Energy Balance Models, II

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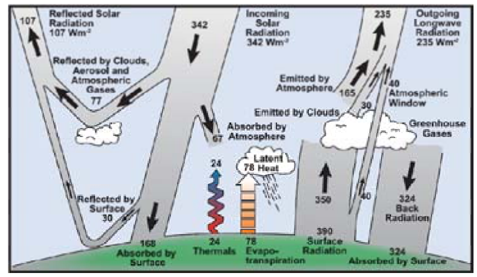
Energy Balance Models

Earth's Energy Balance



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

Energy Balance Models

Budyko-Sellers Model

$$R\frac{\partial T}{\partial t} = \underbrace{Qs(y)}_{\text{insolation}}\underbrace{(1-\alpha(y))}_{\text{albedo}} - \underbrace{(A+BT)}_{\text{re-radiation}} + \underbrace{C(\bar{T}-T)}_{\text{transport}}$$

$T = T(y,t) =$ annual mean surface temperature

$y = \sin(\text{latitude}) \qquad y \in [0,1]$

$Q =$ global annual mean insolation $= 343 \text{ W/m}^2$

$s(y) =$ relative annual mean insolation, $\int_0^1 s(y)\,dy = 1$

$\bar{T}(t) = \int_0^1 T(y,t)\,dy =$ mean annual global temperature

$\alpha(y) =$ surface albedo

$A = 202 \text{ W/m}^2 \quad B = 1.9 \text{ W/m}^2/{}^\circ\text{C} \quad C = 3.04 \text{ W/m}^2/{}^\circ\text{C}$

Energy Balance Models

Budyko-Sellers Model

What about $s(y)$, the relative insolation function?

$$s(y) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\gamma - y\cos\beta\right)^2}\,d\gamma$$

where β = obliquity. (Current value is about 23.5°.)

Tung and North's quadratic approximation:

$$s(y) \approx 1 - 0.241(3y^2-1)$$

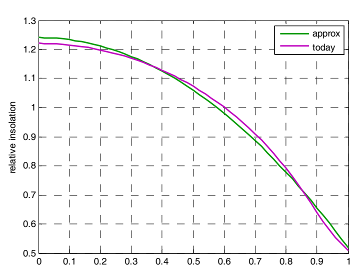
Energy Balance Models

Budyko-Sellers Model

green = quadratic approximation (Tung and North)

mauve = formula using obliquity of 23.5°

Relative Insolation Function



Energy Balance Models

Budyko-Sellers Model

What about $\alpha(y)$, the albedo function?

$\alpha = \alpha_1 = 0.32$ (water and land)
$\alpha = \alpha_2 = 0.62$ (ice)

Suppose that there is a single ice boundary at latitude $\arcsin(\eta)$.

$$\alpha(y) = \alpha(y,\eta) = \begin{cases}\alpha_1, & y < \eta,\\ \alpha_2, & y > \eta,\end{cases}$$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\bar{T}-T)$$

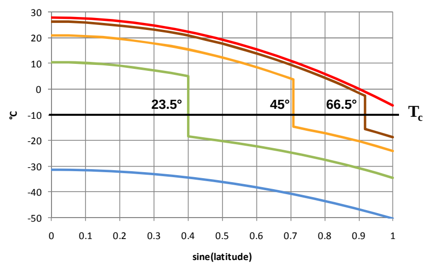
Energy Balance Models

Budyko-Sellers Model

$$R\frac{\partial T}{\partial t}=Qs(y)\left(1-\alpha(y,\eta)\right)-(A+BT)+C\left(\overline{T}-T\right)$$

Lots of equilibrium solutions

The previous argument seems to imply that these are all stable.



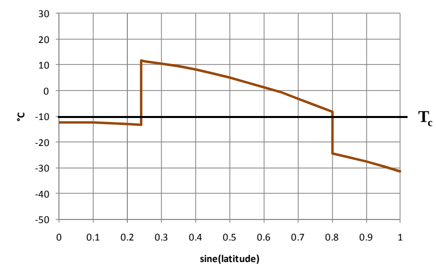
Energy Balance Models

Budyko-Sellers Model

$$R\frac{\partial T}{\partial t}=Qs(y)\left(1-\alpha(y,\eta)\right)-(A+BT)+C\left(\overline{T}-T\right)$$

Ridiculous solution

The standard argument seems to imply that these are all stable.



