UNIVERSITY OF MINNESOTA

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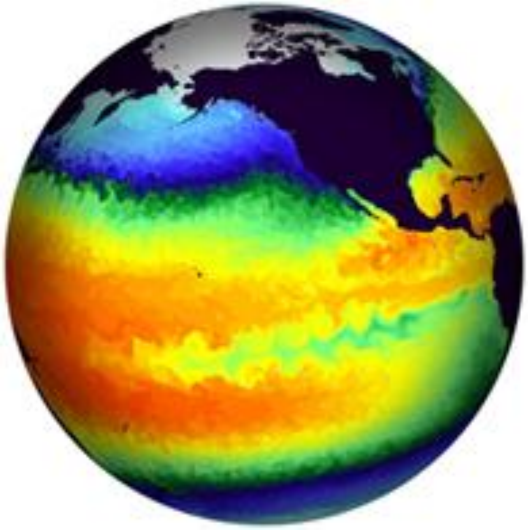
Studying fast/slow dynamics of Shallow-Water Equations and its use for heterogeneous computing

PaleoCarbon Webinar

Samantha Oestreicher

October 4, 2011



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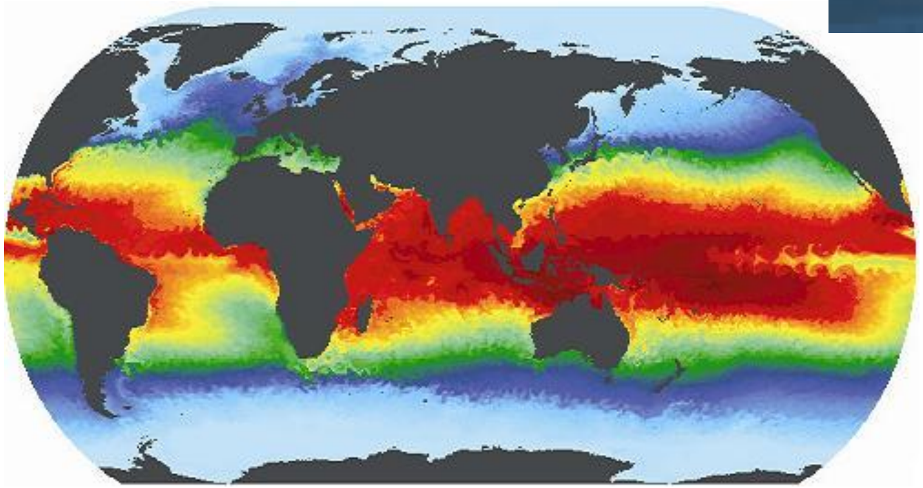
Introduction

World's oceans exhibit a variety of dynamics that vary in both time and space.

These equations have different natural regimes, or limits.

The inherent structure of these limits may allow the equations to be split into their fast/slow dynamics to improve computational integration.



Motivation

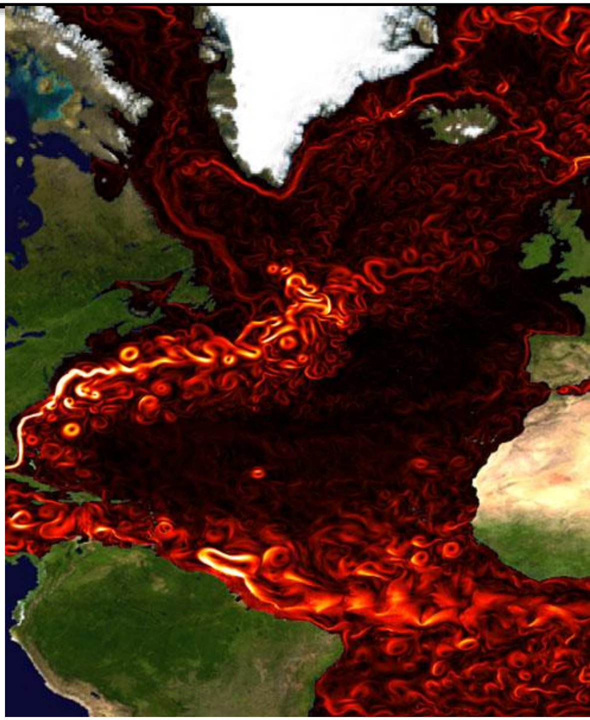
Global climate models (GCMs) make thousands of nonlinear computations for each time step across the ocean.

Scientists improve ocean computation speed with the assumption that the ocean is more active horizontally than vertically.

2-dimensional ocean dynamics equations are used for the horizontal motions.

We will be considering the Parallel Ocean Program (POP) that is developed at Los Alamos National Laboratory (LANL).

Motivation

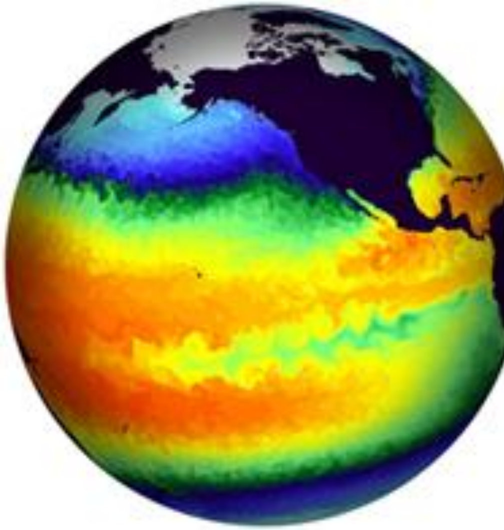


The ocean equations have intrinsic fast/slow dynamics.

