Ducks in the Ocean: Canards and Relaxation Oscillations in Large-scale Ocean Dynamics

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Outline

- Motivation: Dansgaard-Oeschger events
- Bistability and Stommel's model
- Bifurcations vs. relaxation oscillations
- Analysis: ROs, canard cycles, and super-explosion
- An extra dimension
- Conclusion

Dansgaard-Oeschger Events

- Oscillations in North Atlantic climate with an average period of 1.5kyr
- Rapid warming: $\sim 10^{\circ}$ C over a few decades
- Longer cooling period
- Correspond with changes in the Atlantic Meridional Overturning Circulation

Dansgaard-Oeschger Events

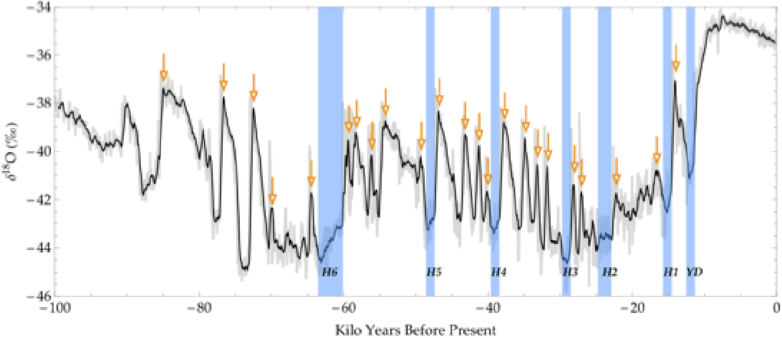


Figure: Oxygen isotope data from Greenland (NGRIP). Orange arrows indicate thermal maxima of Dansgaard-Oeschger cycles over the last 100,000 years. Figure obtained from Saha (2011).

-Rapid transitions indicate relaxation behavior or that the underlying model should have multiple time scales.

-Also indicates system should have two “stable” states.

Stommel's Experiment

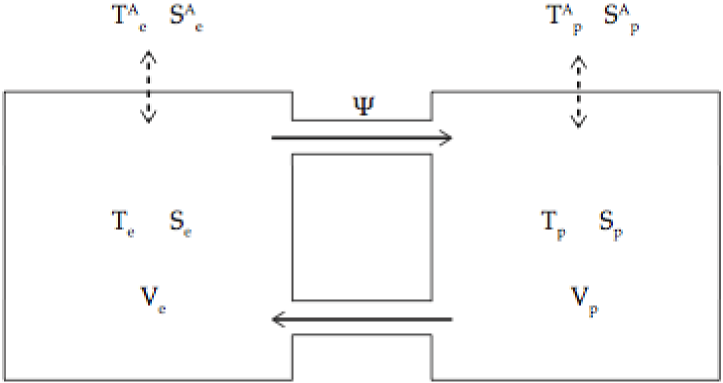


Figure: Schematic of Stommel's model (1961)—from Saha (2011).

Stommel's Model

The equations for the model are:

$$\frac{dT}{dt} = R_T[(T_e^A - T_p^A) - T] - |\psi| T$$
$$\frac{dS}{dt} = R_S[(S_e^A - S_p^A) - S] - |\psi| S,$$

where

- $T = T_e - T_p$,
- $S = S_e - S_p$, and
- $\psi = \rho_0(-\alpha T + \beta S)$ is the circulation variable.

Since $R_S \ll R_T$, we want to capitalize on a separation of time scales.

Stommel's Equations

Glendinning (2009) shows the model has a dimensionless form:

$$\epsilon \dot{x} = 1 - x - \epsilon A|\psi|x$$
$$\dot{y} = \mu - y - A|\psi|y,$$

where

- $\psi = x - y$ is the circulation variable,
- x is scaled temperature difference,
- y is scaled salinity difference, and
- μ is considered the "freshwater flux" parameter (but really it is a ratio of salinity forcing to temperature forcing).

This is a singularly perturbed system that reduces to

$$\dot{y} = \mu - y - A|1 - y|y.$$

The system has a unique stable equilibrium for $A < 1$, but will be bistable for some range of μ values if $A > 1$.

Bistability and Hysteresis

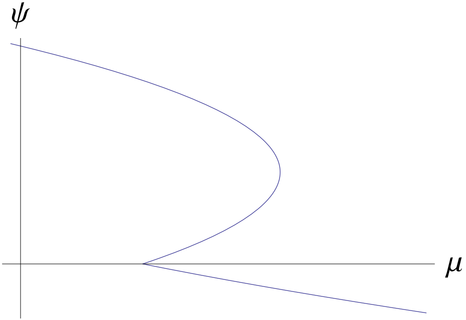


Figure: Bifurcation diagram (with ψ) from Stommel's model for $A > 1$.