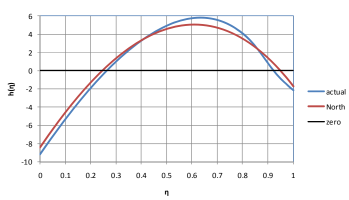
Energy Balance Models

Budyko-Sellers Model

$$R\frac{\partial T}{\partial t}=Qs(y)\left(1-\alpha(y,\eta)\right)-(A+BT)+C\left(\overline{T}-T\right)$$

Additional assumption: At equilibrium, the average temperature across the ice boundary is T_c = -10 °C

$$h(\eta)\equiv\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0)+\frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy\right)\right)-\frac{A}{B}-T_c=0$$



Energy Balance Models

Budyko-Sellers Model

$$R\frac{\partial T}{\partial t}=Qs(y)\left(1-\alpha(y,\eta)\right)-(A+BT)+C\left(\overline{T}-T\right)$$

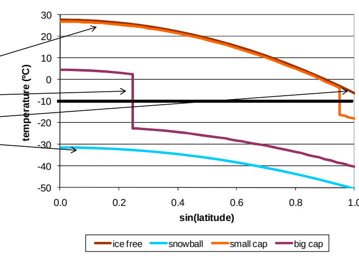
Equilibrium solutions:

small cap

large cap

ice free

snowball



Energy Balance Models

Budyko-Sellers Model

$$R\frac{\partial T}{\partial t}=Qs(y)\left(1-\alpha(y,\eta)\right)-(A+BT)+C\left(\overline{T}-T\right)$$

What about stability?

Is there a dynamical process that picks out the appropriate ice line?

Widiasih's Ice Line Dynamics

$$\frac{d\eta}{dt}=\varepsilon\left(T(\eta)-T_c\right)$$

$$R\frac{\partial T}{\partial t}=Qs(y)\left(1-\alpha(y,\eta)\right)-(A+BT)+C\left(\overline{T}-T\right)$$

State space: $[0,1]\times E$

$$T(\eta)=\frac{1}{2}\left(T(\eta-)+T(\eta+)\right)$$

Energy Balance Models

Budyko-Widiasih Model

$$\frac{d\eta}{dt}=\varepsilon\left(T(\eta)-T_c\right)$$

$$R\frac{\partial T}{\partial t}=Qs(y)\left(1-\alpha(y,\eta)\right)-(A+BT)+C\left(\overline{T}-T\right)$$

Widiasih's Theorem: For an appropriate function space E and for sufficiently small ε, the system has an attracting invariant curve. On the curve, the system is approximated by

$$\boxed{\frac{\partial\eta}{\partial t}=\varepsilon h(\eta)}$$

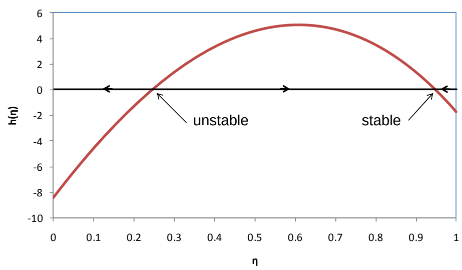
Recall:

$$h(\eta)\equiv\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0)+\frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy\right)\right)-\frac{A}{B}-T_c=0$$

Energy Balance Models

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$



Energy Balance Models

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$$

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T} - T\right)$$

Idea: First the temperature approaches its equilibrium, then the ice line adjusts.

Recall from last time: an equilibrium solution:

$$T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\overline{T}_\eta^*\right),$$

where

$$\overline{T}_\eta^* = \frac{1}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right) - A\right), \quad \overline{\alpha}(\eta) = \alpha_2 - (\alpha_2 - \alpha_1)\int_0^\eta s(y)\,dy.$$

$$T_\eta^*(\eta) = \frac{1}{2}\left(T_\eta^*(\eta-) + T_\eta^*(\eta+)\right) = \frac{1}{B+C}\left(Qs(y)(1-\alpha_0) - A + C\overline{T}_\eta^*\right)$$

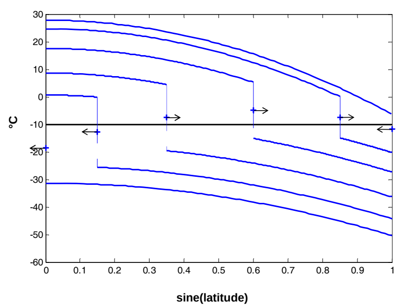
so

$$\boxed{\frac{d\eta}{dt} \approx \varepsilon\left(T_\eta^*(\eta) - T_c\right) = \varepsilon h(\eta)}$$

Energy Balance Models

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$



Energy Balance Models

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta) = \varepsilon\left(\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1 - \alpha_2 + (\alpha_2 - \alpha_1)\int_0^\eta s(y)\,dy\right)\right) - \frac{A}{B} - T_c\right)$$

What about the greenhouse effect?

