An Estimate of the Widiasih Parameter

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Widiasih's Parameter

Budyko-Sellers Model

$$\underbrace{R\frac{\partial T}{\partial t} = Qs(y)}_{\text{insolation}}\underbrace{\left(1-\alpha(y)\right)}_{\text{albedo}}\underbrace{-\left(A+BT\right)}_{\text{re-radiation}}\underbrace{+C\left(\bar{T}-T\right)}_{\text{transport}}$$

$T = T(y,t) =$ annual mean surface temperature

$y = \sin(\text{latitude}) \qquad y \in [0,1]$

$Q =$ global annual mean insolation $= 343\ \text{W/m}^2$

$s(y) =$ relative annual mean insolation, $\int_0^1 s(y)\,dy = 1$

$\bar{T}(t) = \int_0^1 T(y,t)\,dy =$ mean annual global temperature

$\alpha(y) =$ surface albedo

$A = 202\ \text{W/m}^2 \quad B = 1.9\ \text{W/m}^2/{}^\circ\text{C} \quad C = 3.04\ \text{W/m}^2/{}^\circ\text{C}$

Widiasih's Parameter

Budyko-Sellers Model

The Albedo Function

$\alpha = \alpha_1 = 0.32$ (water and land)

$\alpha = \alpha_2 = 0.62$ (ice)

Suppose that there is a single ice boundary at latitude $\arcsin(\eta)$.

$$\alpha(y) = \alpha(y,\eta) = \begin{cases} \alpha_1, & y < \eta, \\ \alpha_2, & y > \eta, \end{cases}$$

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT\right) + C\left(\bar{T}-T\right)$$

Widiasih's Parameter

Widiasih's Ice Line Dynamics

The Widiasih Parameter

$$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$$

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT\right) + C\left(\bar{T}-T\right)$$

State space: $[0,1] \times E$

$$T(\eta) = \frac{1}{2}\left(T(\eta-) + T(\eta+)\right)$$

Widiasih's Parameter

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$$

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT\right) + C\left(\bar{T}-T\right)$$

Widiasih's Theorem: For an appropriate function space E and for sufficiently small ε, the system has an attracting invariant curve. On the curve, the system is approximated by

$$\boxed{\frac{d\eta}{dt} = \varepsilon h(\eta)}$$

Where:

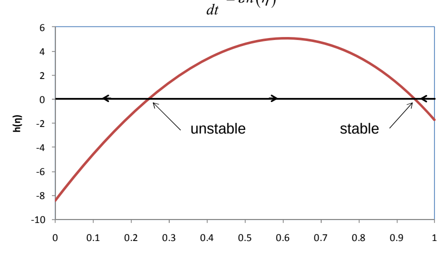
$$h(\eta) \equiv \frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_2 + (\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy\right)\right) - \frac{A}{B} - T_c = 0$$

$$\alpha_0 = (\alpha_1+\alpha_2)/2$$

Widiasih's Parameter

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$



How small is ε?

Widiasih's Parameter

Budyko-Widiasih Model

What about units?

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\bar{T}-T\right)$$

$R\frac{\partial T}{\partial t}$: K/s $Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\bar{T}-T)$: W/m^2 = J/m^2/s

R : J/m^2/K

J = Joules
K = degrees Kelvin
m = meters
s = seconds

heat capacity of liquid water : 4 J/g/K = 4×10^6 J/m^3/K

(1 gram water = 1 cm^3 = 10^{-6} m^3)

assumption: surface is water with depth of 100 m

$$\boxed{R = 4\times10^8 \text{ J/m}^2\text{/K}}$$

Widiasih's Parameter

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$$

Recall: η is the proportion of Earth's surface that is ice-free.
Let σ = surface area of Earth = 5.1x10^{14} m^2 .
Then $\sigma\eta$ is the area of the ice-free surface in square meters.

$$\frac{d(\sigma\eta)}{dt} = \sigma\varepsilon\left(T(\eta) - T_c\right)$$

$\frac{d(\sigma\eta)}{dt}$: m^2/s $T(\eta) - T_c$: K

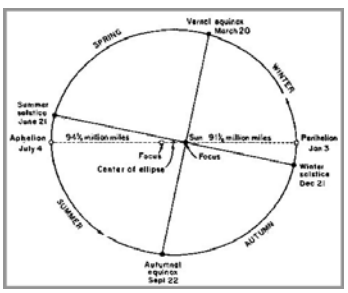
$\sigma\varepsilon$ has units of m^2/s/K

It is not clear how to choose ε from first principles.

