A Dynamical Systems Approach to a Conceptual Climate Model

Energy Balance, Ice Dynamics, and the Carbon Cycle

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Collaborators

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- **Dick McGehee (UMN)**
- Samantha Oestreicher (UMN)
- Mary Lou Zeeman (Bowdoin)
- Emma Cutler (Bowdoin)
- Amanda Gartside (Bowdoin)
- Julie Leifeld (UMN)
- Jim Walsh (Currently at UMN)

PaleoCarbon

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The Components of the Conceptual Model

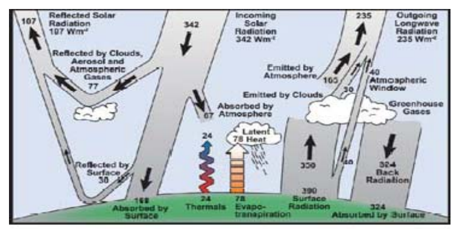
1) An **Energy Balance Model** describing the evolution of **latitudinal temperature** (Budyko-Sellers)
2) A **dynamic (moving) iceline** (Widiasih)
3) Forcing due to **Greenhouse Gases** (Collaborators)

Assumption: Symmetry across the equator

"Earth"

Motto: "Make everything as simple as possible, but not simpler**." -Einstein**

Motivation for Energy Balance Models Component (1)



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

In Equilibrium:
Energy absorbed = Energy emitted

Budyko-Sellers Model

$$R\frac{\partial T}{\partial t} = \underbrace{Qs(y)}_{\text{insolation}}\big(1-\underbrace{\alpha(y,\eta)}_{\text{albedo}}\big) - \underbrace{(A+BT)}_{\text{re-radiation}} + \underbrace{C(\bar{T}-T)}_{\text{transport}}$$

$T = T(y,t) =$ annual mean surface temperature

$y = \sin(\text{latitude}) \qquad y \in [0,1]$

$Q =$ global annual mean insolation

$s(y) =$ relative annual mean insolation, $\int_0^1 s(y)\,dy = 1$

$y = \eta$: ice boundary

$$\alpha(y,\eta) = \begin{cases} \alpha_1, & y < \eta, \\ \alpha_2, & y > \eta. \end{cases} \quad \text{albedo}$$

$\bar{T}(t) = \int_0^1 T(y,t)\,dy =$ mean annual global temperature

K. K. Tung, *Topics in Mathematical Modeling*, Princeton University Press, 2007.

Slide from Prof. McGehee, 2010

Mathematically, the Budyko-Sellers EBM is a bit complicated…

$$\frac{\partial T}{\partial t} = \frac{k}{R}\left(Qs(y)(1-\alpha(y,\eta)) - A - (B+C)T + C\bar{T}\right)$$

$$\bar{T} = \int_0^1 T(y,t)\,dy$$

Its solutions, temperature profiles T(y,t), live in a **function space**, … are typically infinite-dimensional.

Luckily, a **reduction method** has been developed by McGehee and Widiasih:

1. Assume the profile is piecewise smooth across the iceline
2. Make an infinite expansion of each "piece" of the profile
3. This reduces the system to an infinite number of ODE…
4. **An infinite number of which we can reasonably ignore.**

McGehee-Widiasih Reduction

1. Assume the profile is piecewise smooth across the iceline and even **(in y)**

$$T(y,t) = \begin{cases} U(y,t) & : |y| < \eta \\ V(y,t) & : |y| > \eta \\ \frac{U(\eta)+V(\eta)}{2} & : |y| = \eta \end{cases}$$

2. Make an infinite expansion of each "piece" of the profile **(in Fourier-Legendre series)**

$$U(y,t) = \sum_{m=0}^{\infty} u_{2m}(t) p_{2m}(y)$$
$$V(y,t) = \sum_{m=0}^{\infty} v_{2m}(t) p_{2m}(y)$$

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3. Since the basis functions p_{2j} are orthogonal in $L^2[-1,1]$, we can find equations for the coefficients u_{2j} via projection, and this reduces the system to an infinite number of ODE.