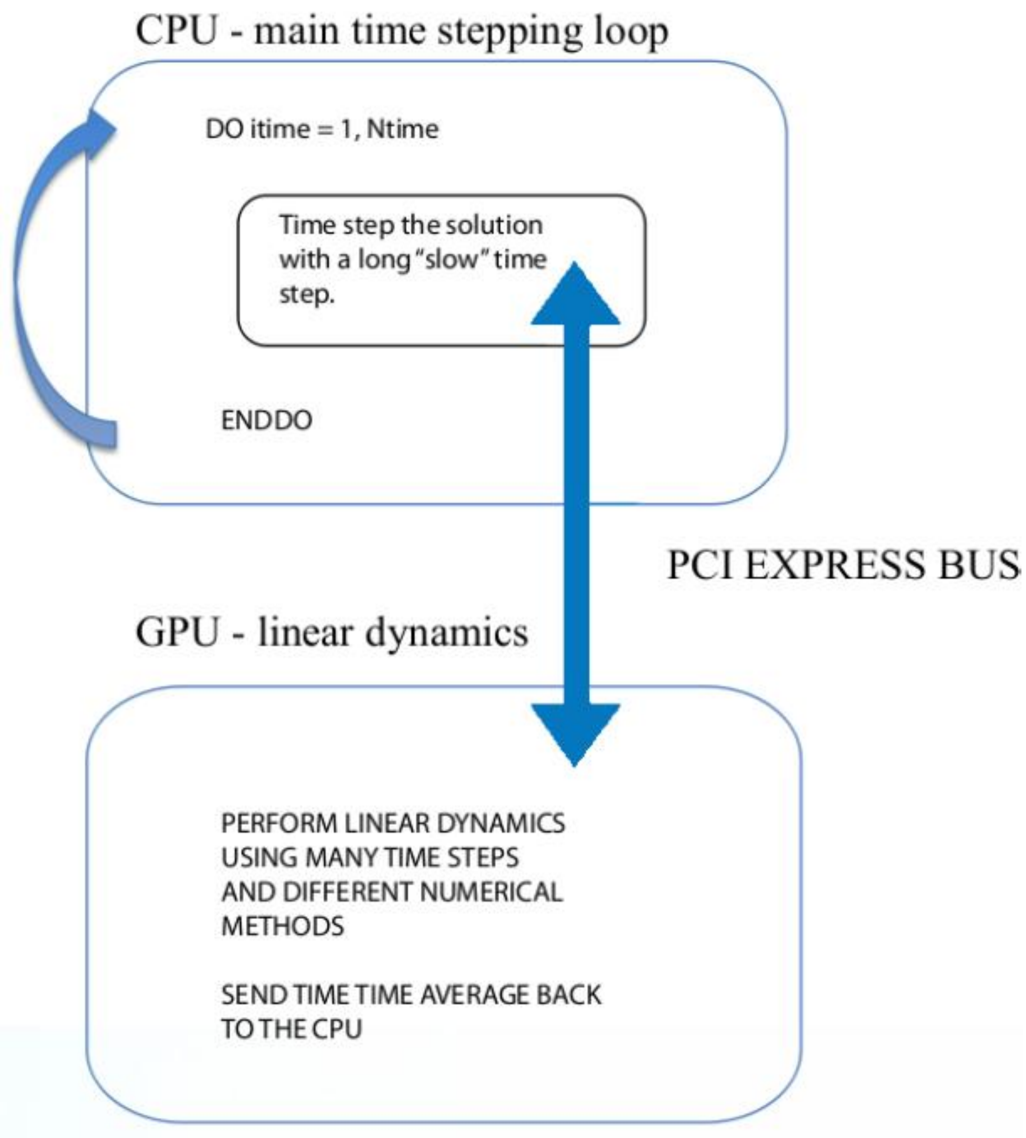
The goal is to improve the 2D computational times by capitalizing on the fast/slow dyamics.

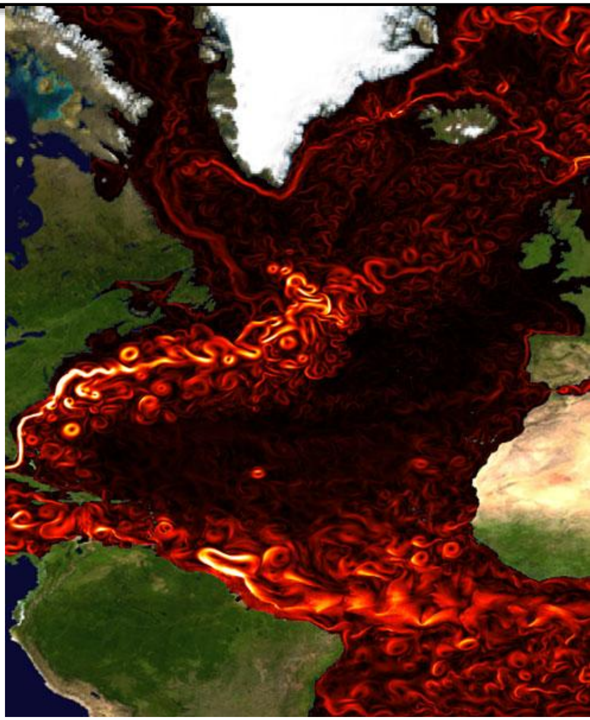
If the fast dynamics can be easily split from the slow, then they may be integrated on a separate processor for faster integration times.

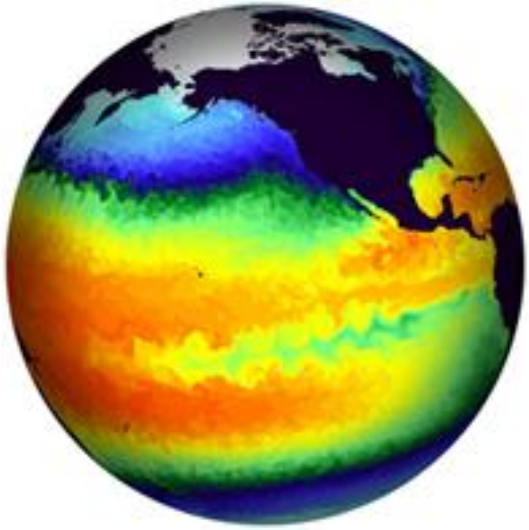
Initial research shows that the fast dynamics can be decoupled from the slow and furthermore the fast dynamics are **always linear**.