Bistability and Hysteresis

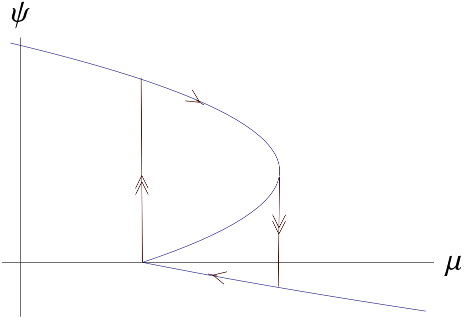


Figure: Bifurcation diagram (with ψ) from Stommel's model for $A > 1$.

Hysteresis

Definition

Hysteresis is the dependence of a system not only on its current environment but also on its past environment. This dependence arises because the system can be in more than one internal state.

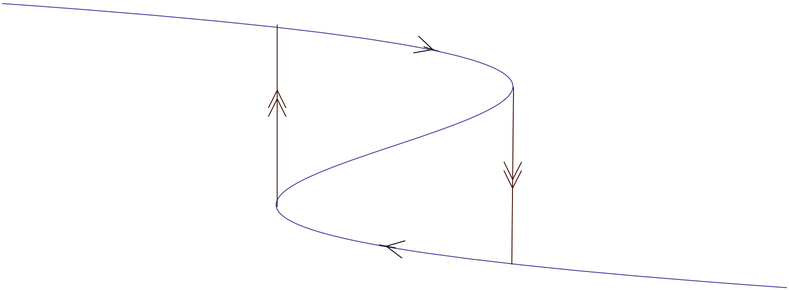
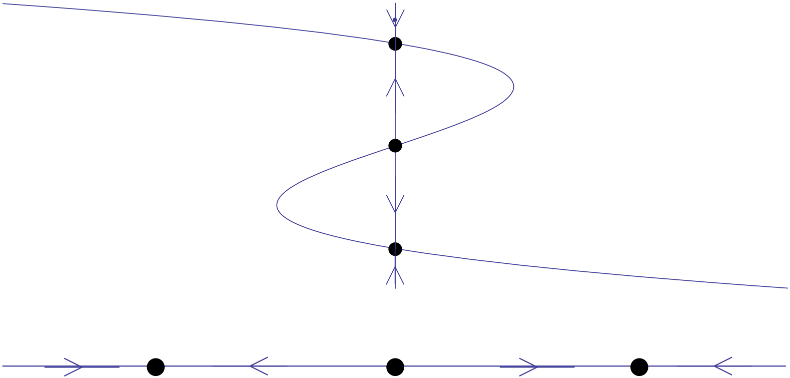


Figure: Hysteresis loop.

Abrupt Changes in Dynamical Systems

Simple, classic example: Saddle-node bifurcation



Abrupt Changes in Dynamical Systems

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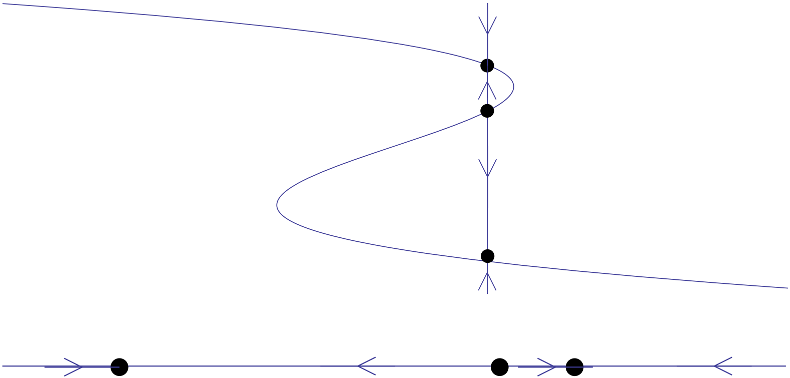