What can we learn from paleoclimate data?

Widiasih's Parameter

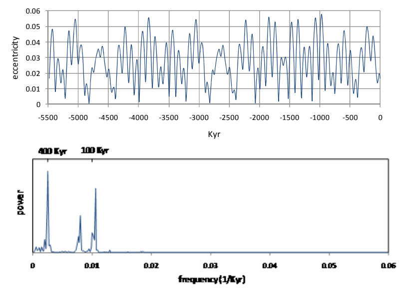
Eccentricity



Imbrie, John & Imbrie, Katherine Palmer, *Ice Ages: Solving the Mystery*, Harvard Univ. Press, 1979.

Widiasih's Parameter

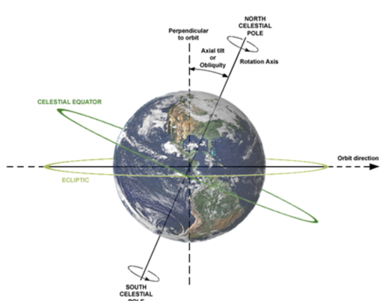
Eccentricity



J. Laskar, *et al.* (2004), *Astronomy & Astrophysics* **428**, 261–285.

Widiasih's Parameter

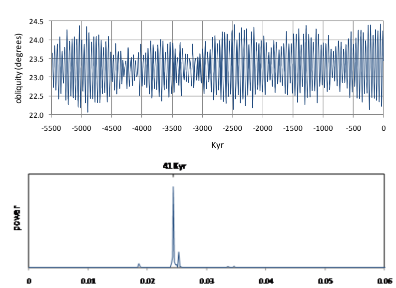
Obliquity



`http://upload.wikimedia.org/wikipedia/commons/6/61/AxialTiltObliquity.png`

Widiasih's Parameter

Obliquity



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Widiasih's Parameter

Dependence on Milankovitch Cycles

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T}-T\right)$$

depends only on obliquity β

depends only on eccentricity e

$$Q(e) = \frac{Q_0}{\sqrt{1-e^2}}$$

$$s(y,\beta) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\gamma - y\cos\beta\right)^2}\,d\gamma$$

Widiasih's Parameter

Dependence on Milankovitch Cycles

L^2 Approximation using Legendre Polynomials

$$s(y,\beta) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\gamma - y\cos\beta\right)^2}\,d\gamma$$

s is an even function of y, so a quadratic L^2 approximation is

$$s(y,\beta) \approx 1 + s_2(\beta)\,p_2(y)$$

where $p_2(y) = \frac{1}{2}\left(3y^2-1\right)$

One can compute $s_2(\beta) = \frac{5}{16}\left(-2+3\sin^2\beta\right)$

The current value of obliquity is about 23.5°.

$$s(y) \approx 1 - 0.238\left(3y^2-1\right)$$

Compare with Tung and North's quadratic approximation:

$$s(y) \approx 1 - 0.241\left(3y^2-1\right)$$

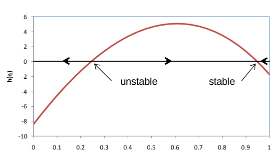
Widiasih's Parameter

Widiasih Ice Line Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$

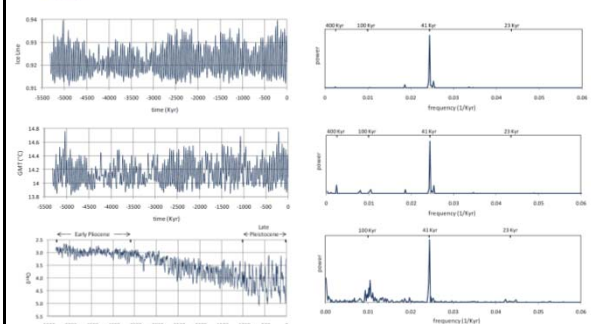
$$h(\eta,e,\beta) \equiv \frac{Q(e)}{B+C}\left(\underbrace{s(\eta,\beta)}_{\text{quadratic in }\eta}(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)\underbrace{\int_0^\eta s(y,\beta)\,dy}_{\text{cubic in }\eta}\right)\right) - \frac{A}{B} - T_c = 0$$

We can solve $h(\eta,e,\beta) = 0$ for $\eta = f(e,\beta)$ at the stable rest point. We can then use Laskar's paleoclimate computations of $e(t)$ and $\beta(t)$ to get the ice line as a function of time $\eta(t) = f(e(t), \beta(t))$. One can also compute the global mean temperature (GMT) as a function of time.