Motivation


CPU - main time stepping loop
DO itime = 1, Ntime
Time step the solution with a long "slow" time step.
ENDDO
PCI EXPRESS BUS
GPU - linear dynamics
PERFORM LINEAR DYNAMICS USING MANY TIME STEPS AND DIFFERENT NUMERICAL METHODS
SEND TIME TIME AVERAGE BACK TO THE CPU





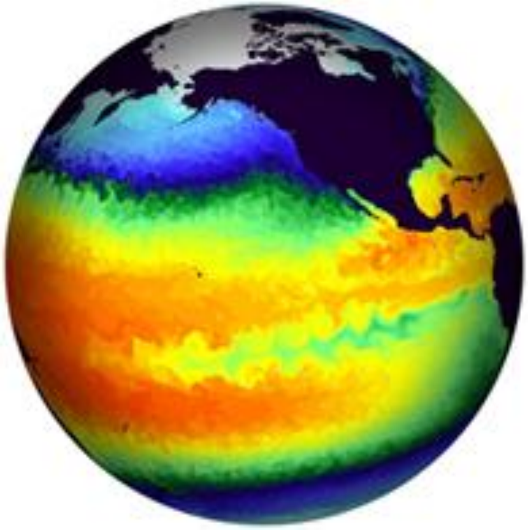
The Equations

One of the simplest models which displays the characteristic fast/slow dynamics of the ocean is the Shallow-Water Equations:

$$\frac{D\vec{v}}{Dt} + Ro^{-1}\vec{v}^{\perp} + (Fr)^{-2}\theta \bigtriangledown h = 0$$

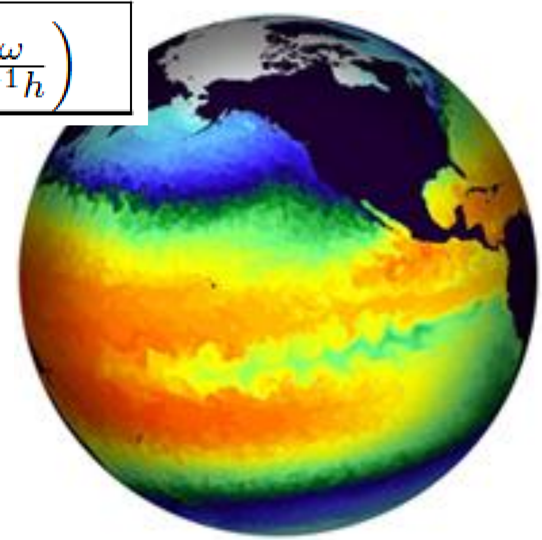
$$\frac{Dh}{Dt} + \theta^{-1} \bigtriangledown \vec{v} + h \bigtriangledown \vec{v} = 0$$

Ro = Rossby Number (rotational forces)
Fr = Froude Number (a measure of resistance)
θ = Height Ratio



The Limits

QG	Low Froude	Low Rossby
$\frac{Ro}{Fr} = \text{constant}$ $Ro \to 0$ $Fr \to 0$	$Fr \to 0$ Ro finite	$Ro \to 0$ Fr finite
$Fr = \sqrt{F}\epsilon$ $\theta = F\epsilon$ $Ro = \epsilon$	$Fr = \epsilon$ $\theta = \epsilon$ $Ro = O(1)$	$Fr = O(1)$ $\theta \models \frac{1}{\epsilon}$ $Ro = \epsilon$
PV: $\frac{D}{Dt}\left(\frac{1+\epsilon\omega}{1+\epsilon Fh}\right)$	PV: $\frac{D}{Dt}\left(\frac{1+Ro\omega}{1+\epsilon h}\right)$	PV: $\frac{D}{Dt}\left(\frac{1+\epsilon\omega}{1+\epsilon^{-1}h}\right)$



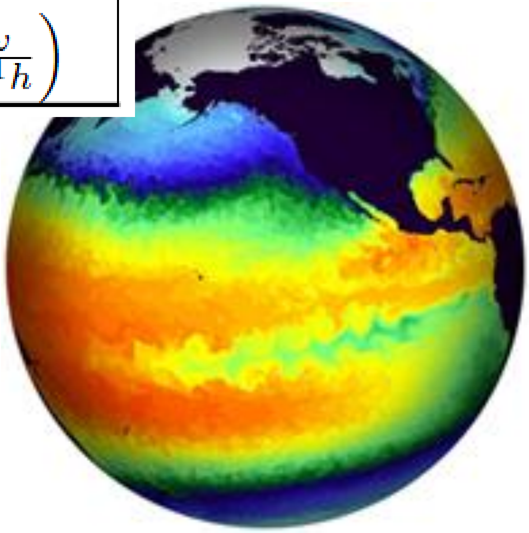
Fr -> 0 represents hydrostatic balance
Ro -> represents geostrophic balance
LF regime is classified by strong stratification and flat top water.
LR regime is classified by strong rotation and wavy top water.

The Quasi-Geostrophic Limit

QG	Low Froude	Low Rossby
$\frac{Ro}{Fr} = \text{constant}$ $Ro \to 0$ $Fr \to 0$	$Fr \to 0$ Ro finite	$Ro \to 0$ Fr finite
$Fr = \sqrt{F}\epsilon$ $\theta = F\epsilon$ $Ro = \epsilon$	$Fr = \epsilon$ $\theta = \epsilon$ $Ro = O(1)$	$Fr = O(1)$ $\theta \models \frac{1}{\epsilon}$ $Ro = \epsilon$
PV: $\frac{D}{Dt}\left(\frac{1+\epsilon\omega}{1+\epsilon F h}\right)$	PV: $\frac{D}{Dt}\left(\frac{1+Ro\omega}{1+\epsilon h}\right)$	PV: $\frac{D}{Dt}\left(\frac{1+\epsilon\omega}{1+\epsilon^{-1} h}\right)$

$$\frac{D\vec{v}}{Dt} + Ro^{-1}\vec{v}^{\perp} + (Fr)^{-2}\theta \bigtriangledown h = 0$$

$$\frac{Dh}{Dt} + \theta^{-1} \bigtriangledown \vec{v} + h \bigtriangledown \vec{v} = 0$$



