Milankovitch Cycles

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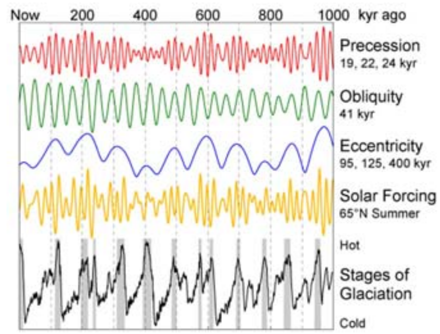
http://www.tqnyc.org/NYC052141/beginningpage.html

Milankovitch Cycles

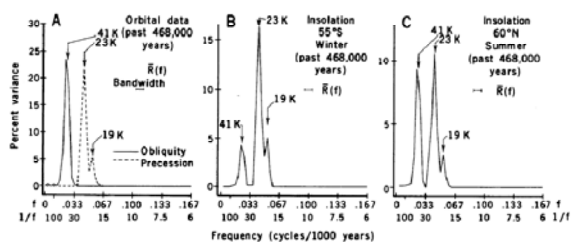
Solar Forcing



http://en.wikipedia.org/wiki/Milankovitch_cycles

Milankovitch Cycles

Solar Forcing (Hays, et al)



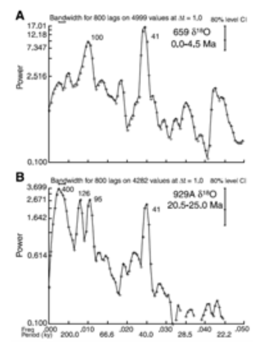
Hays, *et al*, *Science* **194** (1976), p. 1125

Milankovitch Cycles

Climate Response (Zachos, et al)

A. Power spectrum of climate for the last 4.5 Myr. Note the peaks at 41Kyr and 100 Kyr.

B. Power spectrum of climate for the period 25 Myr bp to 20.5 Myr bp. Note the new peak at 400 Kyr and the "split" peaks at 126Kyr and 95 Kyr.



Zachos, *et al*, *Science* **292** (2001), p. 689

Milankovitch Cycles

Budyko's Ice Line Model

$$R\frac{dT}{dt} = \boxed{Qs(y)}\left(1-\alpha(T)(y)\right) - I(T)(y) + H(T)(y)$$

The annual global average insolation is Q.
The annual average insolation as a function of latitude θ, where $y = \sin\theta$, is $Qs(y)$.
Q is largely determined by the eccentricity, but $s(y)$ is determined from a combination of the other orbital elements.

What is $s(y)$ as a function of obliquity and precession?

Milankovitch Cycles

Insolation Function

A point on the surface of the Earth

$\varphi = \text{latitude}, \quad \gamma = \text{longitude}$

$$s = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} \cos\varphi\cos\gamma \\ \cos\varphi\sin\gamma \\ \sin\varphi \end{bmatrix}$$

Orthogonal matrix to obliquity angle

$$S_2(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

Orthogonal matrix to precession angle

$$S_3(\rho) = \begin{bmatrix} \cos\rho & -\sin\rho & 0 \\ \sin\rho & \cos\rho & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In solar coordinates: $S_3(\rho)S_2(\beta)s$

Milankovitch Cycles

Instantaneous Insolation Function

The Earth's position with respect to the Sun, in the plane of the ecliptic

$$(r,\theta)$$

Instantaneous insolation at the point s on the Earth's surface:

$$I=\left[-\frac{K}{4\pi r^2}\begin{bmatrix}\cos\theta & \sin\theta & 0\end{bmatrix}S_3(\rho)S_2(\beta)s\right]^+$$

Doing the math:

$$I(\beta,\rho,r,\theta,\varphi,\gamma)=$$

$$\frac{K}{4\pi r^2}\left[-\cos\varphi\left(\cos\beta\cos(\theta-\rho)\cos\gamma+\sin(\theta-\rho)\sin\gamma\right)-\sin\varphi\sin\beta\cos(\theta-\rho)\right]^+$$

