A Finite Dimensional Model of Ice-Albedo Feedback

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Ice-Albedo Feedback

Budyko-Sellers Model

$$R\frac{\partial T}{\partial t} = \underbrace{Qs(y)}_{\text{insolation}}\underbrace{\left(1-\alpha(y,\eta)\right)}_{\text{albedo}} - \underbrace{(A+BT)}_{\text{re-radiation}} + \underbrace{C\left(\bar{T}-T\right)}_{\text{transport}}$$

$T = T(y,t)$: annual mean surface temperature

$y = \sin(\text{latitude})$ $\quad y \in [0,1]$

Q: global annual mean insolation

$s(t)$: relative annual mean insolation $\quad \int_0^1 s(y)\,dy = 1$

$y = \eta$: ice boundary

$\alpha(y,\eta) = \begin{cases} \alpha_1, & y < \eta, \\ \alpha_2, & y > \eta. \end{cases}$ albedo

$\bar{T}(t) = \int_0^1 T(y,t)\,dy$: global annual mean temperature

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$Q = 343\ \text{W/m}^2 \qquad A = 202\ \text{W/m}^2$

$B = 1.9\ \text{W/m}^2/{}^\circ\text{C} \quad C = 3.04\ \text{W/m}^2/{}^\circ\text{C}$

$\alpha_1 = 0.32$: (water and land)

$\alpha_2 = 0.62$: (ice)

$y =$ proportion of Earth's surface between latitude $-\arcsin(y)$ and $+\arcsin(y)$,

$\eta =$ proportion of Earth's surface that is ice-free.

If $\sigma =$ surface area of Earth ($\approx 5.1\times 10^{14}\ \text{m}^2$), then

$\sigma(1-\eta) =$ surface area of ice in square meters.

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Heat Capacity

$R =$ heat capacity of Earth's surface (J/m^2/°C).

Heat capacity of water ≈ 4 J/g/°C.

Mass of 1 cm^3 of $H_2O \approx 1$ g.

Assumption: Earth's surface consists of 100 m of H_2O.

Each square meter of Earth's surface consists of

100 m^3 = 10^8 cm^3 of H_2O, or about 10^8 g, so

$\boxed{R \approx 4\times 10^8\ \text{J/m}^2/{}^\circ\text{C}}$.

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Budyko-Sellers Model

$$R\frac{\partial T}{\partial t} = \underbrace{Qs(y)}_{\text{insolation}}\underbrace{\left(1-\alpha(y,\eta)\right)}_{\text{albedo}} - \underbrace{(A+BT)}_{\text{re-radiation}} + \underbrace{C\left(\bar{T}-T\right)}_{\text{transport}}$$

Time Scale

A Watt is a Joule per **second**, but all quantities are **annual** averages.
One year is approximately κ = 3.16x10^7 seconds.
If we measure time in years instead of seconds, the equation becomes

$$\boxed{\frac{\partial T}{\partial t} = \frac{\kappa}{R}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\bar{T}-T\right)\right)}$$

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Budyko-Sellers-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon\kappa\left(T_b - T_c\right)$$

$$\frac{\partial T}{\partial t} = \frac{\kappa}{R}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\bar{T}-T\right)\right)$$

$$T_b = \frac{T(\eta-)+T(\eta+)}{2}$$

$$\bar{T} = \bar{T}(t) = \int_0^1 T(y,t)\,dy$$

What about ε ?

The parameter ε should be determined from climate data.
(good project!)

Best current guess:

$\varepsilon \approx 10^{-12}$

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Budyko-Sellers-Widiasih Model

Heat of Fusion

$$\frac{d\eta}{dt} = \varepsilon\kappa\left(T_b - T_c\right)$$

To move the ice line, we must melt or freeze water.
Heat of fusion of water: 334 J/g , or 3.34×10^8 J/m^3

Assumption: Average thickness of ice is 450 m.
Energy to melt 1 m^2 of ice: Ω = 1.5×10^{11} Joules
Energy to move ice line from η to $\eta + \Delta\eta$:

$\Omega\sigma\Delta\eta$ Joules, or $\Omega\Delta\eta$ J/m^2
(σ = surface area of Earth.)

