Welander's Ocean Box Model as a Nonsmooth System

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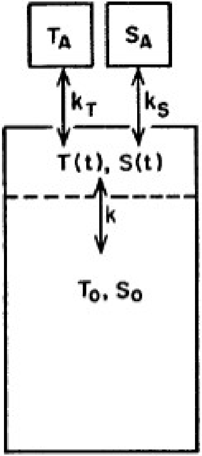
Outline

- Welander's analysis of his model
- Strangeness in the nonsmooth version
- Nonsmooth Analysis basics
- The "Blow Up" Method

Welander's Model

The Model:

$$\begin{aligned}\dot{T} &= k_T(T_A - T) - k(\rho)T \\ \dot{S} &= k_S(S_A - S) - k(\rho)S \\ \rho &= -\alpha T + \gamma S\end{aligned}$$



Welander, Pierre. "A simple heat-salt oscillator." Dynamics of Atmospheres and Oceans 6.4 (1982): 233-242.

Welander's Model

- This is a two dimensional system. Any rectangle with $|T| > T_A$ and $|S| > S_A$ is invariant under the flow.

Welander's Model

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- Therefore, if there exists only one equilibrium, and if it is unstable, Poincare-Bendixson allows us to conclude the existence of a periodic solution.

Welander's Model

The linearized system is

$$\begin{bmatrix} w' \\ z' \end{bmatrix} = \begin{bmatrix} -(k_T + k(\overline{\rho})) + \overline{T}\alpha k'(\overline{\rho}) & -\overline{T}\gamma k'(\overline{\rho}) \\ \overline{S}\alpha k'(\overline{\rho}) & -(k_S + k(\overline{\rho})) - \overline{S}\gamma k'(\overline{\rho}) \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}$$

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We want an equilibrium that is completely unstable, so we'd like the trace and determinant of the matrix to be positive. Parameters can be chosen to satisfy this condition.

Welander's Model

Welander then gives a specific example:

$$k = \frac{2k_1}{\pi}[\arctan m(\rho + \rho_1) + \pi/2]$$

Nondimensionalizing and letting

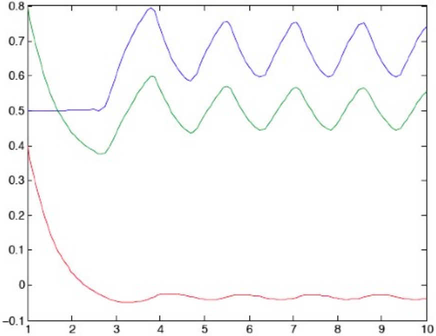
$$\frac{\alpha T_A}{\gamma S_A} = \frac{4}{5}, \quad \frac{k_T}{k_S} = 2, \quad \frac{k_1}{k_T} = \frac{1}{2}, \quad \frac{\rho_1}{\gamma S_A} = \frac{1}{30}, \quad m\gamma S_A = 500$$

We get the equations

$$\begin{aligned} \dot{T}^* &= 1 - T^* - k^*(\rho^*)T^* \\ \dot{S}^* &= 0.5(1 - S^*) - k^*(\rho^*)S^* \\ \rho^* &= -0.8T^* + S^* \end{aligned}$$

Welander's Model

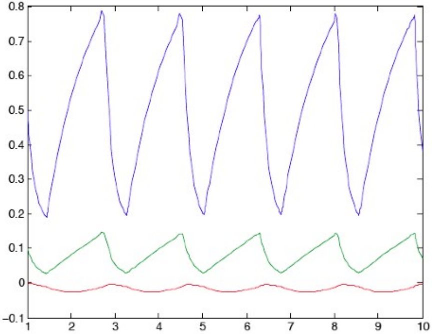
Numerically:



The top line is temperature, the middle is salinity, and the bottom is density.

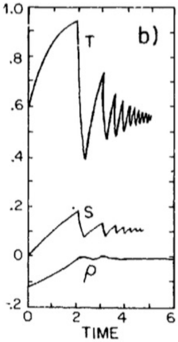
Welander's Model

Welander also looks at the case where $k = \begin{cases} 0 & \rho < \varepsilon \\ 5 & \rho > \varepsilon \end{cases} \quad (\varepsilon \neq 0)$



Welander's Model

But when $\varepsilon = 0$,

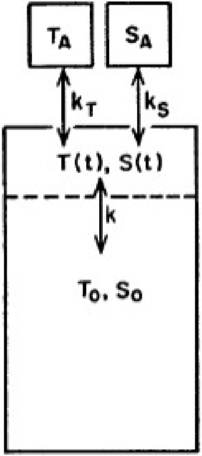


So what's going on in this nonsmooth model?

The Nonsmooth Model

$$
\begin{aligned}
\dot{T} &= 1 - T - k(\rho)T \\
\dot{S} &= \beta(1 - S) - k(\rho)S \\
\rho &= -\alpha T + S
\end{aligned}
$$

where $k(\rho) = \begin{cases} 1 & \rho > \varepsilon \\ 0 & \rho < \varepsilon \end{cases}$



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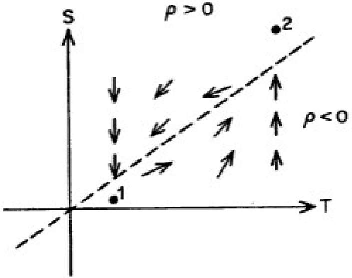
The Nonsmooth Model

The model is linear on each side of the discontinuity. The equilibria don't exist in regions for which they are equilibria, setting up an oscillation.