The Quasi-Geostrophic Limit

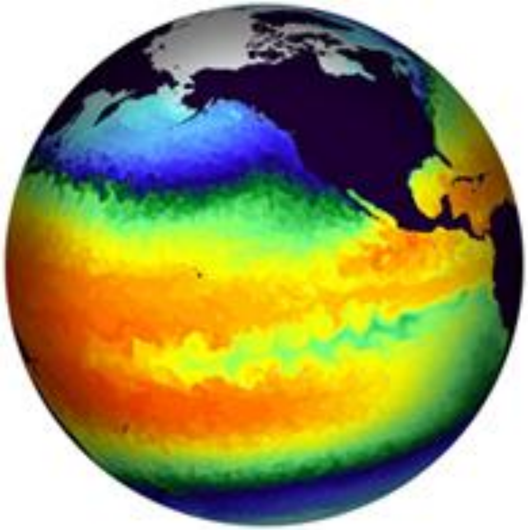
$$\frac{D\vec{v}}{Dt} + \frac{1}{\epsilon}\vec{v}^{\perp} + \frac{1}{\epsilon}\nabla h = 0$$

$$F\frac{Dh}{Dt} + \frac{1}{\epsilon}\nabla \vec{v} + Fh\nabla \vec{v} = 0$$

Apply the following change of variables: $h_{old} = F^{-1/2}h_{new}$

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{\epsilon}\vec{v}^{\perp} + \frac{\nabla h}{\sqrt{F}\epsilon} + \vec{v}\cdot\nabla\vec{v} = 0$$

$$\frac{\partial h}{\partial t} + \frac{\nabla\cdot\vec{v}}{\sqrt{F}\epsilon} + \vec{v}\cdot\nabla h + h\nabla\cdot\vec{v} = 0$$



Embid, Pedro and Andrew Majda. "Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity" Commun. In Partial Differential Equations, 21 (3&4), 619-658. 1996.

The fast/slow operators

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{\epsilon}\vec{v}^{\perp} + \frac{\nabla h}{\sqrt{F}\epsilon} + \vec{v}\cdot\nabla\vec{v} = 0$$

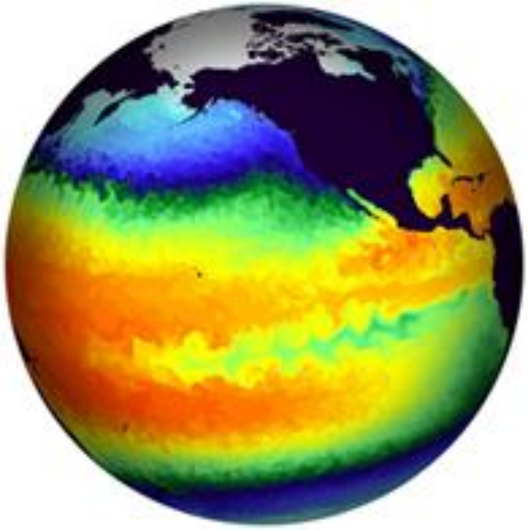
$$\frac{\partial h}{\partial t} + \frac{\nabla\cdot\vec{v}}{\sqrt{F}\epsilon} + \vec{v}\cdot\nabla h + h\nabla\cdot\vec{v} = 0$$

Next we split the equations into two operators depending on their scaling with the following form:

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{\epsilon}\mathcal{L}(\vec{u}) + \mathcal{B}(\vec{u},\vec{u}) = 0$$

$$\mathcal{L}(\vec{u}) = \begin{bmatrix} v^{\perp} + F^{-1/2}\nabla h \\ F^{-1/2}\nabla\cdot\vec{v} \end{bmatrix}$$

$$\mathcal{B}(\vec{u},\vec{u}) = \begin{bmatrix} \vec{v}\cdot\nabla\vec{v} \\ \nabla(h\vec{v}) \end{bmatrix}$$



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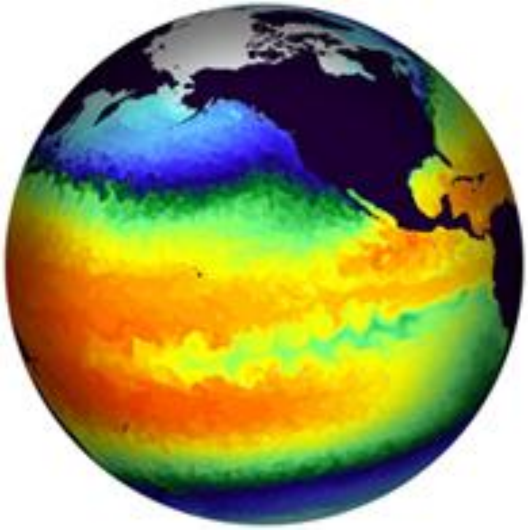
Fourier Modes

$$\mathcal{L}(\vec{u}) = \begin{bmatrix} v^{\perp} + F^{-1/2} \nabla h \\ F^{-1/2} \nabla \cdot \vec{v} \end{bmatrix}$$

We may write the above operator in component form:

$$\begin{aligned} -v + \frac{h_x}{\sqrt{F}} &= 0 \\ u + \frac{h_y}{\sqrt{F}} &= 0 \\ \frac{v_x + v_y}{\sqrt{F}} &= 0 \end{aligned}$$

It's linear!



