Dynamic Oscillations in Paleoclimate Theory.

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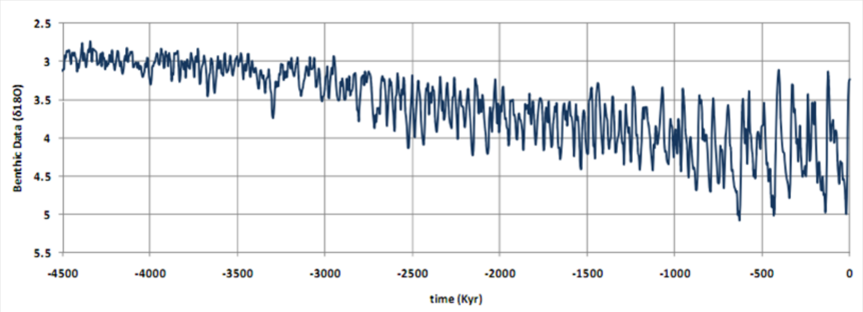
University of Minnesota

Mathematics and Climate Research Network

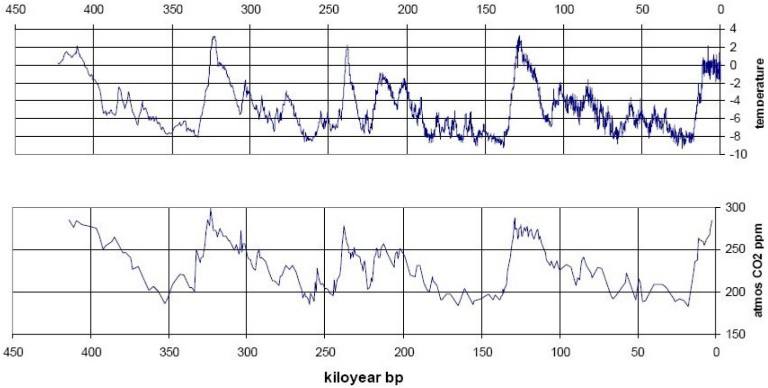
Earth's History

Over the last 4.5 millions years there have been oscillations of Earth's temperature over time. Approximately 1.2 million years ago, Earth moved through a transition into longer and deeper glacial-interglacial cycles.



Glacial-Interglacial Cycles

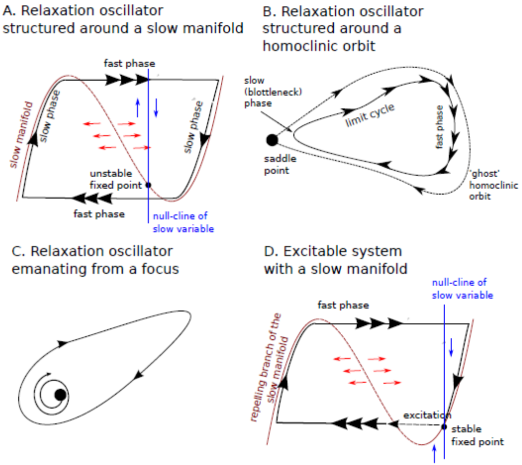
We can use ice core data (here from Vostok) to learn more about the last 800 kyrs.



Despite the eccentricity signal being weak during this time period, the δ^{18}O signal shows a strong $100,000$ year frequency signal.

Using Oscillators to explain Pleistocene

In Crucifix, Michel's "Ocsillations and relaxation phenomena in Pleistocene climate theory." published by the Phil. Trans. Royal Society A in 2012. doi: 10.1098/rsta.2011.0315



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- slow-fast dynamics
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Excitability An excitable system has a stable fixed point- but external forcing can *excite* the system can cause a excursion away from and back to the stable fixed point.

Note There is sometimes a nice connection between excitable systems and slow/fast relaxation oscillators through a particular parameter. (Example forthcoming)

(A) Oscillator Structured around a Slow Manifold

The first example we'll consider is a coupled system of two variables.

$$\begin{aligned} \dot{a} &= b - \frac{1}{40}(a^3 - 25a^2 + 80) = b - \frac{1}{40}(a(a-5)(a+5)) + 2 \\ \dot{b} &= \epsilon(a_c - a) \end{aligned} \tag{1}$$

for $0 < \epsilon \ll 1$

This is a slow/fast system. a is the fast variable while b is the slow variable. We will analyse this system by first finding the null-clines of each variable and plotting the phase portrait of the system.

