Energy Balance Models

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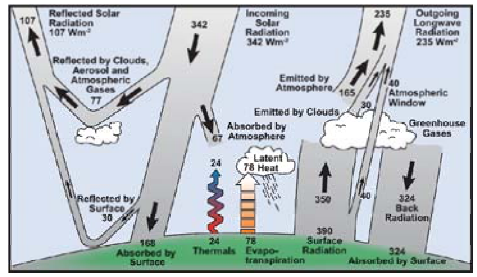
Energy Balance Models

Heat Balance



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

Energy Balance Models

References

Classic Papers:

M. I. Budyko, The effect of solar radiation variation on the climate of the Earth, *Tellus* **21** (1969), 611-619.

W. D. Sellers, A Global Climatic Model Based on the Energy Balance of the Earth-Atmosphere System, *Journal of Applied Meteorology* **8** (1969), 392-400.

Recent Interpretation:

K.K. Tung, Topics in Mathematical Modeling, Princeton University Press, 2007. (Chapter 8)

Energy Balance Models

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

T = global mean temperature (°C)
Q = mean solar input (W/m²)
α = mean albedo
$A+BT$ = outward radiation (linear approximation)
R = heat capacity of Earth's surface

Tung's values:

T = global mean temperature (°C)
Q = 343 W/m²
A = 202 W/m²
B = 1.9 W/(m² °C)
$\alpha = \alpha_1$ = 0.32 (water and land)
$\alpha = \alpha_2$ = 0.62 (ice)

Energy Balance Models

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium temperature

$$T_{eq} = \frac{Q(1-\alpha) - A}{B}$$

ice free Earth: $\alpha = \alpha_1$, T_{eq} = 16.4 °C
snowball Earth: $\alpha = \alpha_2$, T_{eq} = -37.7 °C

According to Tung, glaciers form if $T < T_c$ = -10 °C and melt if $T > T_c$.

Since 16.4 > -10, no glacier would form on an ice free Earth.
Since -37.7 < -10, no glacier would melt on a snowball Earth.

Energy Balance Models

Inhomogeneous Earth

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T} - T\right)$$

Now the annual average surface temperature T is a function of y = sine(latitude).

The albedo α is a function of y and the location η of the ice boundary.

The outward radiation $A+BT$ is as before.

Heat transport across latitudes is assumed to be linear and is given by
$C\left(\overline{T} - T\right)$
where C = 3.04 W/m²

The global annual average insolation is Q, with the same value as above, while $s(y)$ is the relative insolation, normalized to satisfy

$$\int_0^1 s(y)\,dy$$

Energy Balance Models

Inhomogeneous Earth

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T}-T\right)$$

The variable y is chosen instead of the latitude, because the global annual mean temperature is given by

$$\overline{T}(t) = \int_0^1 T(y,t)\,dy$$

We assume symmetry with respect to the equator, so the variable y takes on values between 0 and 1.

We assume an ice boundary at $y = \eta$, with ice toward the pole and no ice toward the equator. The albedo is therefore

$$\alpha(y,\eta) = \begin{cases} \alpha_1, & y < \eta, \\ \alpha_2, & y > \eta. \end{cases}$$

Rate of solar energy absorption at y = sine(latitude):

$$Qs(y)\left(1-\alpha(y,\eta)\right)$$

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Inhomogeneous Earth

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T}-T\right)$$

Look for an equilibrium solution having an ice line at $y = \eta$

$$T = T_\eta^*(y)$$

This equilibrium satisfies

$$\boxed{Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT_\eta^*(y)\right) + C\left(\overline{T}_\eta^* - T_\eta^*(y)\right) = 0}$$

Next step: Solve for the equilibrium temperature profile, assuming we know the ice boundary.

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$$\boxed{Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT_\eta^*(y)\right) + C\left(\overline{T}_\eta^* - T_\eta^*(y)\right) = 0}$$

Integrate:

$$\int_0^1\left(Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT_\eta^*(y)\right) + C\left(\overline{T}_\eta^* - T_\eta^*(y)\right)\right)dy = 0,$$

$$Q\left(1-\overline{\alpha}(\eta)\right) - A - B\overline{T}_\eta^* = 0$$

where

$$\overline{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)\,dy = \int_0^\eta \alpha_1 s(y)\,dy + \int_\eta^1 \alpha_2 s(y)\,dy$$

$$= \alpha_1 S(\eta) + \alpha_2\left(1-S(\eta)\right) = \alpha_2 - (\alpha_2-\alpha_1)S(\eta),$$

and where

$$S(\eta) = \int_0^\eta s(y)\,dy$$

Given the ice line η, the global mean temperature is

$$\boxed{\overline{T}_\eta^* = \frac{1}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right) - A\right)}$$

