} \quad \alpha_0 = \frac{\alpha_1 + \alpha_2}{2} = 0.47$$

Budyko's Model
Albedo



Budyko's Model
Outward Radiation

Assume black body radiation. Stephan-Boltzman: energy radiation is proportional to the 4th power of the absolute temperature (Kelvin). Use linear approximation about the temperature of 0ºC:

$$\boxed{I(T)(y) = a + bT(y)}$$

$$a = 202\,\text{Wm}^{-2}$$
$$b = 1.90\,\text{Wm}^{-2}/^\circ\text{C}$$

Budyko's Model
Dynamic Heat Transport

Simple assumption: the surface tries to equilibrate to the global mean temperature. Heat gain:

$$\boxed{H(T)(y) = c\left(\bar{T} - T(y)\right)}$$

where

$$\bar{T} = \int_0^1 T(y)\,dy$$

is the global mean temperature and where

$$c = 1.6b = 3.04\,\text{Wm}^{-2}/^\circ\text{C}$$

Budyko's Model
Dynamical System

$$\boxed{k\dot{T} = F(T)}$$

$$F(T)(y) = Qs(y)\left(1 - \alpha(T)(y)\right) - I(T)(y) + H(T)(y)$$

$$\alpha(T)(y) = \begin{cases} \alpha_1 & y < y_s(T) \\ \alpha_2 & y > y_s(T) \\ \alpha_0 & y = y_s(T) \end{cases} \qquad T(y_s) = T_c$$

$$I(T)(y) = a + bT(y)$$

$$H(T)(y) = c\left(\bar{T} - T(y)\right)$$

k is the heat capacity of the surface.

Budyko's Model
Equilibrium Solution

Look for an equilibrium solution T^*.

$$F\left(T^*\right)=0$$

$$Qs(y)\left(1-\alpha\left(T^*\right)(y)\right)-I\left(T^*\right)(y)+H\left(T^*\right)(y)=0$$

$$\alpha(T)(y)=\begin{cases}\alpha_1 & y<y_s(T)\\ \alpha_2 & y>y_s(T)\\ \alpha_0 & y=y_s(T)\end{cases}\qquad T(y_s)=T_c$$

$$I(T)(y)=a+bT(y)$$

$$H(T)(y)=c\left(\overline{T}-T(y)\right)$$

Budyko's Model
Ice-free Equilibrium

Look for an equilibrium solution satisying

$$T^*(y)>T_c,\quad \forall y.$$

Then

$$\alpha\left(T^*\right)(y)=\alpha_1,\quad \forall y,$$

and

$$Qs(y)\left(1-\alpha\left(T^*\right)(y)\right)-I\left(T^*\right)(y)+H\left(T^*\right)(y)=0$$

becomes

$$Qs(y)(1-\alpha_1)-a-bT^*(y)+c\left(\overline{T}^*-T^*(y)\right)=0.$$

Therefore,

$$(b+c)T^*(y)=(1-\alpha_1)Qs(y)-a+c\overline{T}^*.$$

Budyko's Model
Ice-free Equilibrium

Integrate both sides:

$$(b+c)\overline{T}^*=(b+c)\int_0^1 T^*(y)\,dy=\int_0^1\left((1-\alpha_1)Qs(y)-a+c\overline{T}^*\right)dy$$
$$=(1-\alpha_1)Q-a+c\overline{T}^*$$

Therefore

$$\overline{T}^*=\frac{(1-\alpha_1)Q-a}{b}$$

and

$$\boxed{T^*(y)=\frac{(1-\alpha_1)Q}{b+c}\left(s(y)+\frac{c}{b}\right)-\frac{a}{b}}$$

Budyko's Model
Ice-free Equilibrium

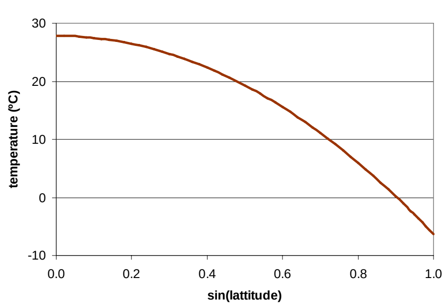
$$T^*(y)=\frac{(1-\alpha_1)Q}{b+c}\left(s(y)+\frac{c}{b}\right)-\frac{a}{b}$$