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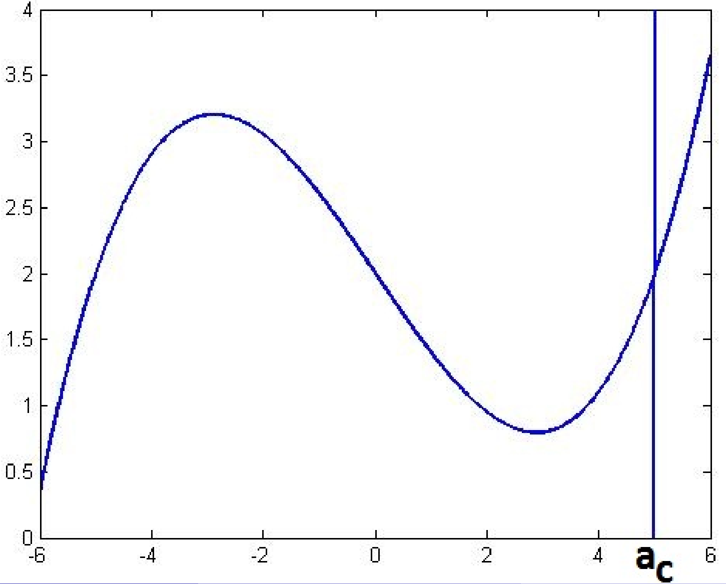
MCRN Webinar: Fast/Slow Dynamics on Thursdays at 1pm eastern time.

Fast/Slow Stable System



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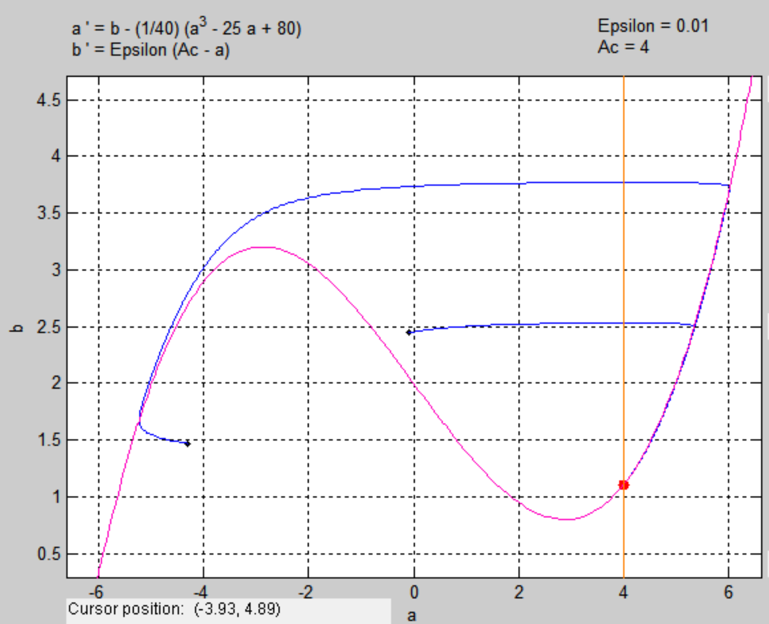
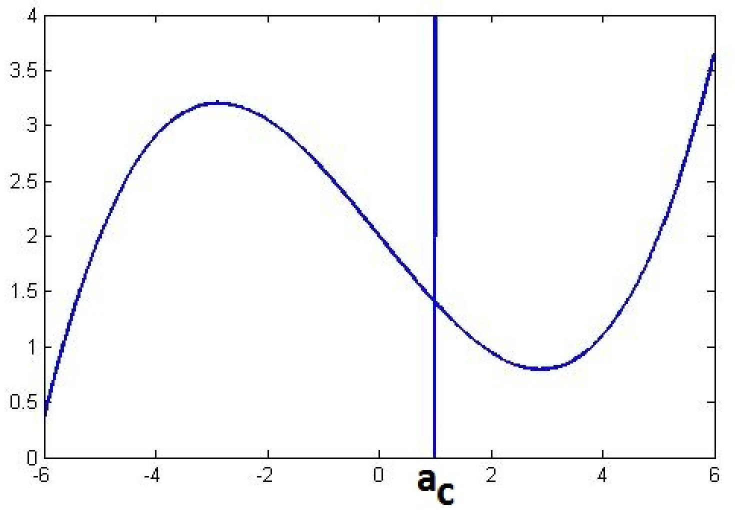