$$\frac{du_0}{dt} = \frac{k}{R}\left(Qs_0(1-\alpha_1) - A - (B+C)u_0 + C\overline{T}\right)$$
$$\frac{dv_0}{dt} = \frac{k}{R}\left(Qs_0(1-\alpha_2) - A - (B+C)v_0 + C\overline{T}\right)$$
$$\frac{du_2}{dt} = \frac{k}{R}\left(Qs_2(1-\alpha_1) - (B+C)u_2\right)$$
$$\frac{dv_2}{dt} = \frac{k}{R}\left(Qs_2(1-\alpha_2) - (B+C)v_2\right)$$
$$\left.\begin{aligned}\frac{du_{2m}}{dt} &= -\frac{k}{R}(B+C)u_{2m} \\ \frac{dv_{2m}}{dt} &= -\frac{k}{R}(B+C)v_{2m}\end{aligned}\right\} \quad m \geq 2$$

where $T(y,t)$ is given by

$$T(y,t) = \begin{cases} \sum_{m=0}^{\infty} u_{2m}(t)p_{2m}(y) & : y < \eta \\ \sum_{m=0}^{\infty} v_{2m}(t)p_{2m}(y) & : y > \eta \\ \frac{1}{2}\left(\sum_{m=0}^{\infty} u_{2m}(t)p_{2m}(y) + \sum_{m=0}^{\infty} v_{2m}(t)p_{2m}(y)\right) & : y = \eta \end{cases}$$

Why is this better??

We can show

$$\frac{\partial T}{\partial t} = \frac{k}{R}\left(Qs(y)(1-\alpha(y,\eta)) - A - (B+C)T + C\overline{T}\right)$$

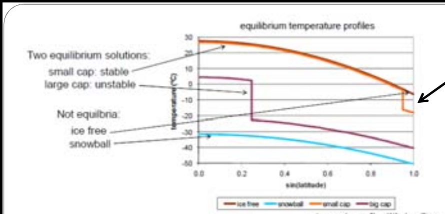
Is equivalent to

$$\frac{du_0}{dt} = \frac{k}{R}\left(Qs_0(1-\alpha_1) - A - (B+C)u_0 + C\overline{T}\right)$$
$$\frac{dv_0}{dt} = \frac{k}{R}\left(Qs_0(1-\alpha_2) - A - (B+C)v_0 + C\overline{T}\right)$$
$$\frac{du_2}{dt} = \frac{k}{R}\left(Qs_2(1-\alpha_1) - (B+C)u_2\right)$$
$$\frac{dv_2}{dt} = \frac{k}{R}\left(Qs_2(1-\alpha_2) - (B+C)v_2\right)$$
$$\left.\begin{aligned}\frac{du_{2m}}{dt} &= -\frac{k}{R}(B+C)u_{2m} \\ \frac{dv_{2m}}{dt} &= -\frac{k}{R}(B+C)v_{2m}\end{aligned}\right\} \quad m \geq 2$$

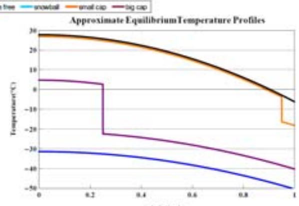
Exponential decay!

4. An infinite number of (ODE) which we can reasonably ignore.

$$T(y,t) \approx \begin{cases} u_0(t)p_0(y) + u_2(t)p_2(y) & : y < \eta \\ v_0(t)p_0(y) + v_2(t)p_2(y) & : y > \eta \\ \frac{1}{2}\left(u_0(t)p_0(y) + u_2(t)p_2(y) + v_0(t)p_0(y) + v_2(t)p_2(y)\right) & : y = \eta \end{cases}$$



Computed from Budyko-Sellers model (D. McGehee)



Computed using the low-dimensional approximation

Actually, the equilibrium solutions are exactly the same in both frames!

Component (2): Dynamic Iceline

- Observation: In Budyko-Sellers, the ice boundary is static.
- Q: How can it become dynamic in the simplest possible way?
- Idea (Due to **E. Widiasih**): **Ice melts when the temperature at the iceline is too high, and water freezes when the temperature is too low.**

$$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$$

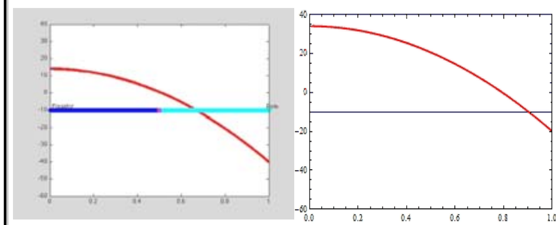
The (reduced) coupled EBM and iceline system

$$\frac{du_0}{dt} = \frac{k}{R}\left(Qs_0(1-\alpha_1) - A - (B+C)u_0 + CT\right)$$
$$\frac{dv_0}{dt} = \frac{k}{R}\left(Qs_0(1-\alpha_2) - A - (B+C)v_0 + CT\right)$$
$$\frac{du_2}{dt} = \frac{k}{R}\left(Qs_2(1-\alpha_1) - (B+C)u_2\right)$$
$$\frac{dv_2}{dt} = \frac{k}{R}\left(Qs_2(1-\alpha_2) - (B+C)v_2\right)$$
$$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$$