This works as long as the temperature is everywhere above -10º C. Since s is decreasing,

$$T^*(y)\geq T^*(1)=\frac{(1-\alpha_1)Q}{b+c}\left(s(1)+\frac{c}{b}\right)-\frac{a}{b}\approx -6.32$$

for the current values of the parameters.

Budyko's Model
Ice-free Equilibrium



Budyko's Model
Ice-free Equilibrium

We will think of Q as a parameter. The ice-free solution will exist as long as

$$T^*(1)=\frac{(1-\alpha_1)Q}{b+c}\left(s(1)+\frac{c}{b}\right)-\frac{a}{b}>T_c$$

For the current values of the other parameters,

$$Q>330\,Wm^{-2}$$

Budyko's Model

Snowball Equilibrium

Look for an equilibrium solution satisying

$$T^*(y) < T_c, \quad \forall y.$$

This is exactly the same as before, except that

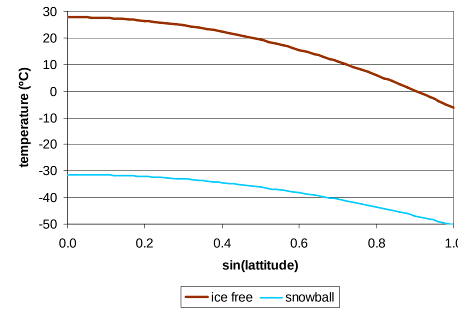
$$\alpha(T^*)(y) = \alpha_2, \quad \forall y.$$

$$T^*(y) = \frac{(1-\alpha_2)Q}{b+c}\left(s(y) + \frac{c}{b}\right) - \frac{a}{b}$$

$$T^*(y) \le T^*(0) = \frac{(1-\alpha_2)Q}{b+c}\left(s(0) + \frac{c}{b}\right) - \frac{a}{b} \approx -31.36$$

Budyko's Model

Snowball Equilibrium



Budyko's Model

Snowball Equilibrium

Again thinking of Q as a parameter, the snowball solution will exist as long as

$$T^*(0) = \frac{(1-\alpha_1)Q}{b+c}\left(s(0) + \frac{c}{b}\right) - \frac{a}{b} < T_c$$

For the current values of the other parameters,

$$Q < 441 Wm^{-2}$$

Budyko's Model

Ice Cap Equilibrium

Recall:

$$Qs(y)\left(1-\alpha(T^*)(y)\right) - I(T^*)(y) + H(T^*)(y) = 0$$

$$\alpha(T)(y) = \begin{cases} \alpha_1 & y < y_s(T) \\ \alpha_2 & y > y_s(T) \\ \alpha_0 & y = y_s(T) \end{cases} \qquad T(y_s) = T_c$$

$$I(T)(y) = a + bT(y) \qquad H(T)(y) = c\left(\overline{T} - T(y)\right)$$

Evaluate at ice boundary: $y = y_s^* = y_s(T^*), \quad T^*(y_s^*) = T_c$

$$I(T^*)(y_s^*) = a + bT_c \qquad H(T^*)(y_s^*) = c\left(\overline{T}^* - T_c\right)$$

$$\alpha(T^*)(y_s^*) = \alpha_0$$

Budyko's Model

Ice Cap Equilibrium

Therefore:

$$Qs(y_s^*)\left(1-\alpha(T^*)(y_s^*)\right) - I(T^*)(y_s^*) + H(T^*)(y_s^*) = 0$$

becomes

$$Qs(y_s^*)(1-\alpha_0) - a - bT_c + c\left(\overline{T}^* - T_c\right) = 0$$

$$(1-\alpha_0)Qs(y_s^*) - a + c\overline{T}^* - (b+c)T_c = 0$$

Goal: We think of Q and y_s^* as parameters. The above equation can be solved for Q as a function of y_s^*.