Fast/Slow Relaxation Oscillator



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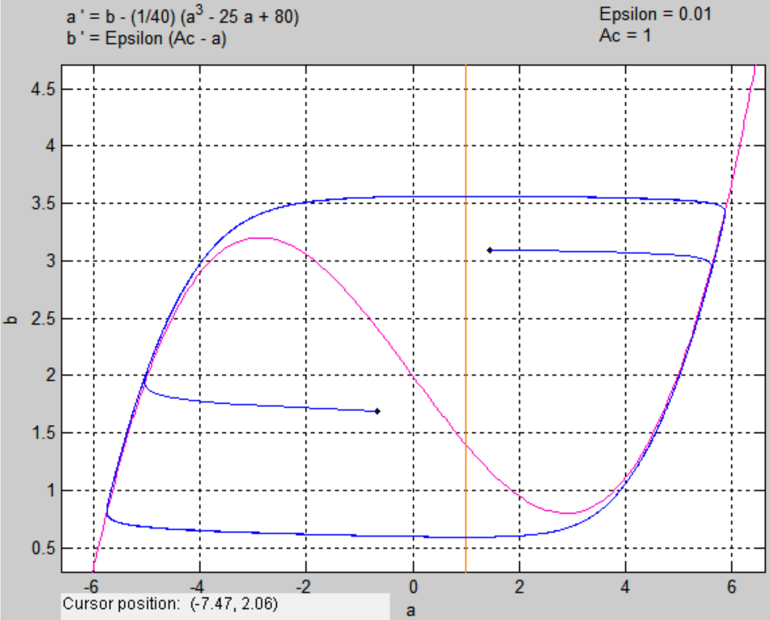
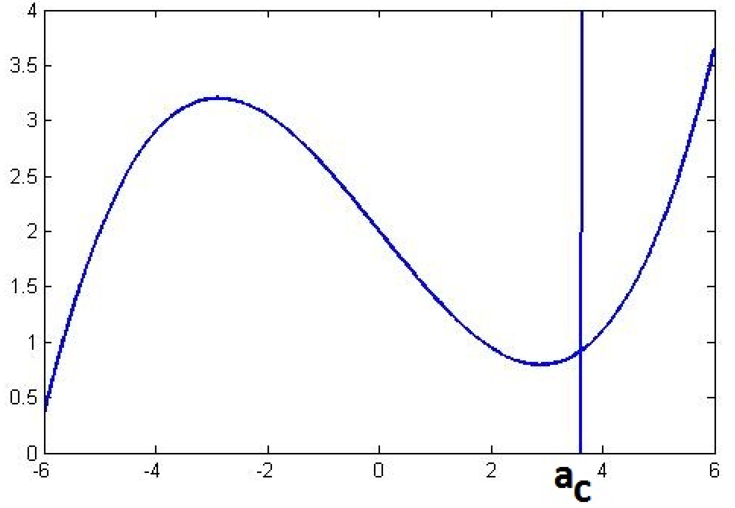


(D) Fast/Slow Excitable System



(C) Relaxation Oscillator emanating from a focus

Van der Pol Equations

$$\ddot{x} + \mu(x^2 - a)\dot{x} + x = 0 \tag{2}$$

OR

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \mu y(a - x^2) \end{aligned} \tag{3}$$

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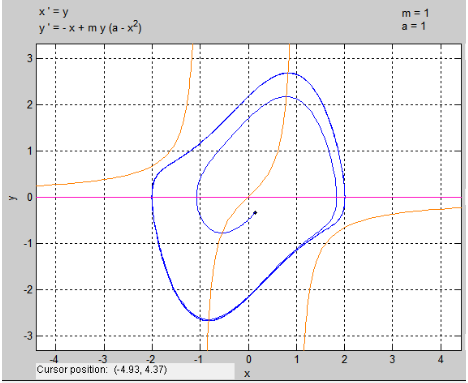
2D limit cycle can't exist in

- simple harmonic oscillator
- gradient system ($\dot{x} = -\bigtriangledown V(x)$)
- Lyaponov systems

When do limit cycles exist in 2D?