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Fourier Modes

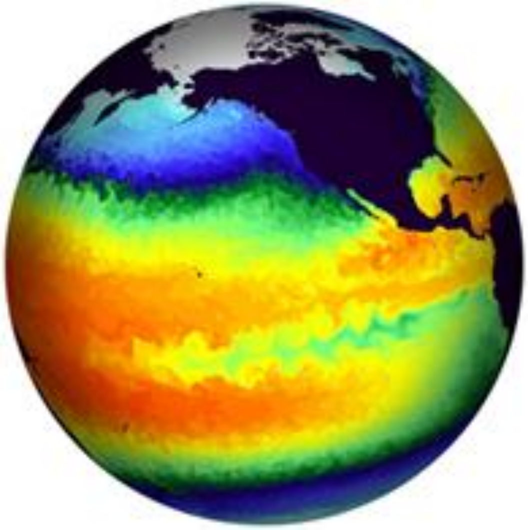
Now we will just analyze the fast dynamics. We will assume the variables are 2 pi periodic. This simplification allows us to write the associated eigenfunctions explicitly in Fourier Modes. Below is the identity that must be satisfied for a multidimensional Fourier transform.

$$\begin{bmatrix} u \\ v \\ h \end{bmatrix} = \begin{bmatrix} \hat{u_k} \\ \hat{v_k} \\ \hat{h_k} \end{bmatrix} e^{i(kx+ly-\omega t)}$$

We can now find the appropriate form of the partial derivatives. For example,

$$h_x = \frac{\partial}{\partial x}\hat{h_k}e^{i(kx+ly-\omega t)} = ik\hat{h_k}e^{i(kx+ly-\omega t)}$$

$$h_y = il\hat{h_k}e^{i(kx+ly-\omega t)}$$



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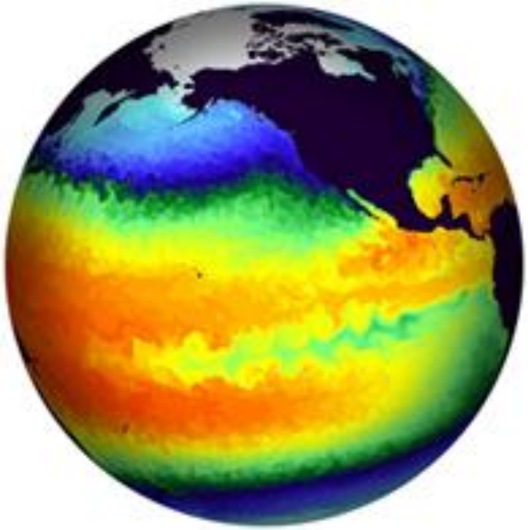
Fourier Modes

$$\begin{aligned} -v + \frac{h_x}{\sqrt{F}} &= 0 \\ u + \frac{h_y}{\sqrt{F}} &= 0 \\ \frac{v_x + v_y}{\sqrt{F}} &= 0 \end{aligned} \quad + \quad \begin{bmatrix} u \\ v \\ h \end{bmatrix} = \begin{bmatrix} \hat{u}_k \\ \hat{v}_k \\ \hat{h}_k \end{bmatrix} e^{i(kx+ly-\omega t)} \quad =$$

$$\begin{aligned} -\hat{v}_k e^{i(kx+ly-\omega t)} + \frac{ik}{\sqrt{F}} \hat{h}_k e^{i(kx+ly-\omega t)} &= 0 \\ \hat{u}_k e^{i(kx+ly-\omega t)} + \frac{il}{\sqrt{F}} \hat{h}_k e^{i(kx+ly-\omega t)} &= 0 \\ \frac{1}{\sqrt{F}} \left(ik\hat{u}_k e^{i(kx+ly-\omega t)} + il\hat{v}_k e^{i(kx+ly-\omega t)} \right) &= 0 \end{aligned}$$



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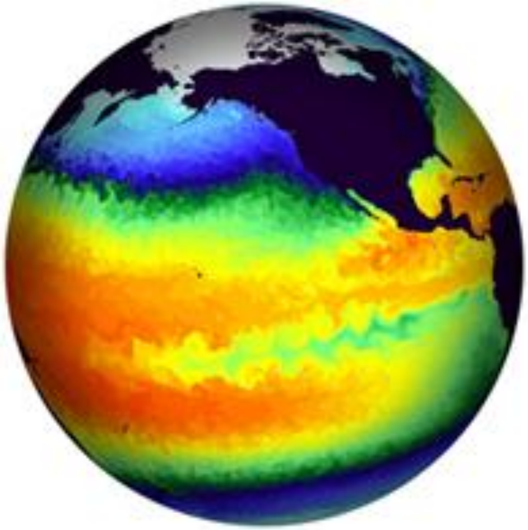
Fourier Modes

$$\begin{aligned} -v + \frac{h_x}{\sqrt{F}} &= 0 \\ u + \frac{h_y}{\sqrt{F}} &= 0 \\ \frac{v_x + v_y}{\sqrt{F}} &= 0 \end{aligned} \quad + \quad \begin{bmatrix} u \\ v \\ h \end{bmatrix} = \begin{bmatrix} \hat{u}_k \\ \hat{v}_k \\ \hat{h}_k \end{bmatrix} e^{i(kx+ly-\omega t)} \quad =$$

$$\begin{aligned} -\hat{v}_k \cancel{e^{i(kx+ly-\omega t)}} + \frac{ik}{\sqrt{F}} \hat{h}_k \cancel{e^{i(kx+ly-\omega t)}} &= 0 \\ \hat{u}_k \cancel{e^{i(kx+ly-\omega t)}} + \frac{il}{\sqrt{F}} \hat{h}_k \cancel{e^{i(kx+ly-\omega t)}} &= 0 \\ \frac{1}{\sqrt{F}} \left(ik\hat{u}_k \cancel{e^{i(kx+ly-\omega t)}} + il\hat{v}_k \cancel{e^{i(kx+ly-\omega t)}} \right) &= 0 \end{aligned}$$

$$\mathcal{L}(ik) = \begin{bmatrix} 0 & -1 & \frac{ik}{\sqrt{F}} \\ 1 & 0 & \frac{il}{\sqrt{F}} \\ \frac{ik}{\sqrt{F}} & \frac{il}{\sqrt{F}} & 0 \end{bmatrix}$$