Problem: $\overline{T}^*$ depends on y_s^*

Budyko's Model

Ice Cap Equilibrium

Computation of global mean temperature at equilibrium

Recall:

$$Qs(y)\left(1-\alpha(T^*)(y)\right) - I(T^*)(y) + H(T^*)(y) = 0$$

$$\alpha(T)(y) = \begin{cases} \alpha_1 & y < y_s(T) \\ \alpha_2 & y > y_s(T) \\ \alpha_0 & y = y_s(T) \end{cases} \qquad T(y_s) = T_c$$

$$I(T)(y) = a + bT(y) \qquad H(T)(y) = c\left(\overline{T} - T(y)\right)$$

Integrate:

$$\int_0^1 \left[Qs(y)\left(1-\alpha(T^*)(y)\right) - I(T^*)(y) + H(T^*)(y)\right] dy = 0$$

Budyko's Model
Ice Cap Equilibrium

$$\int_0^1 H\left(T^*\right)(y)\,dy = \int_0^1 c\left(\overline{T}^* - T^*(y)\right)dy = 0$$

$$\int_0^1 I\left(T^*\right)(y)\,dy = \int_0^1 \left(a + bT^*(y)\right)dy = a + b\overline{T}^*$$

Therefore,

$$\int_0^1 \left[Qs(y)\left(1-\alpha\left(T^*\right)(y)\right) - I\left(T^*\right)(y) + H\left(T^*\right)(y)\right]dy = 0$$

becomes

$$Q\left(1-\overline{\alpha}\left(T^*\right)\right) - a - b\overline{T}^* = 0,$$

where

$$\overline{\alpha}\left(T^*\right) = \int_0^1 s(y)\alpha\left(T^*\right)(y)\,dy.$$

Budyko's Model
Ice Cap Equilibrium

Compute:

$$\begin{aligned}\overline{\alpha}\left(T^*\right) &= \int_0^1 s(y)\alpha\left(T^*\right)(y)\,dy \\ &= \int_0^{y_s^*} s(y)\alpha\left(T^*\right)(y)\,dy + \int_{y_s^*}^1 s(y)\alpha\left(T^*\right)(y)\,dy \\ &= \alpha_1 S\left(y_s^*\right) + \alpha_2\left(1 - S\left(y_s^*\right)\right),\end{aligned}$$

where

$$S(y) = \int_0^y s(\eta)\,d\eta.$$

Abuse notation:

$$\overline{\alpha}\left(y_s^*\right) = \alpha_1 S\left(y_s^*\right) + \alpha_2\left(1 - S\left(y_s^*\right)\right)$$

Budyko's Model
Ice Cap Equilibrium

Recall:

$$Q\left(1-\overline{\alpha}\left(T^*\right)\right) - a - b\overline{T}^* = 0,$$

which yields

$$\overline{T}^* = \frac{1}{b}\left(Q\left(1-\overline{\alpha}\left(y_s^*\right)\right) - a\right).$$

I.e., the global mean temperature for a equilibrium solution depends only on the location of the ice boundary.

Budyko's Model
Ice Cap Equilibrium

Recall:

$$\left(1-\alpha_0\right)Qs\left(y_s^*\right) - a + c\overline{T}^* - (b+c)T_c = 0$$

$$\overline{T}^* = \frac{1}{b}\left(Q\left(1-\overline{\alpha}\left(y_s^*\right)\right) - a\right)$$

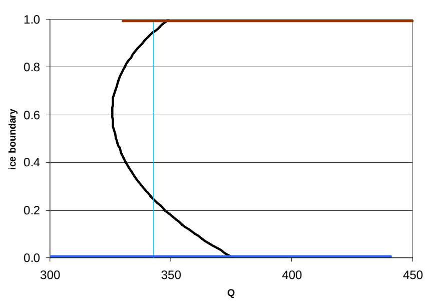
Combining:

$$\left(1-\alpha_0\right)Qs\left(y_s^*\right) - a + \frac{c}{b}\left(Q\left(1-\overline{\alpha}\left(y_s^*\right)\right) - a\right) - (b+c)T_c = 0$$

$$Q\left(\left(1-\alpha_0\right)s\left(y_s^*\right) + \frac{c}{b}\left(1-\overline{\alpha}\left(y_s^*\right)\right)\right) = (b+c)\left(\frac{a}{b} + T_c\right)$$

The above equation can be solved for Q as a function of y_s^*.