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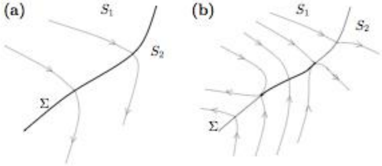
Nonsmooth Analysis

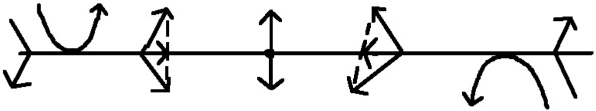
There is an intuitive way to piece together dynamics in a nonsmooth model with a line of discontinuity, which was originally formulated by Filippov.



Di Bernardo, Mario, et al. "Bifurcations in nonsmooth dynamical systems." SIAM review (2008): 629-701.

Nonsmooth Analysis

Taking a convex combination of the systems on either side of the discontinuity gives a way to find a flow in the sliding region.



Nonsmooth Analysis

The set up:
Consider the system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \lambda)$$

with discontinuity boundary given defined by the zero set of a scalar function $h(\mathbf{x})$.

$$\lambda = \begin{cases} 1 & h(\mathbf{x}) > 0 \\ -1 & h(\mathbf{x}) < 0 \end{cases} \quad \text{On } h(\mathbf{x}) = 0,\ \lambda \in [-1, 1]$$

The standard Filippov formulation would be

$$\dot{\mathbf{x}} = \frac{1}{2}(1+\lambda)f^{+}(\mathbf{x}) + \frac{1}{2}(1-\lambda)f^{-}(\mathbf{x})$$

Nonsmooth Analysis

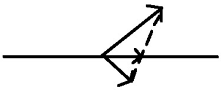
A sliding solution is defined as follows:

If

$$\begin{aligned} 0 &= f(\mathbf{x},\lambda)\cdot\nabla h(\mathbf{x}) \\ 0 &= h(\mathbf{x}) \end{aligned}$$

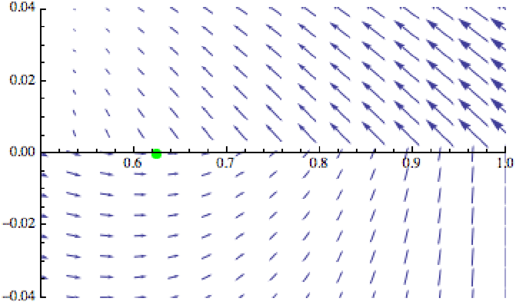
can be solved for some $\lambda^* \in [-1,1]$, then $\dot{\mathbf{x}} = f(\mathbf{x},\lambda^*)$ defines a sliding solution of the system.

Note that no sliding solutions of the Filippov formulation exist in crossing regions.



The Nonsmooth Model

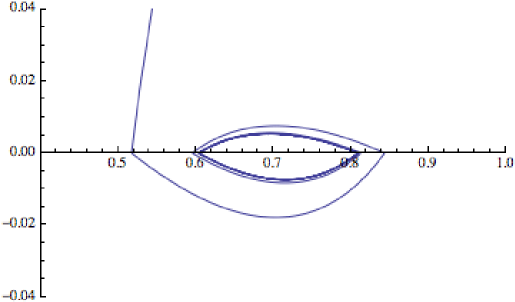
In the nonsmooth model, the sliding region transitions from unstable to stable as $\varepsilon = 0$.



The point at which the tangencies collide was studied by Filippov, and is called a fused focus.

The Nonsmooth Model

We see a nonsmooth version of a supercritical Hopf bifurcation.



Nonlinear Sliding

But the story is more complicated...
One doesn't need to define the vector field on the boundary in terms of the convex combination. If $f(\mathbf{x}, \lambda)$ is already defined in terms of a nonsmooth parameter λ, then solving

$$\begin{aligned} 0 &= f(\mathbf{x}, \lambda) \cdot \nabla h(\mathbf{x}) \\ 0 &= h(\mathbf{x}) \end{aligned}$$

for $\lambda \in [-1, 1]$ gives nonlinear sliding solutions. They don't need to be unique.

Nonlinear Sliding

$$\dot{x} = \begin{pmatrix} 1 \\ 2 - \lambda - x \end{pmatrix} - 2(1 - \lambda^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$f^+ = \begin{pmatrix} 1 \\ 1 - x \end{pmatrix}$$

$$f^- = \begin{pmatrix} 1 \\ 3 - x \end{pmatrix}$$

so the Filippov sliding region is $1 \leq x \leq 3$

Nonlinear sliding solves $f \cdot \nabla h = 0$, giving a condition on λ

$$\lambda = \frac{1 \pm \sqrt{1 + 8x}}{4}$$

so a sliding solution exists on $x \geq -\frac{1}{8}$

A Question

Does the fused focus bifurcation correspond to a smooth supercritical Hopf bifurcation?