Energy needed to move the ice line:

$$\Omega\frac{d\eta}{dt} \text{ J/yr/m}^2$$

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Budyko-Sellers-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon\kappa\left(T_b - T_c\right)$$

$$\frac{\partial T}{\partial t} = \frac{\kappa}{R}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T} - T\right) - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

State Space

η lives in [0,1].
T lives in a space of functions on [0,1].

What space of functions?

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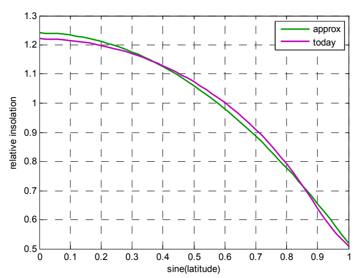
green = quadratic approximation (Tung and North)

mauve = formula using obliquity of 23.5°

Relative Insolation Function



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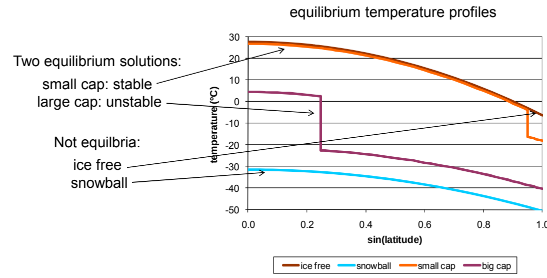
If s is assumed to be quadratic, then the equilibrium solutions are piecewise quadratic with a discontinuity at the ice boundary.



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Budyko-Sellers-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon\kappa\left(T_b - T_c\right)$$

$$\frac{\partial T}{\partial t} = \frac{\kappa}{R}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T} - T\right) - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

State Space

What space of functions?
Assume that T lives in the space of functions on [0,1] which are piecewise quadratic with a discontinuity at the ice boundary.

$$T(y) = \begin{cases} U(y), & y < \eta, \\ V(y), & y > \eta, \\ \left(U(\eta) + V(\eta)\right)/2, & y = \eta, \end{cases}$$

where U and V are quadratic on [0,1] .

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Budyko-Sellers-Widiasih Model

Legendre Polynomials

Since s is even, we assume that U and V are even and expand using the first two even Legendre polynomials:

$$p_0(y) = 1$$

$$p_2(y) = \frac{1}{2}\left(3y^2 - 1\right)$$

$$U(y,t) = u_0(t)p_0(y) + u_2(t)p_2(y) = u_0(t) + u_2(t)p_2(y),$$
$$V(y,t) = v_0(t)p_0(y) + v_2(t)p_2(y) = v_0(t) + v_2(t)p_2(y),$$
$$s(y) = s_0 p_0(y) + s_2 p_2(y) = 1 + s_2 p_2(y).$$

$$\frac{\partial U}{\partial t} = \dot{u}_0 + \dot{u}_2 p_2(y),$$
$$\frac{\partial V}{\partial t} = \dot{v}_0 + \dot{v}_2 p_2(y).$$

$$\left(\dot{u} = \frac{du}{dt}\right)$$

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Five Dimensional Budyko-Sellers-Widiasih Model

Summary

$$\dot{\eta} = \varepsilon\kappa\left(T_b - T_c\right)$$

$$\dot{u}_0 = \frac{\kappa}{R}\left(Q\left(1-\alpha_1\right) - A + C\overline{T} - \left(B+C\right)u_0 - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

$$\dot{v}_0 = \frac{\kappa}{R}\left(Q\left(1-\alpha_2\right) - A + C\overline{T} - \left(B+C\right)v_0 - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

$$\dot{u}_2 = \frac{\kappa}{R}\left(Qs_2\left(1-\alpha_1\right) - \left(B+C\right)u_2\right)$$

$$\dot{v}_2 = \frac{\kappa}{R}\left(Qs_2\left(1-\alpha_2\right) - \left(B+C\right)v_2\right)$$

$$T_b = \frac{u_0+v_0}{2} + \frac{u_2+v_2}{2}p_2\left(\eta\right)$$

$$\overline{T} = \eta u_0 + \left(1-\eta\right)v_0 + P_2\left(\eta\right)\left(u_2 - v_2\right)$$