Milankovitch Cycles

Instantaneous Insolation Function

Earth's surface in the Sun's coordinate system:

$$\hat{s}=\begin{bmatrix}\cos\hat{\varphi}\cos\hat{\gamma}\\ \cos\hat{\varphi}\sin\hat{\gamma}\\ \sin\hat{\varphi}\end{bmatrix}=S_2(\beta)s$$

Instantaneous insolation in these coordinates:

$$I=\left[-\frac{K}{4\pi r^2}\begin{bmatrix}\cos\theta\cos\rho+\sin\theta\sin\rho\\ -\cos\theta\sin\rho+\sin\theta\cos\rho\\ 0\end{bmatrix}^T\begin{bmatrix}\cos\hat{\varphi}\cos\hat{\gamma}\\ \cos\hat{\varphi}\sin\hat{\gamma}\\ \sin\hat{\varphi}\end{bmatrix}\right]^+$$

$$=\left[-\frac{K}{4\pi r^2}\cos\hat{\varphi}\left(\cos(\theta-\rho)\cos\hat{\gamma}+\sin(\theta-\rho)\sin\hat{\gamma}\right)\right]^+$$

$$=\left[-\frac{K}{4\pi r^2}\cos\hat{\varphi}\cos(\theta-\rho-\hat{\gamma})\right]^+$$

Milankovitch Cycles

Annual Insolation Function

Instantaneous insolation:

$$I(\rho,r,\theta,\hat{\varphi},\hat{\gamma})=\frac{K}{4\pi r^2}\cos\hat{\varphi}\left[-\cos(\theta-\rho-\hat{\gamma})\right]^+$$

Annual average:

$$\bar{I}(\rho,\hat{\varphi},\hat{\gamma})=\frac{1}{P}\int_0^P\frac{K}{4\pi r(t)^2}\cos\hat{\varphi}\left[-\cos\left(\theta(t)-\rho-\hat{\gamma}\right)\right]^+dt$$

$$=\frac{K\cos\hat{\varphi}}{4\pi P}\int_0^P\frac{\left[-\cos\left(\theta(t)-\rho-\hat{\gamma}\right)\right]^+}{r(t)^2}dt,$$

P = one year

Milankovitch Cycles

Annual Insolation Function

Specific angular momentum: $\Omega=r(t)^2\frac{d\theta}{dt}$

Annual average:

$$\bar{I}(\rho,\hat{\varphi},\hat{\gamma})=\frac{K\cos\hat{\varphi}}{4\pi P}\int_0^P\frac{\left[-\cos\left(\theta(t)-\rho-\hat{\gamma}\right)\right]^+}{r(t)^2}dt$$

$$=\frac{K\cos\hat{\varphi}}{4\pi P\Omega}\int_0^{2\pi}\left[-\cos(\theta-\rho-\hat{\gamma})\right]^+d\theta$$

$$=\frac{K\cos\hat{\varphi}}{4\pi P\Omega}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta$$

$$\boxed{\bar{I}(\hat{\varphi},\hat{\gamma})=\frac{K\cos\hat{\varphi}}{2\pi P\Omega}}$$

Note the disappearance of ρ (precession angle).

Milankovitch Cycles

Annual Insolation Function

Earth's coordinate system: $\hat{s}=S_2(\beta)s$

$$\sin\hat{\gamma}=-\sin\beta\cos\varphi\cos\gamma+\cos\beta\sin\varphi$$

$$\bar{I}(\varphi,\gamma)=\frac{K\cos\hat{\varphi}}{2\pi P\Omega}=\frac{K\sqrt{1-\left(-\sin\beta\cos\varphi\cos\gamma+\cos\beta\sin\varphi\right)^2}}{2\pi P\Omega}$$

Average over a day:

$$\bar{I}(\varphi)=\frac{K}{2\pi P\Omega}\frac{1}{2\pi}\int_0^{2\pi}\sqrt{1-\left(\sin\beta\cos\varphi\cos\gamma-\cos\beta\sin\varphi\right)^2}\,d\gamma$$

$$\boxed{\bar{I}(\varphi)=\frac{K}{4\pi^2P\Omega}\int_0^{2\pi}\sqrt{1-\left(\sin\beta\cos\varphi\cos\gamma-\cos\beta\sin\varphi\right)^2}\,d\gamma}$$