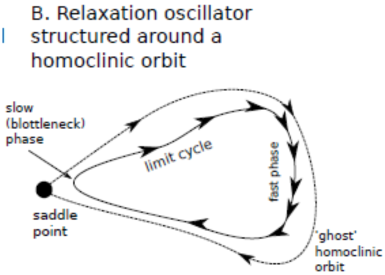
Theorem

Poincare-Bendixson Theorem If the trajectory is confined to a compact region, R, and there are no fixed points in R then there exists a limit cycle, C, somewhere in R.



(B) Oscillations with unstable homoclinic Orbit

The best common example of something like this that I could find was Smale's Horseshoe map. But that's too complicated to discuss here. There are also examples in three species biological models.



Back to the Pleistocene: 4 Models for glacial-interglacial

1. Saltzman
2. Palliard
3. Palliard-Parrenin
4. Crucifix

Saltzman and Maasch Model

$M =$ Milankovitch forcing at 65^oN at summer solstice.
$X =$ ice volume
$Y =$ atmospheric CO_2.
$Z =$ North Atlantic Deepwater Formation.
$\dot{} =$ time derivative.
All variables are deviations from the mean.

$$\begin{aligned} \dot{X} &= -X - Y - uM(t) \\ \dot{Y} &= -pZ + rY + sZ^2 - Z^2 Y \\ \dot{Z} &= -q(X + Z) \end{aligned} \tag{4}$$

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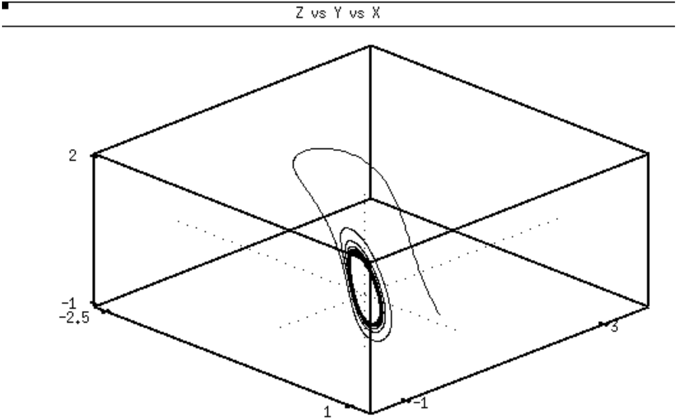
Notes: Maasch and Saltzman show there exists a parameter shift which induces a change from a stable equilibrium solution to 100kyr cycles. The parameter shift is

$$p = 0.8 \to 1,\ q = 1.2,\ r = 0.7 \to 0.8,\ s = 0.8,\ \text{and}\ u = 0.7$$

where p and r vary linearly in time.

Saltzman and Maasch



Saltzman and Maasch

$$
\begin{cases}
\frac{\mathrm{d}I}{\mathrm{d}t} = \alpha_1 - (c\alpha_2)\mu - \alpha_3 I - k_\theta \alpha_2 \theta - k_R \alpha_2 F_I(t) \\
\frac{\mathrm{d}\mu}{\mathrm{d}t} = \beta_1 - (\beta_2 - \beta_3\theta + \beta_4\theta^2)\mu - (\beta_5 - \beta_6\theta)\theta + F_\mu(t) \\
\frac{\mathrm{d}\theta}{\mathrm{d}t} = \gamma_1 - \gamma_2 I - \gamma_3\theta
\end{cases}
\quad \text{(SM90)}
$$