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Five Dimensional Budyko-Sellers-Widiasih Model

Quadratic Modes

$$\dot{u}_2 = \frac{\kappa}{R}\left(Qs_2\left(1-\alpha_1\right) - \left(B+C\right)u_2\right)$$

$$\dot{v}_2 = \frac{\kappa}{R}\left(Qs_2\left(1-\alpha_2\right) - \left(B+C\right)v_2\right)$$

The dynamics of u_2 and v_2 are independent of each other and of the other variables (including η).
These variables decay exponentially to

$$u_2^* = \frac{Qs_2\left(1-\alpha_1\right)}{\left(B+C\right)} \qquad v_2^* = \frac{Qs_2\left(1-\alpha_2\right)}{\left(B+C\right)}$$

Exercise: All higher order modes behave the same way.

Ice-Albedo Feedback

Three Dimensional Budyko-Sellers-Widiasih Model

Globally Attracting 3D Subspace
Summary

$$\dot{\eta} = \varepsilon\kappa\left(T_b - T_c\right)$$

$$\dot{u}_0 = \frac{\kappa}{R}\left(Q\left(1-\alpha_1\right) - A + C\overline{T} - \left(B+C\right)u_0 - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

$$\dot{v}_0 = \frac{\kappa}{R}\left(Q\left(1-\alpha_2\right) - A + C\overline{T} - \left(B+C\right)v_0 - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

Ice Boundary Temperature

$$T_b = \frac{u_0+v_0}{2} + p_2\left(\eta\right)\frac{Qs_2\left(1-\alpha_0\right)}{B+C}$$

Global Mean Temperature

$$\overline{T} = \eta u_0 + \left(1-\eta\right)v_0 + P_2\left(\eta\right)\frac{Qs_2\left(\alpha_2-\alpha_1\right)}{B+C}$$

Ice-Albedo Feedback

Three Dimensional Budyko-Sellers-Widiasih Model

$$\dot{u}_0 = \frac{\kappa}{R}\left(Q\left(1-\alpha_1\right) - A + C\overline{T} - \left(B+C\right)u_0 - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

$$\dot{v}_0 = \frac{\kappa}{R}\left(Q\left(1-\alpha_2\right) - A + C\overline{T} - \left(B+C\right)v_0 - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

Further Reduction

$$w = \frac{u_0+v_0}{2}, \qquad z = u_0 - v_0$$

$$\dot{w} = \frac{\dot{u}_0+\dot{v}_0}{2} = \frac{\kappa}{R}\left(Q\left(1-\alpha_0\right) - A + C\overline{T} - \left(B+C\right)w - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

$$\dot{z} = \dot{u}_0 - \dot{v}_0 = \frac{\kappa}{R}\left(Q\left(\alpha_2-\alpha_1\right) - \left(B+C\right)z\right)$$

Ice-Albedo Feedback

Three Dimensional Budyko-Sellers-Widiasih Model

$$\dot{\eta} = \varepsilon\kappa\left(T_b - T_c\right)$$

$$\dot{w} = \frac{\kappa}{R}\left(Q\left(1-\alpha_0\right) - A + C\overline{T} - \left(B+C\right)w - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

$$\dot{z} = \frac{\kappa}{R}\left(Q\left(\alpha_2-\alpha_1\right) - \left(B+C\right)z\right)$$

Note that z converges exponentially to its equilibrium value

$$z^* = \frac{Q\left(\alpha_2-\alpha_1\right)}{B+C}$$

Attracting invariant 2D subspace

$$\boxed{\begin{aligned}\dot{\eta} &= \varepsilon\kappa\left(T_b - T_c\right)\\ \dot{w} &= \frac{\kappa}{R}\left(Q\left(1-\alpha_0\right) - A + C\overline{T} - \left(B+C\right)w - \varepsilon\Omega\left(T_b - T_c\right)\right)\end{aligned}}$$