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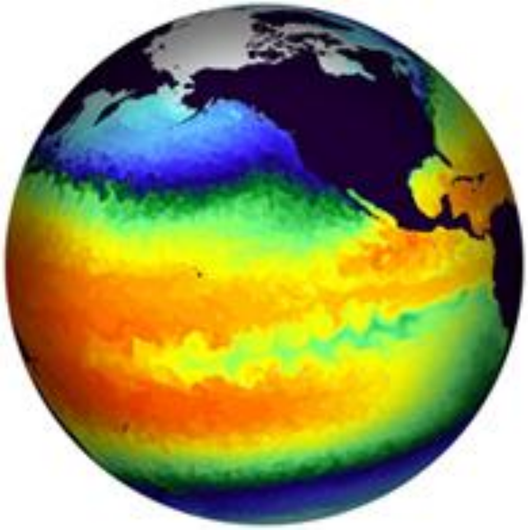
Fourier Modes

$$\mathcal{L}(ik) = \begin{bmatrix} 0 & -1 & \frac{ik}{\sqrt{F}} \\ 1 & 0 & \frac{il}{\sqrt{F}} \\ \frac{ik}{\sqrt{F}} & \frac{il}{\sqrt{F}} & 0 \end{bmatrix}$$

This matrix is skew-hermitian! Ie: $a_{ij} = -\overline{a_{ji}}$ Thus is has an orthonormal basis of eigenvectors with purely imaginary eigenfrequencies.

Next we find the eigenfrequencies:

$$det(i\omega I + \mathcal{L}(i\vec{k}) = i\omega(\omega^2 - (1 + F^{-1}|\vec{k}|^2))$$



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Fourier Modes

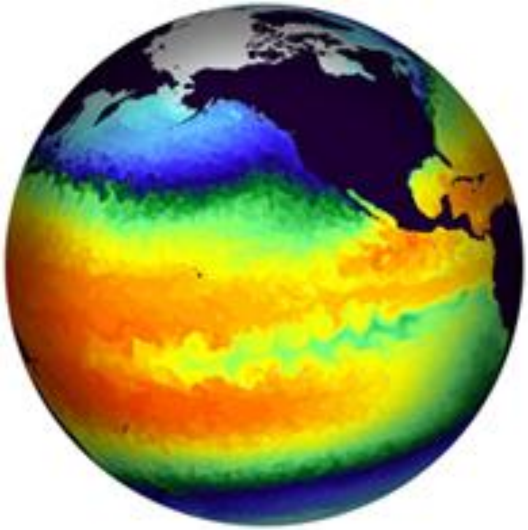
$$det(i\omega I + \mathcal{L}(i\vec{k}) = i\omega(\omega^2 - (1 + F^{-1}|\vec{k}|^2))$$

The eigenfrequencies are $\omega = 0, \pm\sqrt{1 + \frac{|k_H|^2}{F}}$ Where $\vec{k} = (k, l)$ and $|k_H|^2 = k^2 + l^2$

The eigenfunction and eigenfrequencies can be found to be:

$$\omega = 0 \qquad \omega = \sqrt{1 + \frac{|k_H|^2}{F}} \qquad \omega = -\sqrt{1 + \frac{|k_H|^2}{F}}$$

$$\begin{bmatrix} \frac{il}{\sqrt{F}}\omega(\vec{k}) \\ \frac{ik}{\sqrt{F}\omega(\vec{k})} \\ \frac{1}{\omega(\vec{k})} \end{bmatrix} \qquad \begin{bmatrix} \frac{\omega(\vec{k})k + il}{\sqrt{2}\omega(\vec{k})|\vec{k}|} \\ \frac{\omega(\vec{k})l - ik}{\sqrt{2}\omega(\vec{k})|\vec{k}|} \\ \frac{|k_H|^2}{\sqrt{2F}\omega(\vec{k})} \end{bmatrix} \qquad \begin{bmatrix} \frac{-\omega(\vec{k})k + il}{\sqrt{2}\omega(\vec{k})|\vec{k}|} \\ \frac{-\omega(\vec{k})l - ik}{\sqrt{2}\omega(\vec{k})|\vec{k}|} \\ \frac{|k_H|^2}{\sqrt{2F}\omega(\vec{k})} \end{bmatrix}$$



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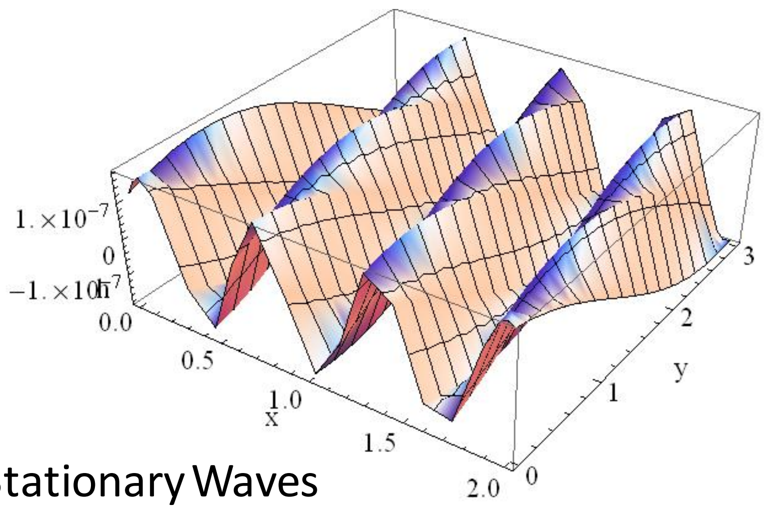
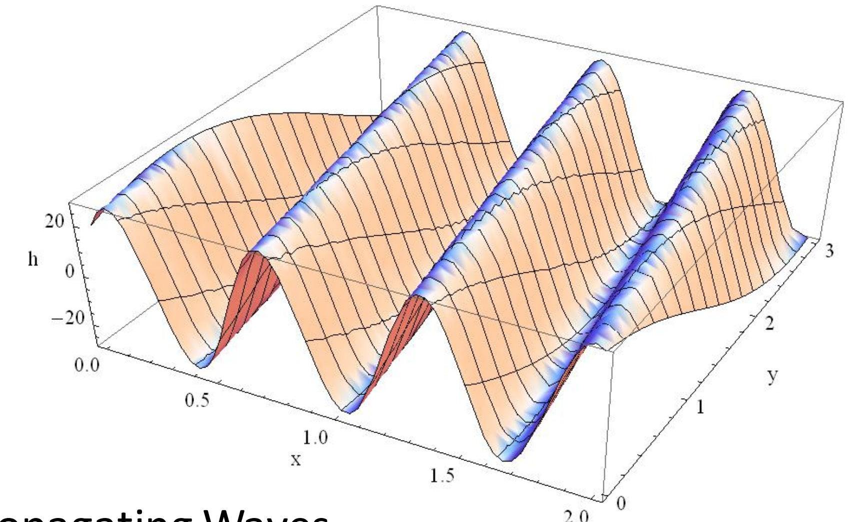
Quasigeostrophic: Wave Solution