Abrupt Changes in Dynamical Systems

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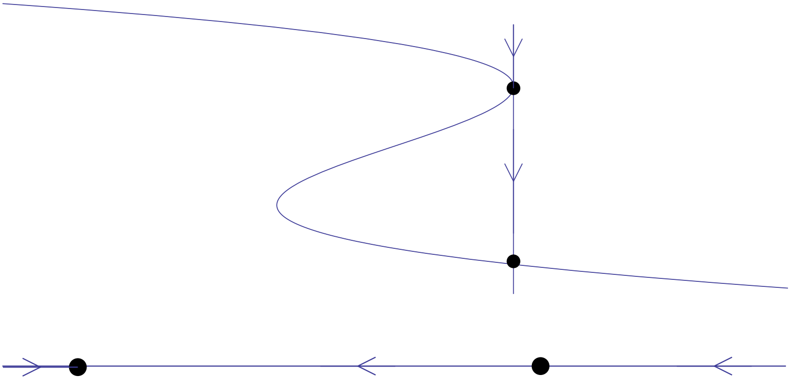
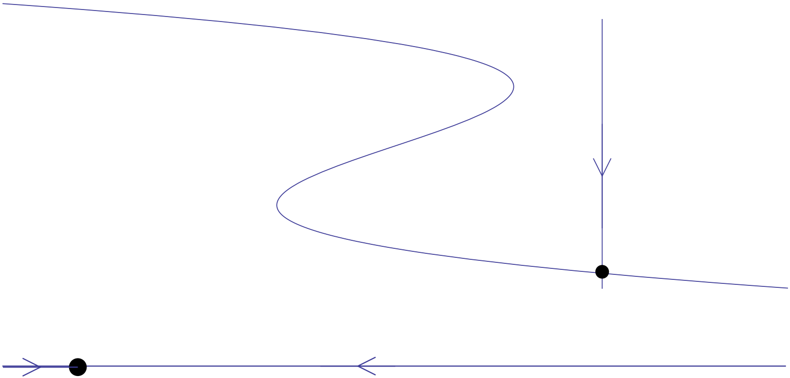


Abrupt Changes in Dynamical Systems

Simple, classic example: Saddle-node bifurcation



Fast/Slow Dynamics

$$\dot{x} = f(x; \lambda)$$

- x is the state variable
- λ is the bifurcation parameter
- λ varies independent of x

$$\begin{aligned} x' &= f(x, y, \epsilon) \\ y' &= \epsilon g(x, y, \epsilon) \end{aligned} \qquad\qquad \begin{aligned} \epsilon\dot{x} &= f(x, y, \epsilon) \\ \dot{y} &= g(x, y, \epsilon) \end{aligned}$$

- x is the fast variable
- y is the slow variable
- $\epsilon \ll 1$ (fixed) is a small parameter
- y variation prescribed—depends on state variables

Relaxation Oscillations

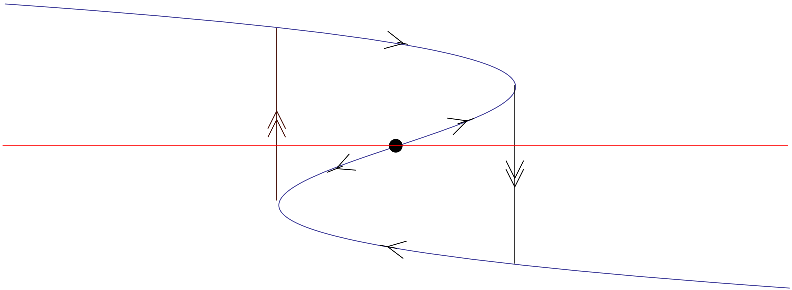


Figure: Blue curve is the fast nullcline—called the *critical manifold*.
Red line is the slow nullcline

Behavior depends on location of the slow nullcline.

Globally Attracting Equilibrium

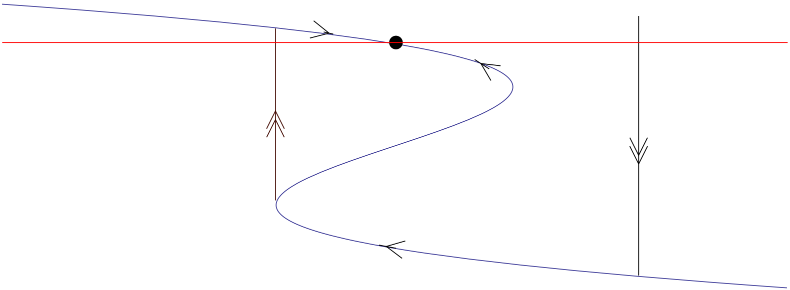


Figure: Blue curve is the fast nullcline—called the *critical manifold*.
Red line is the slow nullcline

In this case, the critical point is attracting.