and

$$
\begin{cases}
\frac{\mathrm{d}I}{\mathrm{d}t} = \alpha_1 - (c\alpha_2)\mu - \alpha_3 I - k_\theta \alpha_2 \theta - k_R \alpha_2 F_I(t) \\
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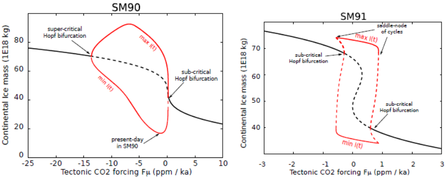
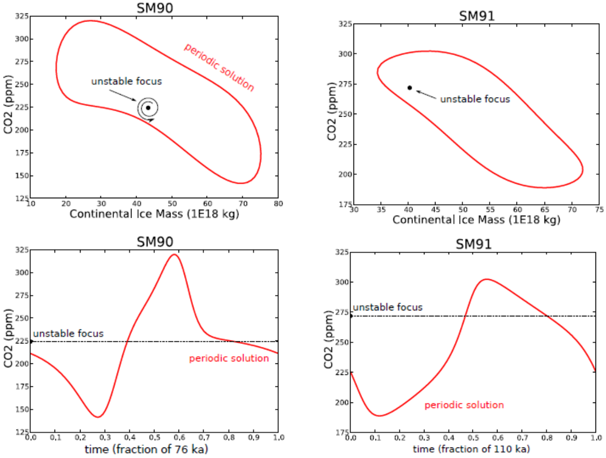


Notes Crucifix argues that this model uses variations in F_μ to capture the Mid-Pliestocene Transition. Perhaps this is what Saltzman eventually presents in his books. But it is not what is presented in the 1990 paper.

Saltzman and Maasch



Paillard's ice age model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x_R(y) - x}{\tau_R(y)} - \frac{F(t)}{\tau_f}$$

$y=$ discrete variable. $i \to g \to G$. (i)- Interglacial. (g)-mild glacial. (G)-deep glacial. Transitions are determined by the values of the astronomical forcing, $F(t)$, or the value of x.

$x_R(y)$ and $\tau_R(y)$ $=$ characteristic relaxation values and time constants respectively. They depend on the discrete value of y.

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Notes

- Physical meaning of y has to do with the Atlantic ocean circulation state. Deep sinking water, intermediate over-turning and shut-down of circulation.
- Contains concepts of fast/slow, but isn't an oscillator b/c shift from g to G is due to external forcing.
- Mid-Pleistocene transition is show to exist by adding to the tectonic forcing terms.

Palliard-Parrenin

$$
\begin{cases}
\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{1}{\tau_I}(-a\mu - bF(t) + c - I) \\
\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{\tau_A}(I - A) \\
\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{1}{\tau_\mu}(dF|(t) - eI + fH(-D) + g - \mu) \\
D = hI - iA + j
\end{cases}
$$

I = Ice Volume
A = Antarctic continental ice sheet
μ = atmospheric CO_2.
τ_A and τ_I are slow time constants.
τ_μ is fast time constant. $H(-D)$ is a Heaviside function.

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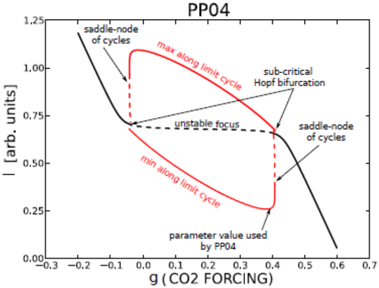
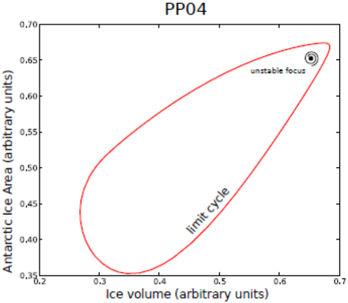
Note: H physically representing ventilation in Southern ocean. CO_2 is released into atmosphere when $D < 0$, which drives deglaciation.

Palliard-Parrenin





Oscillations are structurally created by sub-critical Hopf bifurcations.

As a result we get a relaxation oscillator that is structured around an unstable equilibrium point without being a fast/slow system.

