Energy Balance Models

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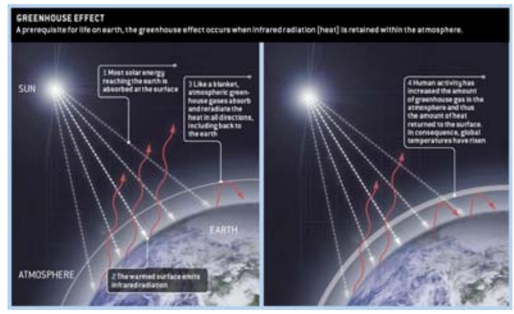
Energy Balance Models

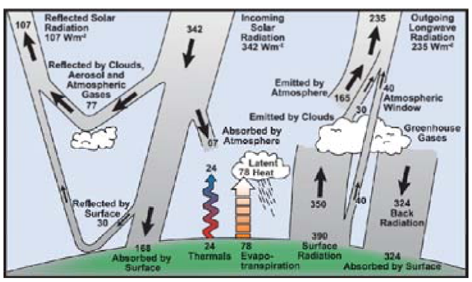
Earth's Energy Balance



Gary Stix, *Scientific American* September 2006, pp.46-49

Energy Balance Models

Earth's Energy Balance



Historical Overview of Climate Change Science, IPCC AR4, p.96
`http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf`

Energy Balance Models

Insolation

Insolation = **In**coming **sol**ar radi**ation**

solar intensity at average distance from the sun: 1368 W/m^2

radius of the Earth: ρ meters
cross sectional area: $\pi\rho^2$ m^2
intercepted power: 1368 $\pi\rho^2$ Watts
surface area: $4\pi\rho^2$ m^2

average insolation: 1368/4 W/m^2 = 342 W/m^2

Energy Balance Models

References

Classic Papers:

M. I. Budyko, The effect of solar radiation variation on the climate of the Earth, *Tellus* **21** (1969), 611-619.

W. D. Sellers, A Global Climatic Model Based on the Energy Balance of the Earth-Atmosphere System, *Journal of Applied Meteorology* **8** (1969), 392-400.

Recent Interpretation:

K.K. Tung, Topics in Mathematical Modeling, Princeton University Press, 2007. (Chapter 8)

Energy Balance Models

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

T = global mean temperature (°C)
Q = mean solar input (W/m^2)
α = mean albedo
$A + BT$ = outward radiation (linear approximation)
R = heat capacity of Earth's surface

Tung's values:

T = global mean temperature (°C)
Q = 343 W/m^2
A = 202 W/m^2
B = 1.9 W/(m^2 °C)

$\alpha = \alpha_1 = 0.32$ (water and land)
$\alpha = \alpha_2 = 0.62$ (ice)

Energy Balance Models

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium temperature

$$T_{eq} = \frac{Q(1-\alpha)-A}{B}$$

ice free Earth: $\alpha = \alpha_1$, T_{eq} = 16.4 °C
snowball Earth: $\alpha = \alpha_2$, T_{eq} = -37.7 °C

According to Tung, glaciers form if $T < T_c$ = -10 °C and melt if $T > T_c$.

Since 16.4 > -10, no glacier would form on an ice free Earth.
Since -37.7 < -10, no glacier would melt on a snowball Earth.

Energy Balance Models

Inhomogeneous Earth

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T}-T\right)$$

Now the annual average surface temperature T is a function of y = sine(latitude).

The albedo α is a function of y and the location η of the ice boundary.

The outward radiation $A+BT$ is as before.

Heat transport across latitudes is assumed to be linear and is given by

$$C\left(\overline{T}-T\right)$$

where C = 3.04 W/m²/°C.

The global annual average insolation is Q, with the same value as above, while $s(y)$ is the relative insolation, normalized to satisfy

$$\int_0^1 s(y)\,dy$$

Energy Balance Models

Inhomogeneous Earth

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T}-T\right)$$

The variable y is chosen instead of the latitude, because the global annual mean temperature is given by

$$\overline{T}(t) = \int_0^1 T(y,t)\,dy$$

We assume symmetry with respect to the equator, so the variable y takes on values between 0 and 1.