Canard Point

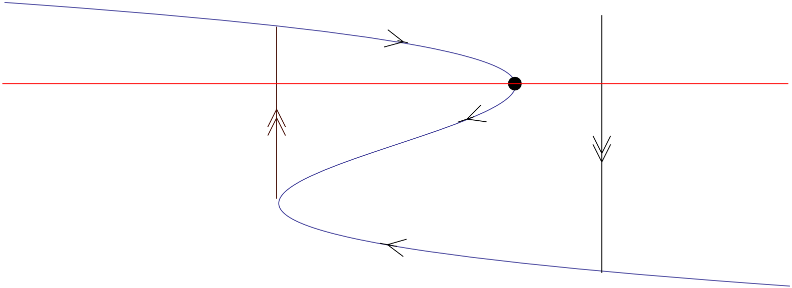


Figure: Blue curve is the fast nullcline—called the *critical manifold*. Red line is the slow nullcline

Here the critical point is a canard point, and the system undergoes a Hopf bifurcation.

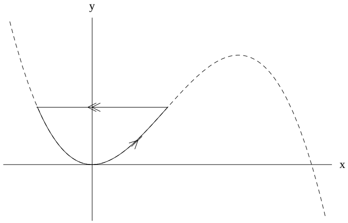
Introduction to Canards

History:

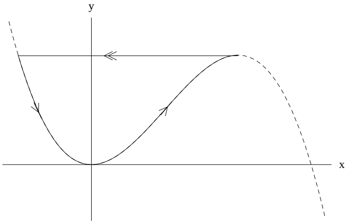
- Discovered by Benoit, Callot, F. Diener, and M. Diener (1981) using non-standard analysis
- Eckhaus (1983) examined canards using matched asymptotics
- Dumortier and Roussarie (1996) used center manifold and blow-up techniques
- Krupa and Szmolyan (2001) generalized the blow-up techniques

Recently, the generalized blow-up techniques have allowed for canards to be examined in higher dimensions (Wechselberger, Krupa, Szmolyan, Brons, Guckenheimer, Desroches, and others).

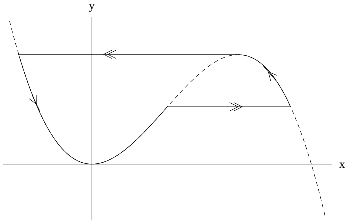
Canards in the Singular Limit



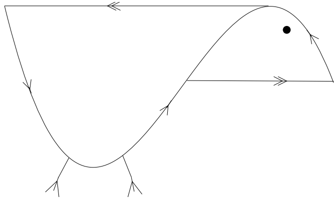
(a) Headless canard cycle.



(b) Maximal canard.



(c) Canard with head.



(d) A duck!

Canard Cycles for $\epsilon > 0$

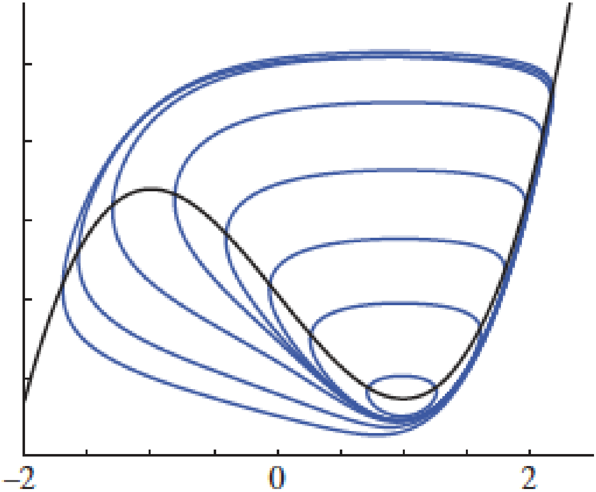


Figure: Image from Desroches *et al.* (2013)

Back to Stommel

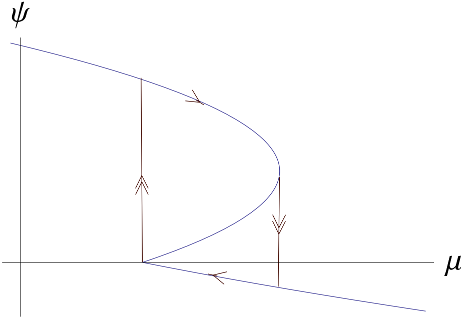


Figure: Bifurcation diagram (with ψ) from Stommel's model for $A > 1$.

Goal

Hypothesis: D-O events relate to hysteresis loop in the bifurcation diagram.