Milankovitch Cycles

Annual Insolation Function

Claim: $I(-\varphi)=I(\varphi)$

Proof:

$$\int_0^{2\pi}\sqrt{1-\left(\sin\beta\cos(-\varphi)\cos\gamma-\cos\beta\sin(-\varphi)\right)^2}\,d\gamma$$

$$=\int_0^{2\pi}\sqrt{1-\left(\sin\beta\cos\varphi\cos\gamma+\cos\beta\sin\varphi\right)^2}\,d\gamma$$

$$=\int_0^{2\pi}\sqrt{1-\left(\sin\beta\cos\varphi\cos(\pi+\gamma)+\cos\beta\sin\varphi\right)^2}\,d\gamma$$

$$=\int_0^{2\pi}\sqrt{1-\left(-\sin\beta\cos\varphi\cos\gamma+\cos\beta\sin\varphi\right)^2}\,d\gamma$$

$$=\int_0^{2\pi}\sqrt{1-\left(\sin\beta\cos\varphi\cos\gamma-\cos\beta\sin\varphi\right)^2}\,d\gamma$$

Milankovitch Cycles

Relation to Budyko

$$R\frac{dT}{dt}=\boxed{Qs(y)}\big(1-\alpha(T)(y)\big)-I(T)(y)+H(T)(y)$$

$$Qs(y)=\bar{I}(\varphi),\quad \text{where } y=\sin\varphi$$

$$Qs(y)=\frac{K}{4\pi^2P\Omega}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\gamma-y\cos\beta\right)^2}\,d\gamma$$

The function s is normalized so that $\int_0^1 s(y)\,dy=1$

$$Q\int_{-1}^{1}s(y)\,dy=\frac{K}{4\pi^2P\Omega}\int_{-1}^{1}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\gamma-y\cos\beta\right)^2}\,d\gamma\,dy$$

$$=\frac{K}{4\pi^2P\Omega}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\sqrt{1-\left(\cos\varphi\sin\beta\cos\gamma-\sin\varphi\cos\beta\right)^2}\,d\gamma\cos\varphi\,d\varphi$$

Milankovitch Cycles

Relation to Budyko

Returning to the Sun's coordinate system, we have

$$\hat{s}=S_2(\beta),\quad \text{hence}\quad \cos\varphi\,d\gamma\,d\varphi=\cos\hat{\varphi}\,d\hat{\gamma}\,d\hat{\varphi}$$

$$Q\int_{-1}^{1}s(y)\,dy=\frac{K}{4\pi^2P\Omega}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\sqrt{1-\left(\cos\varphi\sin\beta\cos\gamma-\sin\varphi\cos\beta\right)^2}\cos\varphi\,d\gamma\,d\varphi$$

$$=\frac{K}{4\pi^2P\Omega}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\cos\hat{\varphi}\cos\hat{\varphi}\,d\hat{\gamma}\,d\hat{\varphi}$$

$$=\frac{K}{2\pi P\Omega}\int_{-\pi/2}^{\pi/2}\cos^2\hat{\varphi}\,d\hat{\varphi}=\frac{K}{2\pi P\Omega}\frac{\pi}{2}$$

$$=\frac{K}{4P\Omega}$$

Since $\int_{-1}^{1}s(y)\,dy=2$ we have $Q=\dfrac{K}{8P\Omega}$

Milankovitch Cycles

Relation to Budyko

Summary

$$Qs(y)=\bar{I}(\varphi)=\frac{K}{4\pi^2P\Omega}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\gamma-y\cos\beta\right)^2}\,d\gamma$$

$$Q=\frac{K}{8P\Omega}$$

$$\boxed{s(y)=\frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\gamma-y\cos\beta\right)^2}\,d\gamma}$$

Note that Q depends only on the eccentricity and that s depends only on the obliquity.