Budyko's Model
Equilibria



Budyko's Model
Ice Cap Equilibrium

Computation of the equilibrium temperature function.
Recall:

$$Qs(y)\left(1-\alpha\left(T^*\right)(y)\right) - I\left(T^*\right)(y) + H\left(T^*\right)(y) = 0$$

$$I(T)(y) = a + bT(y) \qquad H(T)(y) = c\left(\overline{T} - T(y)\right)$$

Substitute:

$$Qs(y)\left(1-\alpha\left(T^*\right)(y)\right) - a - bT^*(y) + c\left(\overline{T}^* - T^*(y)\right) = 0$$

Solve:

$$T^*(y) = \frac{1}{b+c}\left(Qs(y)\left(1-\alpha\left(T^*\right)(y)\right) - a + c\overline{T}^*\right)$$

Budyko's Model

Ice Cap Equilibrium

Summary:

$$T^*(y)=\begin{cases}\frac{1}{b+c}\left(Qs(y)(1-\alpha_1)-a+c\bar{T}^*\right) & \text{for } y<y_s^* \\ \frac{1}{b+c}\left(Qs(y)(1-\alpha_2)-a+c\bar{T}^*\right) & \text{for } y>y_s^*\end{cases}$$

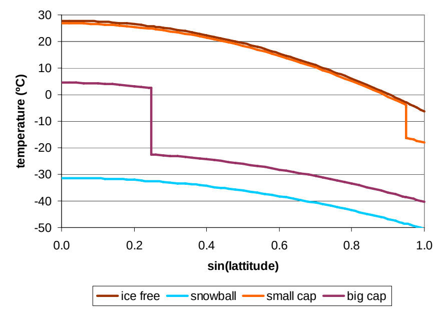
$$Q\left((1-\alpha_0)s(y_s^*)+\frac{c}{b}\left(1-\bar{\alpha}(y_s^*)\right)\right)=(b+c)\left(\frac{a}{b}+T_c\right)$$

$$\bar{T}^*=\frac{1}{b}\left(Q\left(1-\bar{\alpha}(y_s^*)\right)-a\right).$$

$$\bar{\alpha}(y_s^*)=\alpha_1 S(y_s^*)+\alpha_2\left(1-S(y_s^*)\right)$$

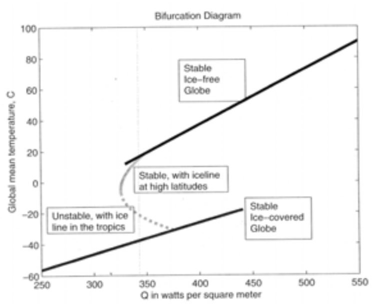
Budyko's Model

Ice Cap Equilibrium



Budyko's Model

Stability



K. K. Tung, Topics in Mathematical Modeling, p. 139

Budyko's Model

Stability

Literature Claim:

Snowball solution is stable.
Ice free solution is stable.
Small polar cap is stable.
Large polar cap is unstable.

What does this mean?

$$k\dot{T}=F(T),\quad T\in\mathcal{F}$$

$$T(t)\to T^* \text{ as } t\to\infty, \text{ in a neighborhood of } T^*$$

What is the topology?

Budyko's Model

Stability

What about linear stability?

Variational equation about an equilibrium:

$$k\dot{u}=DF(T^*)u,\quad u\in T_{T^*}\mathcal{F}$$

Does $\mathcal{F}$ have a differentiable structure? (It is not a linear space.)
Is F differentiable?