


An Introduction to Planetary Energy Balance

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Mathematics of Climate Seminar
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Energy Balance



Conservation of Energy

temperature change ~ energy in - energy out

short wave energy
from the Sun

long wave energy
from the Earth

Everything else is detail.

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Energy Balance

Stefan-Boltzmann Law



power flux (W/m²)

$F = \sigma T^4$

temperature (K)

Stefan-Boltzmann constant
 $\sigma \approx 5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4$

Reasonable approximation:
Every body in the solar system radiates energy according to this law.

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Energy Balance

Stefan-Boltzmann Law

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$F = \sigma T^4$



temperature (K)

Stefan-Boltzmann constant
 $\sigma \approx 5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4$

Example

surface temperature of the Sun: 5780K
power flux: $5.67 \times 10^{-8} \times (5780)^4 =$
 $6.33 \times 10^7 \text{ W/m}^2$

total solar power output: $6.33 \times 10^7 \times 4\pi(r_s)^2$,
where $r_s =$ radius of the sun = $6.96 \times 10^8 \text{ m}$
total solar output: $3.85 \times 10^{26} \text{ W}$

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Energy Balance

Insolation

Solar flux at a distance r from the sun:



$$F = \frac{6.33 \times 10^7 4\pi r_s^2}{4\pi r^2} = 6.33 \times 10^7 \left(\frac{r_s}{r}\right)^2 \text{ W/m}^2$$

$r_s = 6.96 \times 10^8 \text{ m}$
 $r = 1.5 \times 10^{11} \text{ m}$
 $F = 1368 \text{ W/m}^2$ ← solar flux at Earth's orbit

Power intercepted by the Earth: $F \times \pi r_e^2 \text{ W}$

Earth's surface area: $4\pi r_e^2 \text{ m}^2$

Average surface flux: $\frac{F \times \pi r_e^2}{4\pi r_e^2} = \frac{F}{4} = 342 \text{ W/m}^2$

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Energy Balance

Insolation

Global Average Insolation
(Incoming solar radiation)

intercepted flux: $F = 1368 \text{ W/m}^2$
Earth cross-section: πr_e^2
surface area: $4\pi r_e^2$
average flux: $1368/4 = 342 \text{ W/m}^2 = Q$

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

$$Q = \sigma T^4$$

$$T = (Q/\sigma)^{1/4} = (342/5.67 \times 10^{-8})^{1/4}$$



$$= 279 \text{ K} = 6^\circ \text{ C} = 43^\circ \text{ F}$$

Dynamics


heat capacity

$R \frac{dT}{dt} = Q - \sigma T^4$

stable equilibrium

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Energy Balance

Goldilocks Zone

Solar flux at a distance r from the Sun:


$$F = \frac{6.33 \times 10^7 4\pi r_s^2}{4\pi r^2} = 6.33 \times 10^7 \left(\frac{r_s}{r}\right)^2 \text{ W/m}^2$$

$$r_s = 6.96 \times 10^8 \text{ m}$$

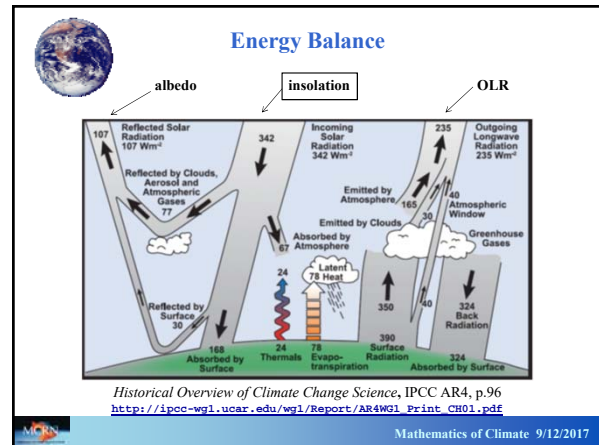
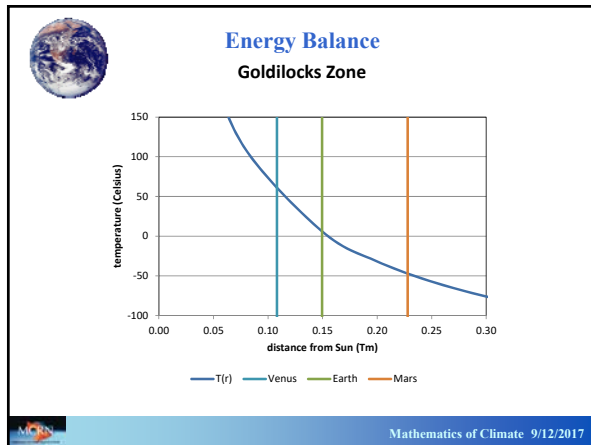
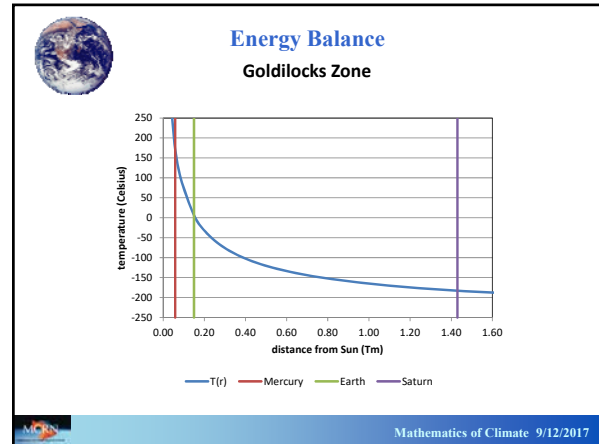

$$F = \frac{3.07 \times 10^{24}}{r^2} \text{ W/m}^2$$

Average surface flux: $\frac{3.07 \times 10^{24}}{4r^2} = \frac{7.67 \times 10^{24}}{r^2} \text{ W/m}^2$