The "Blow Up" System

The fused focus location is highly singular, $y = a = \varepsilon = 0$.

$$\begin{aligned} \dot{x} &= 1 - x - kx \\ \dot{y} &= \beta - \beta\varepsilon - k\varepsilon - \alpha - (\beta + k)y - (\alpha\beta - \alpha)x \end{aligned}$$

$$k(y) = \frac{1}{\pi}\tan^{-1}\left(\frac{y}{a}\right) + \frac{1}{2}.$$

as $a \to 0$

$$k \to \begin{cases} 1 & y > 0 \\ 0 & y < 0 \end{cases}$$

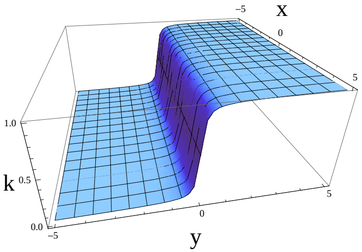
The "Blow Up" System

$$k(y) = \Phi\left(\frac{y}{a}\right)$$

where

$$\Phi(z) = \frac{1}{\pi}\tan^{-1}(z) + \frac{1}{2}$$

$$\dot{k} = \frac{d\Phi}{dz}\frac{dz}{dy}\frac{dy}{dt} = \frac{1}{a}\Phi'(z)\dot{y} = \frac{1}{a}\Phi'(\Phi^{-1}(k))\dot{y}$$



The "Blow Up System"

$$\begin{aligned}\dot{x} &= 1 - x - kx \\ \dot{k} &= \frac{1}{a}\Psi(k)\left[\beta - \beta\varepsilon - k\varepsilon - \alpha + (\beta + k)\,(a\cot(\pi k)) - (\alpha\beta - \alpha)x\right]\end{aligned}$$

Note: for $a \neq 0$, this system is equivalent to the original system, it is not an arbitrary smoothing of the system!

Analysis of the Blow Up System

We use geometric singular perturbation theory to analyze the system.

$$\begin{aligned} \dot{x} &= a(1 - x - kx) \\ \dot{k} &= \Psi(k)\left[\beta - \beta\varepsilon - k\varepsilon - \alpha + (\beta + k)\left(a\cot(\pi k)\right) - (\alpha\beta - \alpha)x\right] \end{aligned}$$

The critical manifold:

$$x = \frac{\beta - \beta\varepsilon - k\varepsilon - \alpha}{\alpha\beta - \alpha}$$

For small $a \neq 0$, and $\varepsilon \neq 0$, the equilibrium perturbs and maintains its stability.

Analysis of the Blow Up System

The critical manifold is a vertical line of fixed points when $\varepsilon = 0$.

However, when $\varepsilon \to 0$, the intersection of the critical manifold with the nullcline $x = \dfrac{1}{1+k}$ is unique.

$$(x, k, a, \varepsilon) = \left(\frac{3}{4}, \frac{1}{3}, 0, 0\right)$$

Analysis of the Blow Up System

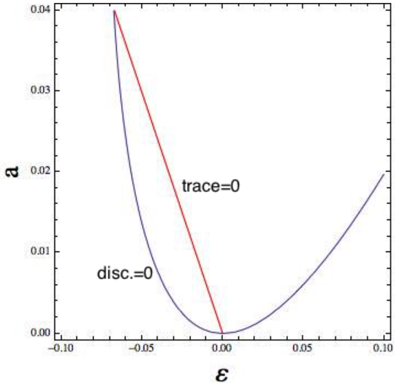
We can calculate Taylor Expansion around the point $(x, k, a, \varepsilon) = \left(\frac{3}{4}, \frac{1}{3}, 0, 0\right)$.

The Jacobian calculated from the expansion, and evaluated at the point is

$$J = \begin{bmatrix} -\frac{4}{3}a & -\frac{3}{4}a \\ \frac{3}{10\pi} & -\left(\frac{5}{4\sqrt{3}} + \frac{3}{4\pi}\right)\varepsilon + \left(-\frac{5}{12} + \frac{\sqrt{3}}{4\pi}\right)a \end{bmatrix}$$

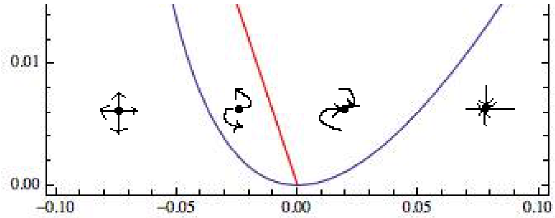
Analysis of the Blow Up System

Plotting the curves where the trace and discriminant of the Jacobian are zero, we get



Analysis of the Blow Up System

The fused focus perturbs to a smooth supercritical Hopf bifurcation.



Remarks and Questions

- Welander's Model exhibits oscillatory behavior in certain parameter regions.
- The nonsmooth model has a fused focus Hopf bifurcation.
- This bifurcation perturbs to a standard smooth supercritical Hopf bifurcation.
- Is there any general theory which will indicate under what conditions the bifurcations structures of smooth systems persist in a limiting nonsmooth system?

References

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