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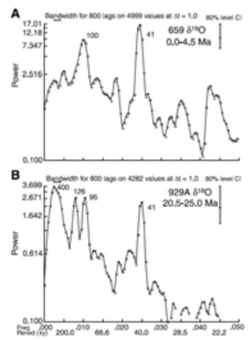


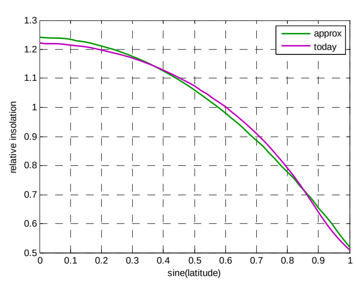
Zachos, *et al*, *Science* **292** (2001), p. 689

Milankovitch Cycles

Relative Insolation Function

green = quadratic approximation (Tung and North)

mauve = formula using obliquity of 23.5°

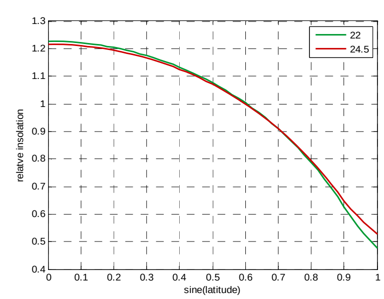


Milankovitch Cycles

Relative Insolation Function

green = obliquity of 22.0°

red = obliquity of 24.5°



Milankovitch Cycles

Widiasih-Budyko Ice Line Model

$$\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A + BT(\eta,t)\right) + C\left(\bar{T}(\eta) - T(\eta,t)\right)$$

$$\frac{\partial \eta}{\partial t} = \varepsilon\left(T(\eta,t) - T_c\right)$$

Assuming there is a single ice line in the northern hemisphere, located at $y=\eta$, the fast variables collapse to a one-dimensional center manifold with equation

$$\frac{\partial \eta}{\partial t} = \varepsilon h(\eta)$$

Milankovitch Cycles

Widiasih-Budyko Ice Line Model

$$\frac{\partial \eta}{\partial t} = \varepsilon h(\eta)$$

where

$$h(\eta) = \frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\bar{\alpha}(\eta)\right)\right) - \frac{A}{B} - T_c$$

$$\bar{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)\,dy = \int_0^{\eta} \alpha_1 s(y)\,dy + \int_{\eta}^1 \alpha_2 s(y)\,dy$$

Solve

$$h(\eta) = 0$$

for the ice line, which will depend on the eccentricity e and the obliquity β.

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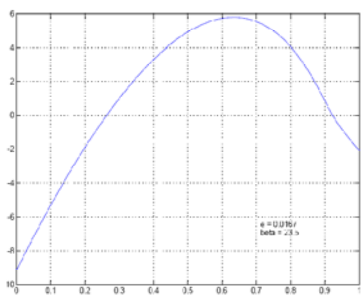
Widiasih-Budyko Ice Line Model

The function h for current eccentricity and obliquity.



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Widiasih-Budyko Ice Line Model

Once we know the ice line, we can solve for the global mean temperature

$$\bar{T} = \frac{1}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right) - A\right)$$

and the temperature at the pole

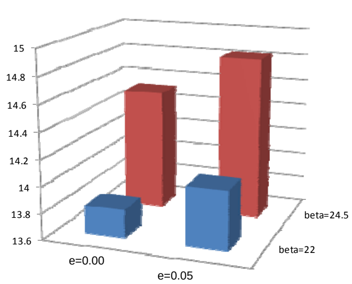
$$T(1) = \frac{1}{B+C}\left(Qs(1)(1-\alpha_2) - A + C\bar{T}\right)$$

and see how these vary with the eccentricity and the obliquity.

Milankovitch Cycles

Global Annual Mean Temperature

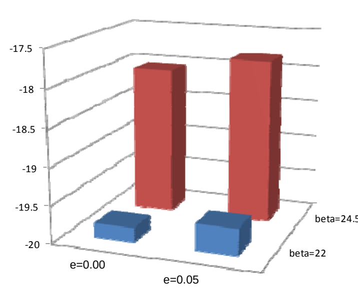
Computed global mean temperature for extremes in eccentricity and obliquity



Milankovitch Cycles

Polar Annual Mean Temperature

Computed polar mean temperature for extremes in eccentricity and obliquity



Milankovitch Cycles

Conclusions

1. Precession doesn't matter.
2. Obliquity is more important than eccentricity.
3. Polar temperatures vary twice as much as global temperatures.