Ice-Albedo Feedback

Two Dimensional Budyko-Sellers-Widiasih Model

Ice Boundary Temperature

$$T_b = \frac{u_0+v_0}{2} + p_2\left(\eta\right)\frac{Qs_2\left(1-\alpha_0\right)}{B+C}$$

Recall

$$w = \frac{u_0+v_0}{2}, \qquad z = u_0 - v_0$$

$$T_b = T_b\left(\eta, w\right) = w + p_2\left(\eta\right)\frac{Qs_2\left(1-\alpha_0\right)}{B+C}$$

$$\boxed{T_b\left(\eta, w\right) = w + \frac{Q\left(1-\alpha_0\right)}{B+C}s_2p_2\left(\eta\right)}$$

Ice-Albedo Feedback

Two Dimensional Budyko-Sellers-Widiasih Model

Global Mean Temperature

$$\bar{T} = \eta u_0 + (1-\eta)v_0 + P_2(\eta)\frac{Qs_2(\alpha_2-\alpha_1)}{B+C}$$

Recall

$$w=\frac{u_0+v_0}{2},\quad z=u_0-v_0,\quad u_0=w+\frac{z}{2},\quad v_0=w-\frac{z}{2}$$

$$\bar{T} = w-\frac{z}{2}+\eta z + P_2(\eta)\frac{Qs_2(\alpha_2-\alpha_1)}{B+C}$$

On invariant 2D subspace: $z=z^*=\dfrac{Q(\alpha_2-\alpha_1)}{B+C}$

$$\begin{aligned}\bar{T}=\bar{T}(w,\eta)&=w+\left(\eta-\frac{1}{2}\right)\frac{Q(\alpha_2-\alpha_1)}{B+C}+P_2(\eta)\frac{Qs_2(\alpha_2-\alpha_1)}{B+C}\\ &=w+\frac{Q(\alpha_2-\alpha_1)}{B+C}\left(\eta-\frac{1}{2}+s_2P_2(\eta)\right)\end{aligned}$$

Ice-Albedo Feedback

Two Dimensional Budyko-Sellers-Widiasih Model

$$\dot{\eta}=\varepsilon\kappa\left(T_b(\eta,w)-T_c\right)$$

$$\dot{w}=\frac{\kappa}{R}\left(Q(1-\alpha_0)-A+C\bar{T}(\eta,w)-(B+C)w-\varepsilon\Omega\left(T_b(\eta,w)-T_c\right)\right)$$

where

$$T_b(\eta,w)=w+\frac{Q(1-\alpha_0)}{B+C}s_2p_2(\eta)$$

$$\bar{T}(w,\eta)=w+\frac{Q(\alpha_2-\alpha_1)}{B+C}\left(\eta-\frac{1}{2}+s_2P_2(\eta)\right)$$

$$p_2(\eta)=\frac{1}{2}\left(3\eta^2-1\right)$$

$$P_2(\eta)=\frac{1}{2}\left(\eta^3-\eta\right)$$

Ice-Albedo Feedback

Two Dimensional Budyko-Sellers-Widiasih Model

$$\dot{\eta}=\varepsilon\kappa\left(T_b(\eta,w)-T_c\right)$$

$$\dot{w}=\frac{\kappa}{R}\left(Q(1-\alpha_0)-A+C\bar{T}(\eta,w)-(B+C)w-\varepsilon\Omega\left(T_b(\eta,w)-T_c\right)\right)$$