We assume an ice boundary at $y = \eta$, with ice toward the pole and no ice toward the equator. The albedo is therefore

$$\alpha(y,\eta) = \begin{cases} \alpha_1, & y<\eta, \\ \alpha_2, & y>\eta. \end{cases}$$

Rate of solar energy absorption at y = sine(latitude):

$$Qs(y)\left(1-\alpha(y,\eta)\right)$$

Energy Balance Models

Inhomogeneous Earth

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T}-T\right)$$

Look for an equilibrium solution having an ice line at $y = \eta$

$$T = T_\eta^*(y)$$

This equilibrium satisfies

$$\boxed{Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT_\eta^*(y)\right) + C\left(\overline{T}_\eta^* - T_\eta^*(y)\right) = 0}$$

Next step: Solve for the equilibrium temperature profile, assuming we know the ice boundary.

Energy Balance Models

Inhomogeneous Earth

$$\boxed{Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT_\eta^*(y)\right) + C\left(\overline{T}_\eta^* - T_\eta^*(y)\right) = 0}$$

Integrate:

$$\int_0^1 \left(Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT_\eta^*(y)\right) + C\left(\overline{T}_\eta^* - T_\eta^*(y)\right)\right)dy = 0,$$

$$Q\left(1-\overline{\alpha}(\eta)\right) - A - B\overline{T}_\eta^* = 0$$

where

$$\overline{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)\,dy = \int_0^\eta \alpha_1 s(y)\,dy + \int_\eta^1 \alpha_2 s(y)\,dy$$

$$= \alpha_1 S(\eta) + \alpha_2\left(1-S(\eta)\right) = \alpha_2 - (\alpha_2-\alpha_1)S(\eta),$$

and where

$$S(\eta) = \int_0^\eta s(y)\,dy$$

Given the ice line η, the global mean temperature is

$$\boxed{\overline{T}_\eta^* = \frac{1}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right) - A\right)}$$

Energy Balance Models

Inhomogeneous Earth

Equilibrium equation (given ice line):

$$Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT_\eta^*(y)\right) + C\left(\overline{T}_\eta^* - T_\eta^*(y)\right) = 0$$

Global mean temperature:

$$\overline{T}_\eta^* = \frac{1}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right) - A\right)$$

Solve for equilibrium temperature profile:

$$\boxed{T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\overline{T}_\eta^*\right)}$$

where

$$\alpha(y,\eta) = \begin{cases} \alpha_1, & y<\eta, \\ \alpha_2, & y>\eta. \end{cases}$$

$$\overline{\alpha}(\eta) = \alpha_2 - (\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy$$

Energy Balance Models

Inhomogeneous Earth

$$T_\eta^*(y)=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right)-A+C\overline{T}_\eta^*\right)$$

Additional assumption: At equilibrium, the average temperature across the ice boundary is T_c = -10 °C

$$T_\eta^*(\eta-)=\frac{1}{B+C}\left(Qs(\eta)(1-\alpha_1)-A+C\overline{T}_\eta^*\right)$$

$$T_\eta^*(\eta+)=\frac{1}{B+C}\left(Qs(\eta)(1-\alpha_2)-A+C\overline{T}_\eta^*\right)$$

(s is continuous)

$$T_c=\frac{T_\eta^*(\eta-)+T_\eta^*(\eta+)}{2}=\frac{1}{B+C}\left(Qs(\eta)(1-\alpha_0)-A+C\overline{T}_\eta^*\right)$$

where

$$\alpha_0=\frac{\alpha_1+\alpha_2}{2}$$

Energy Balance Models

Inhomogeneous Earth

Now we can solve for the ice boundary.

$$\frac{1}{B+C}\left(Qs(y)(1-\alpha_0)-A+C\overline{T}_\eta^*\right)=T_c$$

where

$$\overline{T}_\eta^*=\frac{1}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right)-A\right)$$

Therefore,

$$\frac{1}{B+C}\left(Qs(\eta)(1-\alpha_0)-A+\frac{C}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right)-A\right)\right)=T_c$$

which reduces to

$$\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0)+\frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)\int_0^\eta s(y)dy\right)\right)-\frac{A}{B}-T_c=0$$

which can be solved numerically for η .