In fact, this can be reduced further, resulting in only TWO differential equations! (see Widiasih 2010)

Comparing two versions of the coupled EBM + iceline system

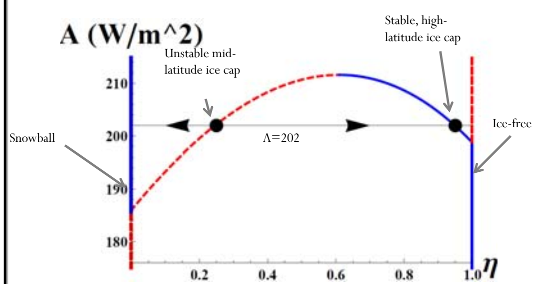
Time evolution of **temperature** at **latitude** y, "jump" denotes location of **iceline.**



Budyko-Sellers + Iceline Equation (From E. Widiasih)

Reduced Budyko-Sellers + Iceline Equation

So, what's really going on here?



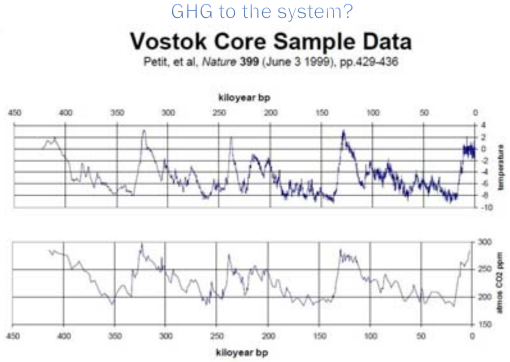
Here A is a fixed number, a bifurcation parameter. What happens if A changes?

Can we use the parameter A as a proxy for GHG, or more specifically, atmospheric CO2?

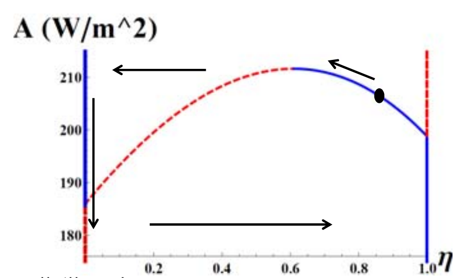
In other words, can we couple "Component (3)" to the existing model?

- **Abbot et. al. (2011):** Yes, think of a reduction in OLR (**A from Budyko**) from the present value as an increase in radiative forcing due to GHG.
- **Hogg (2007), and others**: A is a function of log(pCO2), so one could couple an equation for atmospheric CO2 directly to the Budyko-Iceline model.

Why should we bother to try to couple the effect of GHG to the system?



One option is to try to "move around" in this diagram by allowing A to be dynamic.



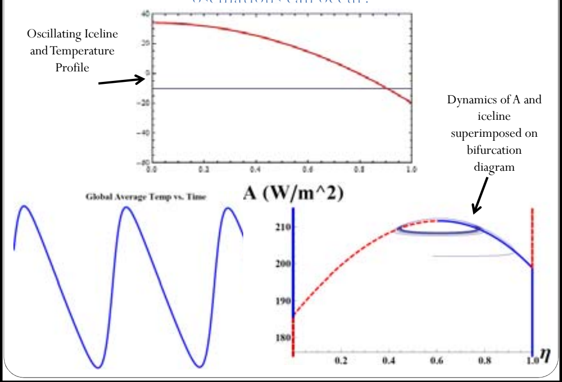
Idea motivated by Abbott et. al. 2011

Another option is to work with A as a function of CO2, a la Hogg

- Motivated by Hogg (2007), Zeeman, Cutler and Gartside (Bowdoin, 2011), focus on a **linear evolution equation** for CO2 with effects from **volcanism** and **weathering of silicate rocks:**

$$\frac{d(pCO_2)}{dt} = V - W\eta$$

Using these simple ideas, what types of glacial oscillations can occur?



Challenges and Future Directions

- Reduction techniques allowed for a much simpler, numerically inexpensive, investigation of a high-dimensional model. **How can we use/improve these for other systems or more general use?**
- Does this model tell us anything about the real world, e.g. **potential mechanisms for onset of glacial oscillations or dramatic bifurcations?**
- If so, **What are the "right" timescales and parameters**, and is it necessary to find them?
- **Math Challenge**: Multiple timescale analysis of a piecewise smooth system (at ice boundaries)- requires **extension of singular perturbation theory**?

Thank you!