Question: What mechanisms can make that hysteresis loop dynamic (i.e., a relaxation oscillation)?
From the literature:

- Intrinsic Ocean Dynamics (de Verdiére)
- Periodic freshwater forcing (Ganopolski and Rhamstorf)
- Stochastic freshwater forcing (Cessi)
- Thermal effects (Saltzman, Sutera, and Evenson)
- Sea-ice feedback mechanism (Saha)

Occam's Razor Approach

Make μ a dynamic slow variable!
We look at the three time scale model:

$$\begin{aligned} x' &= 1 - x - \epsilon A|x-y|x \\ y' &= \epsilon(\mu - y - A|x-y|y) \\ \mu' &= \epsilon\delta f(x, y, \mu, \delta, \epsilon), \end{aligned}$$

Linear μ' Equation

Assuming μ' depends linearly on x and y, we get

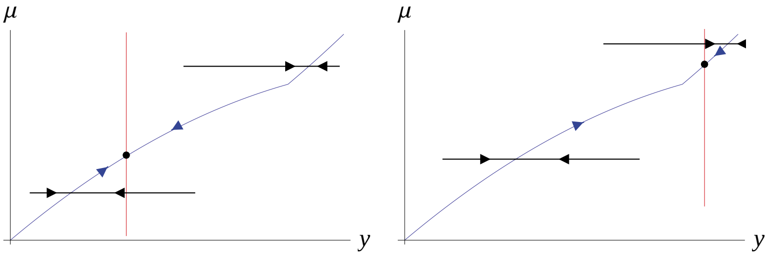
$$\begin{aligned} x' &= 1 - x - \epsilon A|x-y|x \\ y' &= \epsilon(\mu - y - A|x-y|y) \\ \mu' &= \epsilon\delta(1 + ax - by), \end{aligned}$$

where $\epsilon, \delta \ll 1$ are small parameters and $a, b > 0$.

Using GSP (Glendinning's reduction), the equations reduce to:

$$\begin{aligned} \dot{y} &= \mu - y - A|1-y|y \\ \dot{\mu} &= \delta(1 + a - by). \end{aligned}$$

$A < 1$



(a) $\dfrac{1+a}{b} < 1$ (b) $\dfrac{1+a}{b} > 1$

Figure: Possible phase spaces for $A < 1$. The red line is the μ nullcline. The black arrows indicate fast dynamics, and the blue arrows indicate slow dynamics.

Goal for $A > 1$

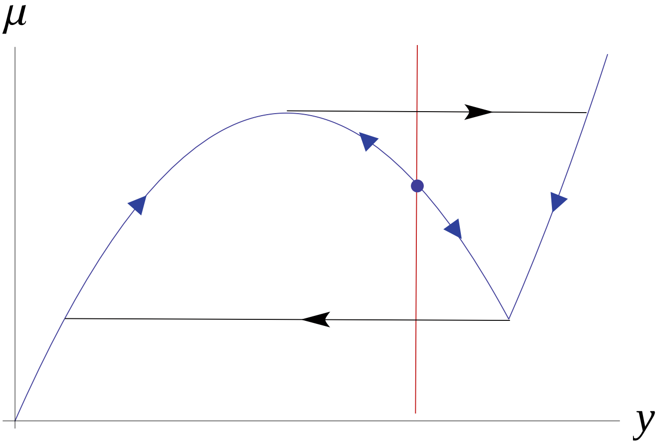


Figure: Limit cycle in the singular ($\delta = 0$) limit.

Question: What happens when the μ-nullcline (red) is close to the fold or the corner?

Canards in Piecewise Smooth Systems

We consider systems of the form

$$\begin{aligned} \dot{x} &= -y + F(x) \\ \dot{y} &= \epsilon(x - \lambda) \end{aligned} \tag{1}$$

where

$$F(x) = \begin{cases} g(x) & x \leq 0 \\ h(x) & x \geq 0 \end{cases}$$

with $g, h \in C^k,\ k \geq 1$, $g(0) = h(0) = 0$, $g'(0) < 0$ and $h'(0) > 0$, and we assume that h has a maximum at $x_M > 0$. The critical manifold

$$N_0 = \{y = F(x)\}$$

is '2'-shaped with a smooth fold at x_M and a corner along the splitting line $x = 0$.

Shadow Systems

We will assume that $h(x)$ can be extended (as far as necessary) into the region where $x < 0$ and define the shadow system to be

$$\begin{aligned} \dot{x} &= -y + h(x) \\ \dot{y} &= \epsilon(x - \lambda). \end{aligned} \tag{2}$$

Lemma (Roberts and Glendinning (2013))

Consider the trajectory $\gamma_n(t) = (x_n(t), y_n(t))$ of (1) that cross the y-axis entering the left half-plane $x < 0$ at $\gamma_n(0) = (0, y_c)$. Also consider the analogous trajectory γ_s of the shadow system (2). Then, the distance from the origin of γ_n is bounded by that of γ_s.