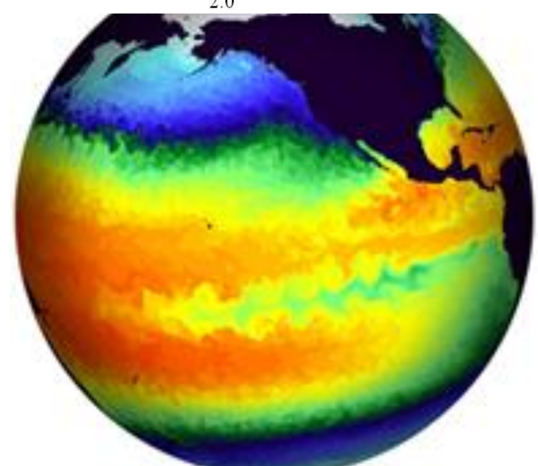
$$\begin{bmatrix} u \\ v \\ h \end{bmatrix} = \sum_k \sum_l a_1 \begin{bmatrix} \frac{-igl}{f} \\ \frac{igk}{f} \\ 1 \end{bmatrix} e^{i(kx+ly)} + a_2 \begin{bmatrix} \frac{k\omega_0 + ilf}{Hk_H} \\ \frac{l\omega_0 - ikf}{Hk_H} \\ 1 \end{bmatrix} e^{i(kx+ly-\omega_0 t)} + a_3 \begin{bmatrix} \frac{-k\omega_0 + ilf}{Hk_H} \\ \frac{-l\omega_0 - ikf}{Hk_H} \\ 1 \end{bmatrix} e^{i(kx+ly+\omega_0 t)}.$$



Stationary Waves
K=10 L=1



Propagating Waves
K=10 L=1



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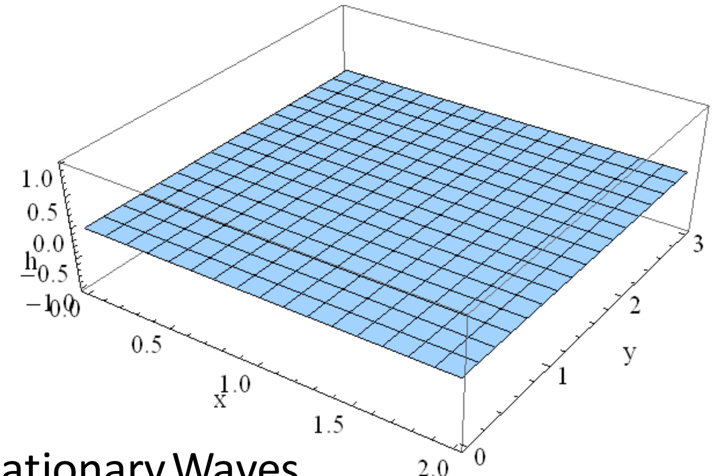
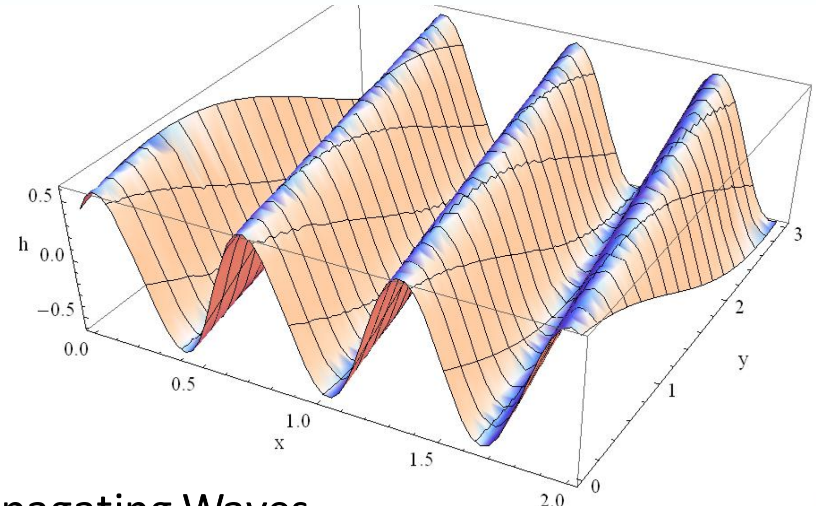
Low Froude: Wave Solution

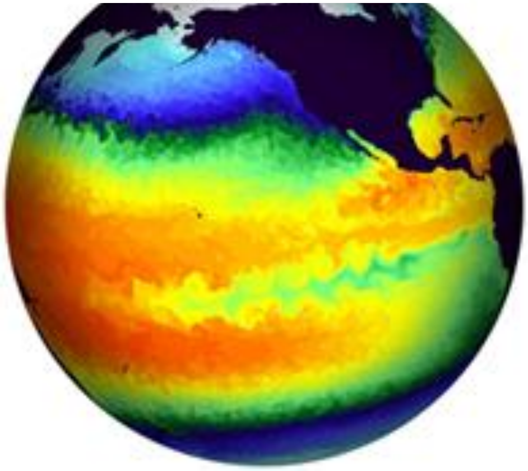
$$\begin{bmatrix} u \\ v \\ h \end{bmatrix} = \sum_k \sum_l a_1 \begin{bmatrix} \frac{-l}{\sqrt{k_H}} \\ \frac{k}{\sqrt{k_H}} \\ 0 \end{bmatrix} e^{i(kx+ly)} + a_2 \begin{bmatrix} \frac{k}{\sqrt{k_H+k^2+1}} \\ \frac{1}{\sqrt{k_H+k^2+1}} \\ \sqrt{\frac{k_H}{k_H+k^2+1}} \end{bmatrix} e^{i(kx+ly-\sqrt{k_H}t)} + a_3 \begin{bmatrix} \frac{-k}{\sqrt{k_H+k^2+1}} \\ \frac{-1}{\sqrt{k_H+k^2+1}} \\ \sqrt{\frac{k_H}{k_H+k^2+1}} \end{bmatrix} e^{i(kx+ly+\sqrt{k_H}t)}.$$