Energy Balance Models

Inhomogeneous Earth

What about $s(y)$, the relative insolation function?

$$s(y)=\frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\gamma-y\cos\beta\right)^2}\,d\gamma$$

where β = obliquity. (Current value is about 23.5°.)

Tung and North's quadratic approximation:

$$s(y)\approx 1-0.241\left(3y^2-1\right)$$

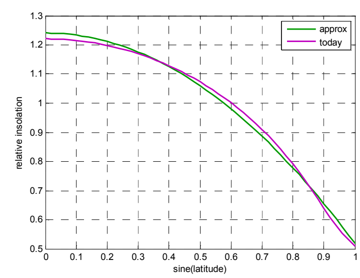
Energy Balance Models

Inhomogeneous Earth

green = quadratic approximation (Tung and North)

mauve = formula using obliquity of 23.5°

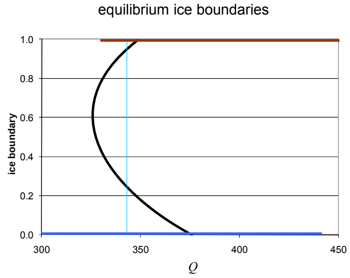
Relative Insolation Function



Energy Balance Models

Inhomogeneous Earth

equilibrium ice boundaries



Energy Balance Models

Inhomogeneous Earth

Tung conclusions:

four equilibrium solutions:

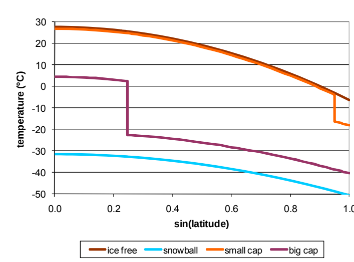
snowball: stable

large cap: unstable

small cap: stable

ice free: ***stable***

equilibrium temperature profiles



Energy Balance Models

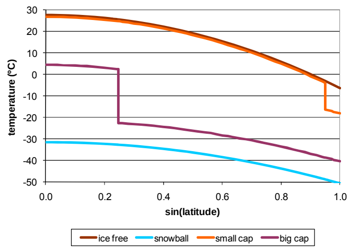
Inhomogeneous Earth

Widiasih conclusions:

four equilibrium solutions:

snowball: stable
large cap: unstable
small cap: stable
ice free: ***unstable***

equilibrium temperature profiles



Energy Balance Models

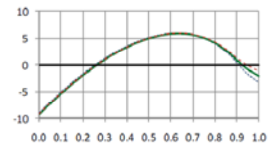
Widiasih Result

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T}-T\right)$$

$$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$$

This infinite dimension system has a one dimensional attracting invariant manifold. On the manifold, the system reduces to

$$\frac{d\eta}{dt} = h(\eta)$$



Energy Balance Models

Paleoclimate

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T}-T\right)$$

We can use the information from the Milankovitch cycles as input to the energy balance model.

Q is determined by eccentricity.
$s(y)$ is determined by obliquity.

We can solve for the ice line as a function of eccentricity and obliquity.

The result correctly predicts that the dominate signal comes from the obliquity.

Not correctly predicted: Amplitude of glacial cycles during the last million years.

Energy Balance Models

Inhomogeneous Earth

Greenhouse Effect

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - \underbrace{(A+BT)} + C\left(\overline{T}-T\right)$$

re-radiation term
(includes greenhouse effect)

Current efforts: Try to incorporate atmospheric CO_2 into the model.

Next week ...