Figure for Shadow System Bound Lemma

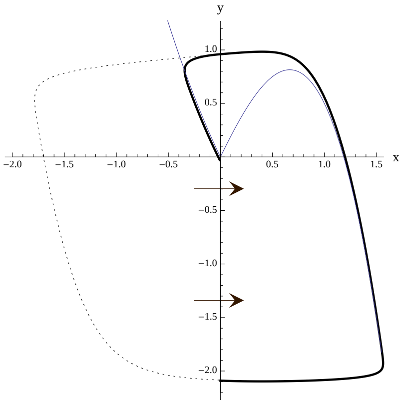


Figure: The dashed curve is a periodic orbit of the shadow system. The bold curve is the trajectory in the nonsmooth system. There is a positively invariant set enclosed by the bold curve and the *y*-axis.

Canards at the Smooth Fold

Theorem (Roberts and Glendinning (2013))

Fix $0 < \epsilon \ll 1$. In system (1), assume $g(0) = 0 = h(0)$, $h'(0) > 0$, and $g'(0) < 0$. Then there is a Hopf bifurcation when $\lambda = x_M$. If the Hopf bifurcation is non-degenerate, then it will produce canard cycles. Furthermore, these canard cycles are bounded by the stable canard orbits of the shadow system.

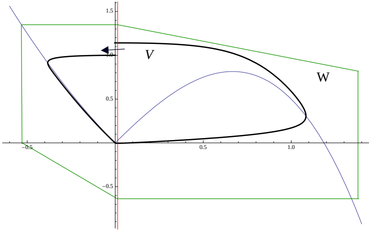
Corner Canards and Super-explosion

Theorem (Roberts and Glendinning (2013))

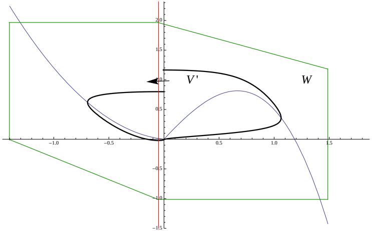
In system (1), assume $g(0) = 0 = h(0)$, $h'(0) > 0$, and $g'(0) < 0$. The system undergoes a bifurcation for $\lambda = 0$ by which a stable periodic orbit $\Gamma^n(\lambda)$ exists for $0 < \lambda < x_M$. There exists an ϵ_0 such that for all $0 < \epsilon < \epsilon_0$ the nature of the bifurcation is described by the following:

- (i) *If $0 < h'(0) < 2\sqrt{\epsilon}$, then canard cycles $\Gamma^n(\lambda)$ are born of a Hopf-like bifurcation as λ increases through 0. The bifurcation is subcritical if $|g'(0)| < |h'(0)|$ and supercritical if $|g'(0)| > |h'(0)|$.*
- (ii) *If $h'(0) > 2\sqrt{\epsilon}$, the bifurcation at $\lambda = 0$ is a super-explosion. The system has a stable periodic orbit $\Gamma^n(\lambda)$, and $\Gamma^n(\lambda)$ is a relaxation oscillation. If $|g'(0)| \geq 2\sqrt{\epsilon}$, the bifurcation is supercritical in that no periodic orbits appear for $\lambda < 0$. However, if $|g'(0)| < 2\sqrt{\epsilon}$ the bifurcation is subcritical, in that a stable periodic orbit and stable critical point coexist simultaneously for some $\lambda < 0$.*

Super-explosion Figures



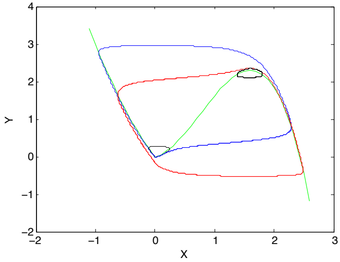
(a) Supercritical super-explosion $\lambda = 0.014$.



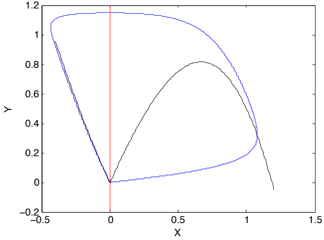
(b) Subcritical super-explosion $\lambda = -0.05$.

Figure: Positively invariant sets demonstrating the existence of attracting periodic orbits for super-explosion.

Possible Canard-related Periodic Orbits



(a) Nonsmooth canard cycles in the supercritical case.



(b) The stable orbit of a super-explosion (blue) for $\epsilon = 0.2$. The line $x = \lambda$ (red) is the slow nullcline. Here $\lambda = 0.001$.