Crucifix's VDP model

The Van der Pol Equation we considered earlier:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \mu y(a - x^2) \end{aligned} \tag{5}$$

Crucifix's Alteration of Van der Pol's Equations:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (-y + \beta + \gamma F(t))/\tau \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -\alpha(y^3/3 - y - x)/\tau \end{cases}$$

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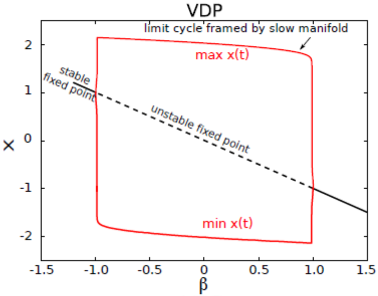
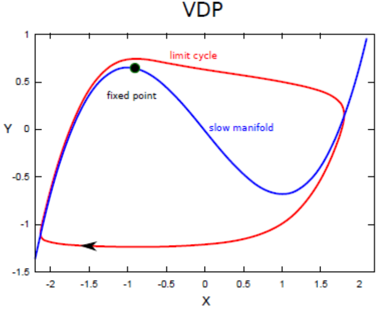
Notes

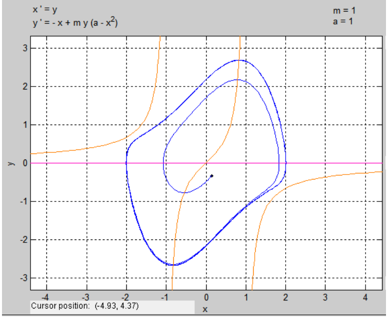
- slow manifold $x = \frac{y^3}{3} - y$.
- β controls position of fixed-point on the slow manifold (and so the ratio of times spent in glacial or interglacial.
- designed to challenge the arguments about predictability of ice ages.

Crucifix VDP vs. VDP

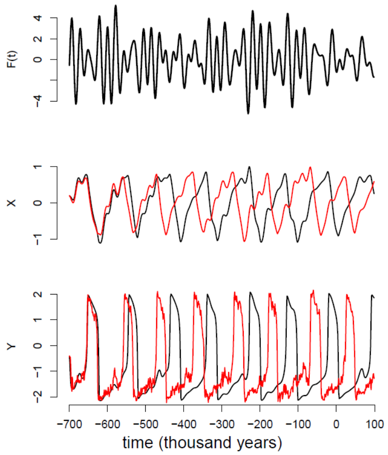




The details of the equation and the size of the parameters makes a big difference in what kind of oscillator we see. There is, perhaps, a close relationship between fast/slow oscillators and sub-critical Hopfs bifurcation oscillators?

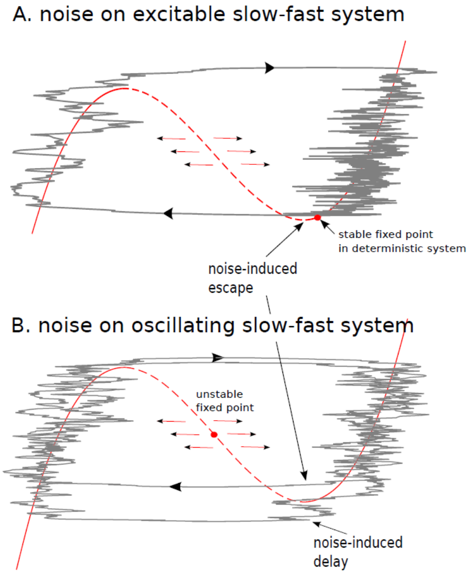


Stochastic effects



- "weak stochastic forcing on an oscillator causes a fading out of the memory of exact initial conditions" - Crucifix pg 16
- This happens a lot with neutral stability in a free oscillator.
- "Stochastic forcing disperses the system states around the different attractors that are compatible with the forcing." -Crucifix pg 17

Stochastic effects



Stochastic forcing may excite an excitable system.

Or delay oscillating slow-fast systems.

My Favorite Conclusions

Question

"Can dynamical systems be used for inference on paleoclimates?"

Challenge

"The modeller's challenge is therefore to operate a model selection on more stringent criteria than just fitting some standard time series."

Conclusion...?

"Whether the process of inference with simple dynamical systems on paleoclimate data will lead new insight in [paleoclimate understanding and modelling] still needs to be demonstrated"