First, let $\varepsilon=0$ and recall that

$$\bar{T}(w,\eta)=w+\frac{Q(\alpha_2-\alpha_1)}{B+C}\left(\eta-\frac{1}{2}+s_2P_2(\eta)\right)$$

$$\begin{aligned}\dot{w}&=\frac{\kappa}{R}\left(Q(1-\alpha_0)-A+C\bar{T}(\eta,w)-(B+C)w\right)\\ &=\frac{\kappa}{R}\left(Q(1-\alpha_0)-A-Bw+\frac{CQ(\alpha_2-\alpha_1)}{B+C}\left(\eta-\frac{1}{2}+s_2P_2(\eta)\right)\right)\\ &=\frac{\kappa B}{R}\left(\Phi_0(\eta)-w\right)\end{aligned}$$

where

$$\Phi_0(\eta)=\frac{1}{B}\left(Q(1-\alpha_0)-A+\frac{CQ(\alpha_2-\alpha_1)}{B+C}\left(\eta-\frac{1}{2}+s_2P_2(\eta)\right)\right)$$

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Two Dimensional Budyko-Sellers-Widiasih Model

For $\varepsilon=0$, the system becomes

$$\dot{\eta}=0$$

$$\dot{w}=\frac{\kappa B}{R}\left(\Phi_0(\eta)-w\right)$$

This system has a curve of fixed point given by

$$w=\Phi_0(\eta)=\frac{1}{B}\left(Q(1-\alpha_0)-A+\frac{CQ(\alpha_2-\alpha_1)}{B+C}\left(\eta-\frac{1}{2}+s_2P_2(\eta)\right)\right)$$

This invariant curve is globally exponentially attracting with exponent

$$-\frac{\kappa B}{R}$$

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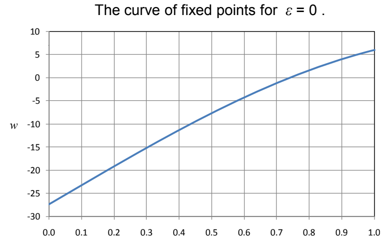
Two Dimensional Budyko-Sellers-Widiasih Model

$$\dot{\eta}=0$$

$$\dot{w}=\frac{\kappa B}{R}\left(\Phi_0(\eta)-w\right)$$

The curve of fixed points for $\varepsilon=0$.



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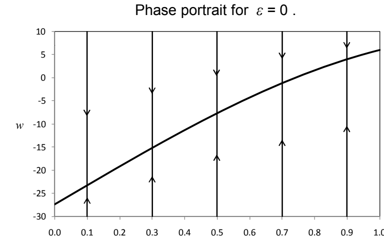
Two Dimensional Budyko-Sellers-Widiasih Model

$$\dot{\eta}=0$$

$$\dot{w}=\frac{\kappa B}{R}\left(\Phi_0(\eta)-w\right)$$

Phase portrait for $\varepsilon=0$.



Ice-Albedo Feedback

Two Dimensional Budyko-Sellers-Widiasih Model

$$\dot{\eta} = \varepsilon\kappa\left(w + \frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) - T_c\right)$$

$$\dot{w} = \frac{\kappa}{R}\left(B\Phi_0(\eta) - Bw - \varepsilon\Omega\left(w + \frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) - T_c\right)\right)$$

If we know η and w, what is the temperature profile $T(y)$?

Recall that $T(y) = \begin{cases} U(y), & y < \eta, \\ V(y), & y > \eta, \end{cases}$ where $U(y) = u_0 + u_2 p_2(y),\quad V(y) = v_0 + v_2 p_2(y).$

Recall also that

$$u_0 = w + \frac{z}{2}, \quad v_0 = w - \frac{z}{2}$$

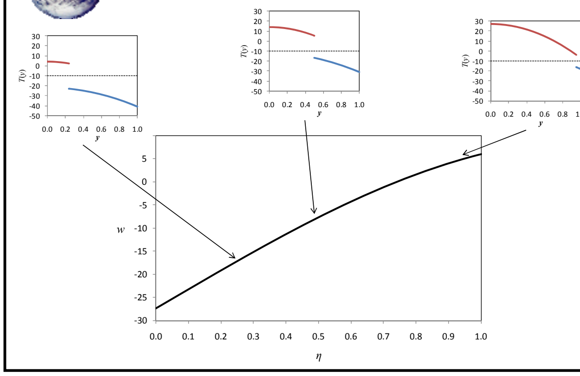
On the invariant 2-space

$$z = z^* = \frac{Q(\alpha_2 - \alpha_1)}{B+C}, \quad u_2 = u_2^* = \frac{Qs_2(1-\alpha_1)}{(B+C)}, \quad v_2 = v_2^* = \frac{Qs_2(1-\alpha_2)}{(B+C)}$$

Given η and w, we can compute the temperature profile $T(y)$.