Figure: Canard orbits and super-explosion in nonsmooth systems.

Back to the Modified Stommel Model

Recall: The equations for the model are

$$\begin{aligned}\dot{y} &= \mu - y - A|1-y|y\\ \dot{\mu} &= \delta(1+a-by).\end{aligned}$$

To simplify the analysis, we reformulate them as

$$\begin{aligned}\dot{y} =& \mu - y - A|1-y|y\\ \dot{\mu} =& \delta_0(\lambda - y),\end{aligned}$$

where $\delta_0 = b\delta$ and $\lambda = (1+a)/b$.

Dynamics in Modified Stommel Model: $A > 1$

Theorem

Assume $A > 1$, $0 < \delta \ll 1$, and $\lambda > 0$ is fixed in the modified Stommel model. Then the following statements hold:

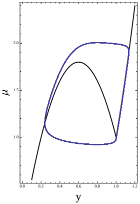
(A) *For $\lambda \geq 1$, there is a globally attracting equilibrium in the haline state.*

(B) *For $(1 + A)/(2A) < \lambda < 1$ the equilibrium is unstable and surrounded by a unique stable periodic orbit created through a non-smooth bifurcation at $\lambda = 1$.*
(i) When $A < 1 + 2\sqrt{\delta}$, the bifurcation creates non-smooth canard cycles.
(ii) When $A > 1 + 2\sqrt{\delta}$, the bifurcation is a super-explosion and the periodic orbit is a relaxation oscillation for

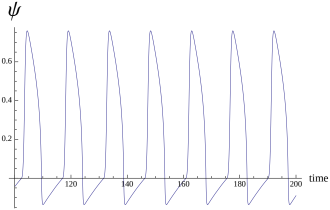
$$\frac{1 + A + 2\sqrt{\delta}}{2A} < \lambda < 1.$$

(C) *For $\lambda \leq (1 + A)(2A)$ there is an attracting equilibrium in the thermal state.*

Oscillatory Behavior in the Modified Stommel Model



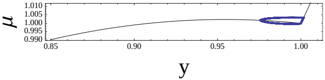
(a) Stable periodic orbit when $A = 5$, $\lambda = 0.8$, and $\delta = 0.1$



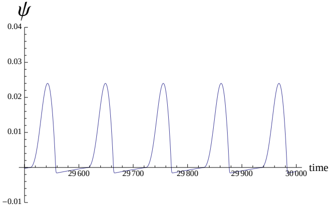
(b) Time series for ψ.

Figure: Relaxation Oscillations

Oscillatory Behavior in the Modified Stommel Model



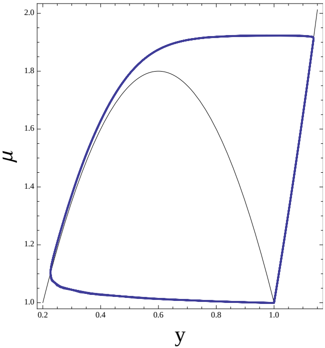
(a) Canard trajectory when $A = 1.1$, $\lambda = 0.995$, and $\delta_0 = 0.01$.



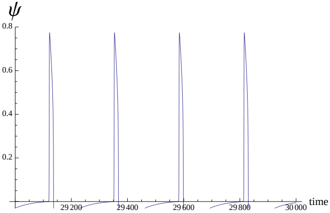
(b) Time series for ψ.

Figure: Canard Cycle

Oscillatory Behavior in the Modified Stommel Model



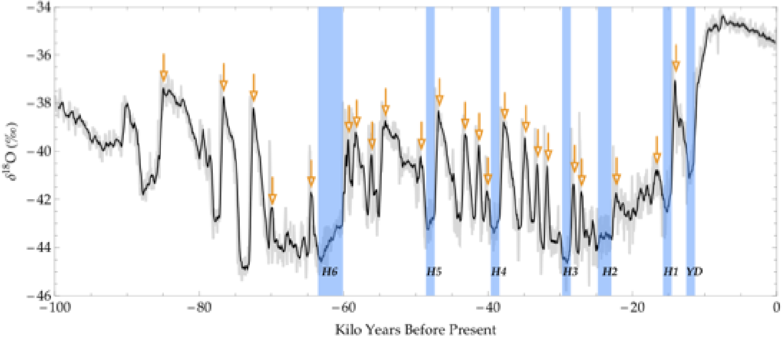
(a) Super-explosion when $A = 5$, $\lambda = 0.995$, and $\delta_0 = 0.1$.



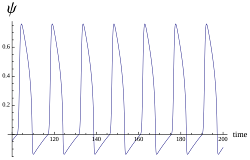
(b) Time series for ψ.