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Inhomogeneous Earth

Equilibrium equation (given ice line):

$$Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT_\eta^*(y)\right) + C\left(\overline{T}_\eta^* - T_\eta^*(y)\right) = 0$$

Global mean temperature:

$$\overline{T}_\eta^* = \frac{1}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right) - A\right)$$

Solve for equilibrium temperature profile:

$$\boxed{T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\overline{T}_\eta^*\right)}$$

where

$$\alpha(y,\eta) = \begin{cases} \alpha_1, & y < \eta, \\ \alpha_2, & y > \eta. \end{cases}$$

$$\overline{\alpha}(\eta) = \alpha_2 - (\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy$$

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Inhomogeneous Earth

$$\boxed{T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\overline{T}_\eta^*\right)}$$

Additional assumption: At equilibrium, the average temperature across the ice boundary is T_c = -10 °C

$$T_\eta^*(\eta-) = \frac{1}{B+C}\left(Qs(\eta)(1-\alpha_1) - A + C\overline{T}_\eta^*\right)$$

$$T_\eta^*(\eta+) = \frac{1}{B+C}\left(Qs(\eta)(1-\alpha_2) - A + C\overline{T}_\eta^*\right)$$

(s is continuous)

$$T_c = \frac{T_\eta^*(\eta-) + T_\eta^*(\eta+)}{2} = \frac{1}{B+C}\left(Qs(\eta)(1-\alpha_0) - A + C\overline{T}_\eta^*\right)$$

where

$$\alpha_0 = \frac{\alpha_1+\alpha_2}{2}$$

Energy Balance Models

Inhomogeneous Earth

Now we can solve for the ice boundary.

$$\frac{1}{B+C}\left(Qs(y)(1-\alpha_0) - A + C\overline{T}_\eta^*\right) = T_c$$

where

$$\overline{T}_\eta^* = \frac{1}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right) - A\right)$$

Therefore,

$$\frac{1}{B+C}\left(Qs(\eta)(1-\alpha_0) - A + \frac{C}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right) - A\right)\right) = T_c$$

which reduces to

$$\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy\right)\right) - \frac{A}{B} - T_c = 0$$

which can be solved numerically for η .

Energy Balance Models

Inhomogeneous Earth

What about $s(y)$, the relative insolation function?

$$s(y)=\frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\gamma-y\cos\beta\right)^2}\,d\gamma$$

where β = obliquity. (Current value is about 23.5°.)

Tung and North's quadratic approximation:

$$s(y)\approx 1-0.241\left(3y^2-1\right)$$

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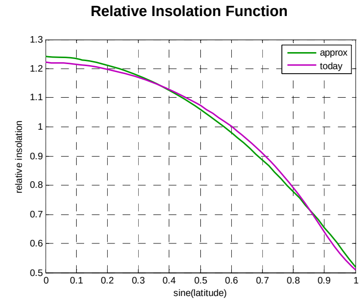
green = quadratic approximation (Tung and North)

mauve = formula using obliquity of 23.5°



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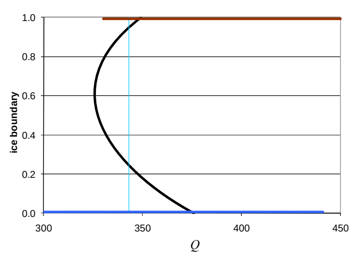
Inhomogeneous Earth



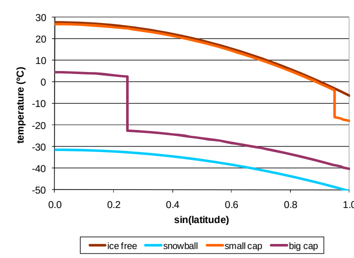
Energy Balance Models

Inhomogeneous Earth

Tung conclusions:

four equilibrium solutions:

snowball: stable
large cap: unstable
small cap: stable
ice free: ***stable***



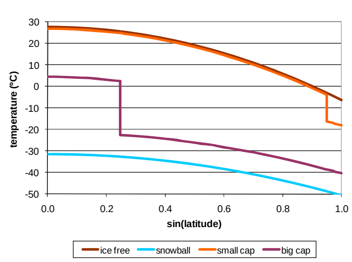
Energy Balance Models

Inhomogeneous Earth

Widiasih conclusions:

four equilibrium solutions:

snowball: stable
large cap: unstable
small cap: stable
ice free: ***unstable***



Energy Balance Models

Inhomogeneous Earth

What's next?

$$R\frac{\partial T}{\partial t}=Qs(y)\left(1-\alpha(y,\eta)\right)-\left(A+BT\right)+C\left(\overline{T}-T\right)$$

What if we use the information from the Milankovitch cycles as input to the energy balance model? Can we model the glacial cycles?

Q is determined by eccentricity.
$s(y)$ is determined by obliquity.

To be continued ... .