Widiasih's Parameter

Results of Model Simulation



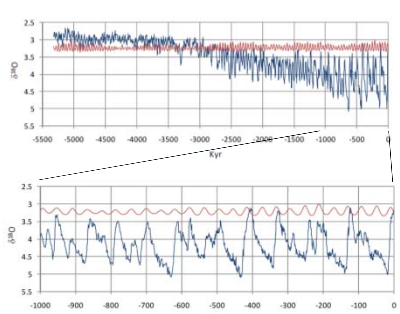
Widiasih's Parameter

Results of Model Simulation

blue = data

red = model

Late Pleistocene (not a good fit)



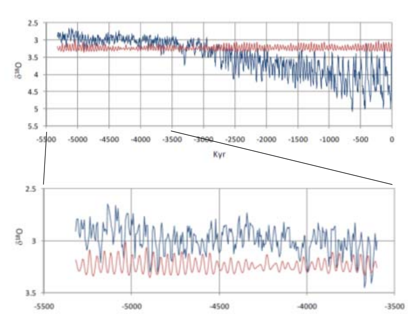
Widiasih's Parameter

Results of Model Simulation

blue = data

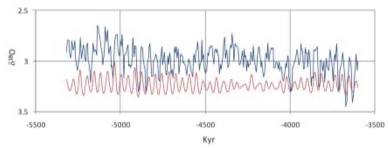
red = model

Early Pliocene (not great, but much better)

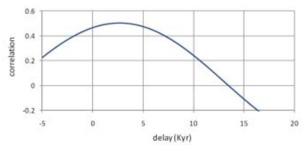


Widiasih's Parameter

Results of Model Simulation



There is a delay of about 2500 years between the model and the data.



Widiasih's Parameter

Widiasih Ice Line Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, e(t), \beta(t))$$

So far we have computed only the moving equilibrium. The actual dynamics will have a delay between the "equilibrium" and the "response." What value of ε will create a delay of 2.5 kiloyears?

Change time units to kiloyears. Let $\kappa = 3.16\text{x}10^{10}$ be the number of seconds in a kiloyear.

$$\frac{d\eta}{dt} = \varepsilon\kappa h(\eta, e(t), \beta(t))$$

Since the deviation from the rest point is relatively small and since the major contribution from the Milankovitch cycles is the obliquity at a period of 41 Kyr, we consider the linear equation

$$\frac{d\xi}{dt} = -\lambda(\xi - a\cos\omega t)$$

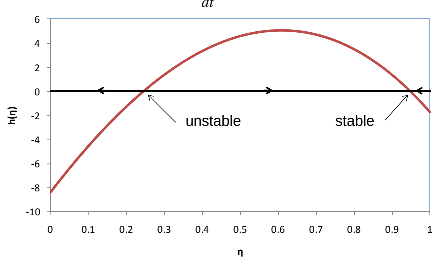
where $\lambda = -\varepsilon\kappa h'(\eta_2)$ and $\omega = 2\pi/41$

and where η_2 is the current value of the ice line (about 0.95).

Widiasih's Parameter

Widiasih Ice Line Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$



Widiasih's Parameter

Linear Approximation

$$\frac{d\xi}{dt} = -\lambda(\xi - a\cos\omega t)$$

Steady-state solution

$$\xi = (a\cos\alpha)\cos(\omega t - \alpha)$$

where $\alpha = \arctan(\omega/\lambda)$

Note that α is the phase shift corresponding to the delay by which the response lags the forcing. A delay of 2.5 Kyr corresponds to $\alpha = 2.5\omega = 5\pi/41$. Hence

$$\lambda = \omega\cot\alpha = -\varepsilon\kappa h'(\eta_2), \text{ so } \varepsilon = -\frac{\omega\cot\alpha}{\kappa h'(\eta_2)}$$

One can show that $h'(\eta_2) \approx -31$ and hence that

$$\boxed{\varepsilon \approx 4\times10^{-13}}$$

An Estimate of the Widiasih Parameter

Conclusion

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$

$$\boxed{\varepsilon \approx 4\times10^{-13}}$$

Aknowledgements

This presentation was all joint work with
Esther Widiasih.
(preprint coming soon)

A similar value for ε was computed by Esther and
Jayna Resman
using direct dynamical simulations of the equation.