$A+BT$ is the outgoing long wave radiation. This term decreases if the greenhouse gases increase.

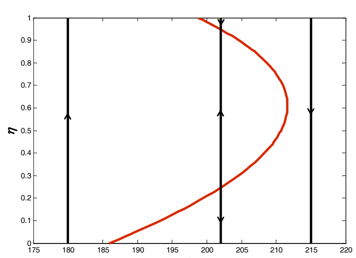
We view A as a parameter.

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

Energy Balance Models

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$



Energy Balance Models

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

What if A is a dynamical variable?

Simple equation:

$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

$$0 < \tilde{\varepsilon} \ll \varepsilon \ll 1$$

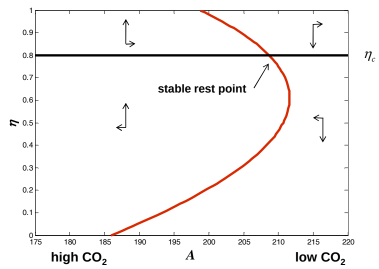
New system:

$$\boxed{\begin{aligned} \frac{dA}{dt} &= \tilde{\varepsilon}(\eta - \eta_c) \\ \frac{d\eta}{dt} &= \varepsilon h(\eta, A) \end{aligned}}$$

Energy Balance Models

Augmented Budyko-Widiasih Model

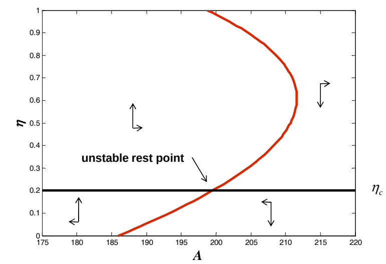
$$\frac{d\eta}{dt} = \varepsilon h(\eta, A), \quad \frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$



Energy Balance Models

Augmented Budyko-Widiasih Model

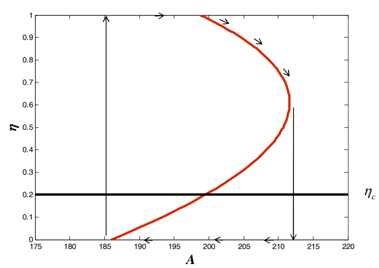
$$\frac{d\eta}{dt} = \varepsilon h(\eta, A), \quad \frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$



Energy Balance Models

Augmented Budyko-Widiasih Model

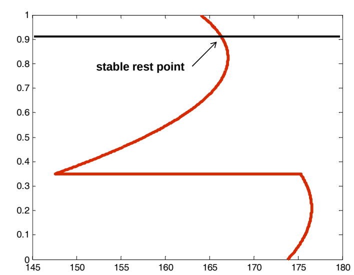
Snowball – Hothouse Oscillations



Energy Balance Models

What about the Jormungand model?

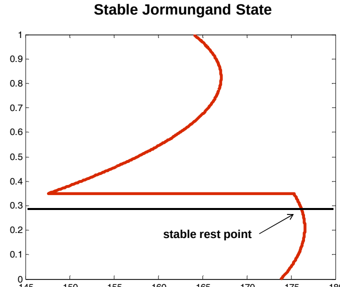
Stable Small Icecap



Energy Balance Models

What about the Jormungand model?

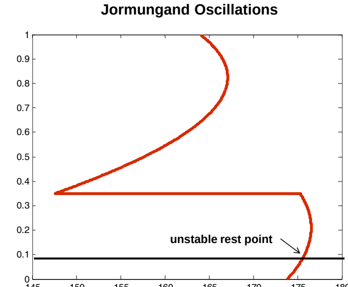
Stable Jormungand State



Energy Balance Models

What about the Jormungand model?

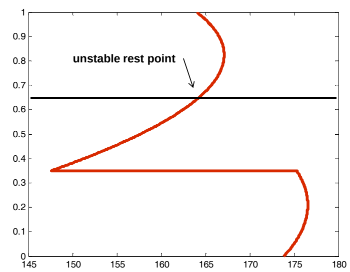
Jormungand Oscillations



Energy Balance Models

What about the Jormungand model?

Big Ice Ages



Energy Balance Models

Lots to Ponder