Stationary Waves
K=10 L=1

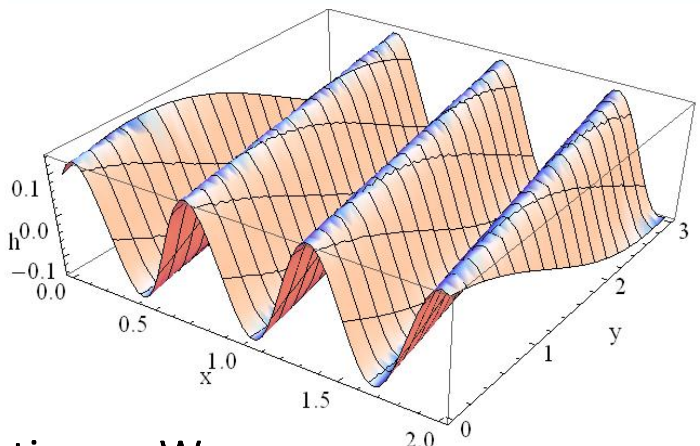


Propagating Waves
K=10 L=1

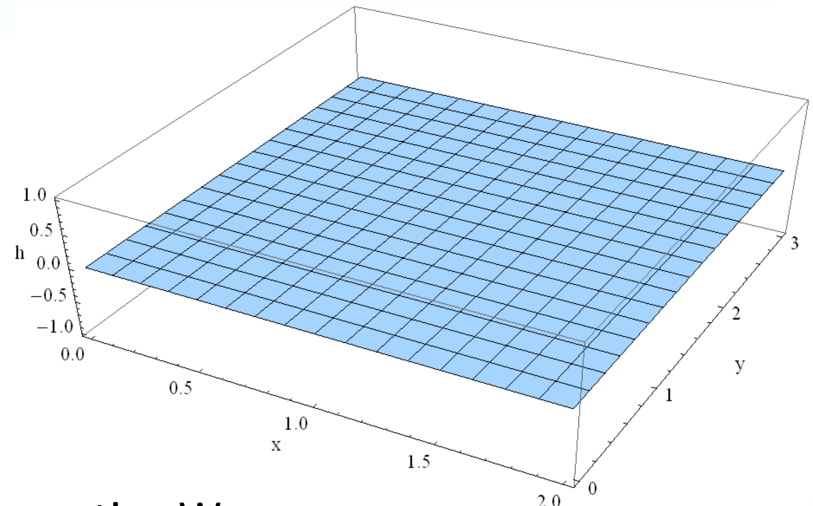


Low Rossby: Wave Solution

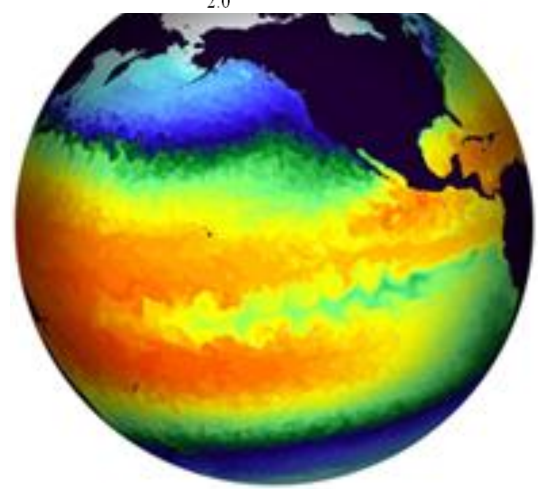
$$\begin{bmatrix} u \\ v \\ h \end{bmatrix} = \sum_k \sum_l a_1 \begin{bmatrix} \frac{-il}{\sqrt{k_H+1}} \\ \frac{ik}{\sqrt{k_H+1}} \\ \frac{1}{\sqrt{k_H+1}} \end{bmatrix} e^{i(kx+ly)} + a_2 \begin{bmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} e^{i(kx+ly-t)} + a_3 \begin{bmatrix} \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} e^{i(kx+ly+t)}.$$



Stationary Waves
K=10 L=1



Propagating Waves
K=10 L=1



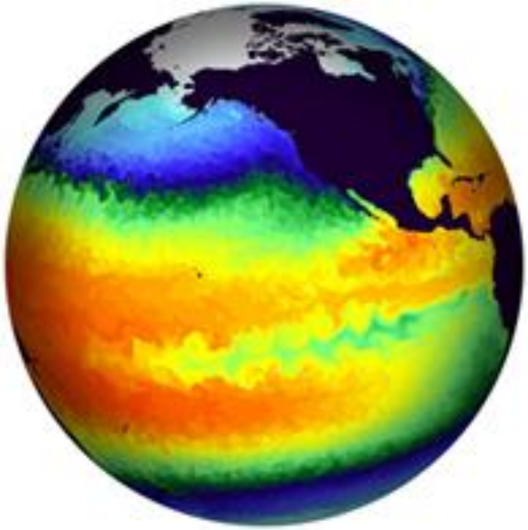
Conclusion

With these three limits of the Shallow-Water Equation, a better understanding may be gained of the separation of slow and fast dynamics.

This work provides insight in how to take advantage of new heterogeneous computer architectures.

The Low Rossby Limit doesn't have the form that we desire. Therefore future inquiry must take place into this limit.

Future work:
Proof of principle numerical algorithms on shallow-water equations that use fast/slow time splitting.



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