Figure: Super explosion - Relaxation Oscillations

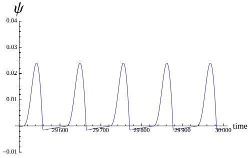
For Comparison



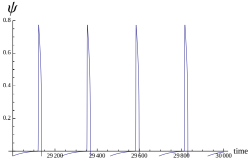
(a) From Saha (2011).



(b) ROs.



(c) Canard Cycles



(d) Super-explosion.

An Extra Dimension

Previously, we examined a modified version of Stommel's model with 1 slow variable (a ratio of forcing terms). Separating the forcing terms produces the model:

$$
\begin{aligned}
\frac{dx}{dt} &= z - x - \epsilon A\left|x - y\right| x \\
\frac{dy}{dt} &= \epsilon(u - y - A\left|x - y\right| y) \\
\frac{dz}{dt} &= \epsilon\delta(ay - bx + c) \\
\frac{du}{dt} &= \epsilon\delta(px - qy + r).
\end{aligned}
$$

This is again a 3 time-scale model with x fast and y intermediate. However, now there are two slow variables z and u.

Goal: Again, we would like to prove that there is an attracting periodic orbit.

Conditions for ROs in $\mathbb{R}^3$

Theorem (Szmolyan and Wechselberger (2004))

Assume a smooth fast/slow system with small parameter $0 < \epsilon \ll 1$ satisfies the following conditions:

(A1) *The critical manifold is 'S'-shaped,*

(A2) *the fold curves $L^{\pm}$ are given as graphs $(y^{\pm}(z), z, u^{\pm}(z))$ for $y \in I^{\pm}$ for certain intervals $I^{\pm}$ where the points on the fold curves $L^{\pm}$ are jump points,*

(A3) *the reduced flow near the fold curves is directed towards the fold curves,*

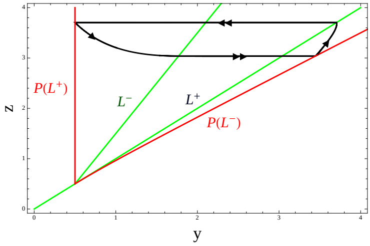
(A4) *the reduced flow is transversal to the curve $P(L^{\pm})|_{I^{\pm}}$, and*

(A5) *there exists a hyperbolic singular periodic orbit Γ.*

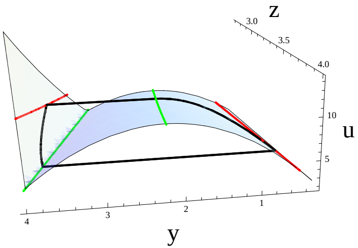
Then there exists a locally unique hyperbolic relaxation orbit close to the singular orbit Γ for ϵ sufficiently small.

An Extra Dimension

Question: Can we prove analogous theorems for ROs in an MMOs in higher dimensional nonsmooth systems?



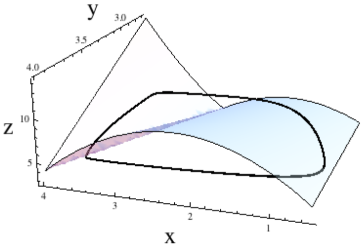
(e) Projection of the singular orbit onto the critical manifold, with projections of the fold lines.



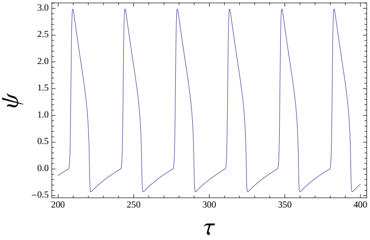
(f) Singular orbit Γ in the full 3D phase space. Colored lines correspond to those in (a).

Figure: Attracting singular periodic orbit in a more complex modification of Stommel's model.

Evidence for ROs in More Complex Model



(a) The stable periodic orbit in phase space.



(b) Time series for ψ for the orbit in (a).

Figure: Example of the stable periodic orbit for $\delta = 0.1$, $\gamma = 1$, $\alpha = 0.5$, $\beta = 1.75$, $m = 2$, $\rho = 0.5$, and $k = 1.5$.

Conclusion

Accomplished:

- Analyzed a large-scale ocean circulation model to find conditions for ROs
- Developed a theory for canards and super-explosion in nonlinear piecewise-smooth planar systems
- Nonsmooth nature of the model plays a role in the asymmetry between warming/cooling

Future Motivation:

- Generalize theorem for ROs in $\mathbb{R}^3$ to nonsmooth systems.

Thank you!!!