Ice-Albedo Feedback

Two Dimensional Budyko-Sellers-Widiasih Model



Ice-Albedo Feedback

Two Dimensional Budyko-Sellers-Widiasih Model

$$\dot{\eta} = \varepsilon\kappa\left(w + \frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) - T_c\right)$$

$$\dot{w} = \frac{\kappa}{R}\left(B\Phi_0(\eta) - Bw - \varepsilon\Omega\left(w + \frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) - T_c\right)\right)$$

What about $\varepsilon > 0$?

Rest points:

$$w = -\frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) + T_c$$
$$w = \Phi_0(\eta)$$

Solve for η:

$$h(\eta) = \Phi_0(\eta) + \frac{Qs_2(1-\alpha_0)}{B+C} - T_c = 0$$

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Two Dimensional Budyko-Sellers-Widiasih Model

$$\dot{\eta} = \varepsilon\kappa\left(w + \frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) - T_c\right)$$

$$\dot{w} = \frac{\kappa}{R}\left(B\Phi_0(\eta) - Bw - \varepsilon\Omega\left(w + \frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) - T_c\right)\right)$$

Note that rest points occur on the curve:

$$w = \Phi_0(\eta)$$

and that near that curve,

$$\frac{d\eta}{dt} \approx \varepsilon\kappa h(\eta)$$

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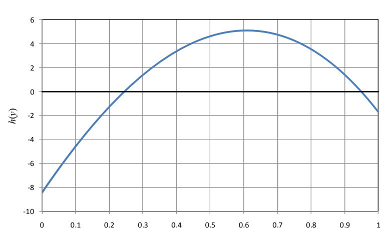
Two Dimensional Budyko-Sellers-Widiasih Model

$$\frac{d\eta}{dt} \approx \varepsilon\kappa h(\eta)$$

The function h.



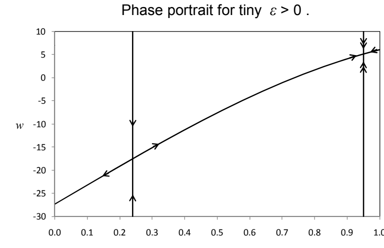
Ice-Albedo Feedback

Two Dimensional Budyko-Sellers-Widiasih Model

$$\dot{\eta} = \varepsilon\kappa\left(T_b(\eta,w) - T_c\right)$$

$$\dot{w} = \frac{\kappa}{R}\left(Q(1-\alpha_0) - A + C\overline{T}(\eta,w) - (B+C)w - \varepsilon\Omega\left(T_b(\eta,w) - T_c\right)\right)$$

Phase portrait for tiny $\varepsilon > 0$.



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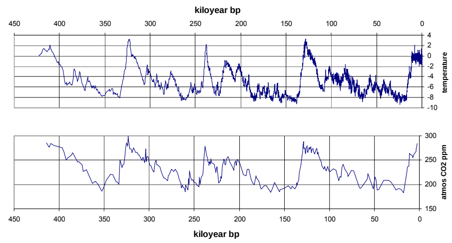
Two Dimensional Budyko-Sellers-Widiasih Model

$$\dot{\eta} = \varepsilon\kappa\left(T_b\left(\eta, w\right) - T_c\right)$$

$$\dot{w} = \frac{\kappa}{R}\left(Q\left(1-\alpha_0\right) - A + C\overline{T}\left(\eta, w\right) - \left(B + C\right)w - \varepsilon\Omega\left(T_b\left(\eta, w\right) - T_c\right)\right)$$

Can the parameters be estimated from the ice core data?



Petit, et al, *Nature* **399** (June 3 1999), pp.429-436