Black body temperature: $\sigma T^4 = \frac{7.67 \times 10^{24}}{r^2} \text{ W/m}^2$

$$T = \left(\frac{7.67 \times 10^{24}}{\sigma r^2}\right)^{1/4} = \frac{1.078 \times 10^8}{\sqrt{r}}$$


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Energy Balance

Insolation

Global Average Insolation
 (Incoming solar radiation)

intercepted flux: $F = 1368 \text{ W/m}^2$

Earth cross-section: πr_e^2

surface area: $4\pi r_e^2$

average flux: $1368/4 = 342 \text{ W/m}^2 = Q$

Simple Model

Assume that Earth is a perfectly thermally conducting black body.

$$Q = \sigma T^4$$


$$T = (Q/\sigma)^{1/4} = (342/5.67 \times 10^{-8})^{1/4}$$

$$= 279\text{K} = 6\text{ C} = 43\text{ F}$$

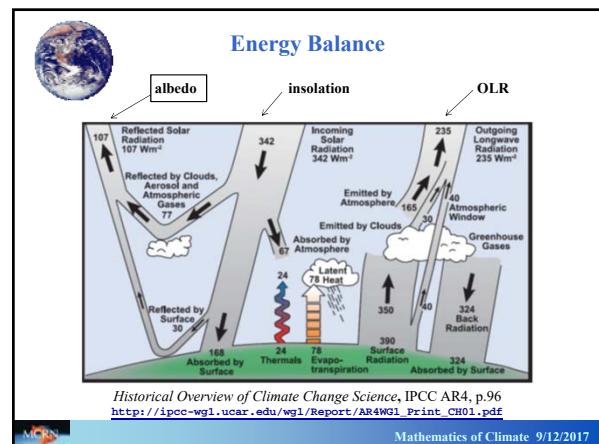
Dynamics

$$R \frac{dT}{dt} = Q - \sigma T^4$$

heat capacity \rightarrow $R \frac{dT}{dt}$ \leftarrow stable equilibrium



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Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α .
For Earth, $\alpha \approx 0.3$.

Simple Model

Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.


$$T = (0.7 \cdot F / \sigma)^{1/4} = (0.7 \cdot 342 / 5.67 \times 10^{-8})^{1/4}$$

$$= 255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}$$

Dynamics

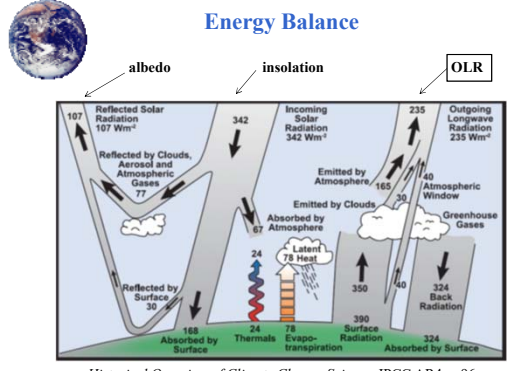
$$R \frac{dT}{dt} = Q(1 - \alpha) - \sigma T^4$$

stable equilibrium



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Energy Balance



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

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Energy Balance

OLR as a Function of Surface Temperature (Outgoing Longwave Radiation)

$$\text{OLR} \approx A + BT$$

A and B are determined from satellite observations.
 T is surface temperature (in Celsius).


$$A = 202 \text{ W/m}^2$$

$$B = 1.90 \text{ W/m}^2\text{K}$$

Dynamics

Kelvin \rightarrow $R \frac{dT}{dt} = Q(1 - \alpha) - \sigma T^4$ **photosphere temperature**

Celsius \rightarrow becomes $R \frac{dT}{dt} = Q(1 - \alpha) - (A + BT)$ **global mean surface temperature**



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Energy Balance

OLR as a Function of Surface Temperature

$$\text{OLR} \approx A + BT$$


Important:
 $A + BT$ is **not** a linear approximation to the Stefan-Boltzmann equation.

Dynamics

Kelvin \rightarrow $R \frac{dT}{dt} = Q(1 - \alpha) - \sigma T^4$ **photosphere temperature**

Celsius \rightarrow becomes $R \frac{dT}{dt} = Q(1 - \alpha) - (A + BT)$ **global mean surface temperature**

different



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Energy Balance

Homogeneous Earth

$$R \frac{dT}{dt} = Q(1 - \alpha) - (A + BT)$$


Equilibrium Temperature: $Q(1 - \alpha) - A - BT_{eq} = 0$

$$T_{eq} = \frac{Q(1 - \alpha) - A}{B}$$

Stable, since $B > 0$.

Ice-free planet: $\alpha = 0.32$, $T_{eq} = 16^\circ\text{C}$
Snowball planet: $\alpha = 0.62$, $T_{eq} = -38^\circ\text{C}$
No glacier would form on an ice-free Earth.
No glacier would melt on a snowball Earth.

Easy question:
Why do we have ice caps?
Hard question:
If Earth was ever a snowball, how did we get out?



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Energy Balance

Latitude Dependence

Make T depend on $y = \sin(\text{latitude})$

$$R \frac{\partial T(y,t)}{\partial t} = Qs(y)(1 - \alpha) - (A + BT(y,t))$$

insolation distribution


Q = global annual average insolation = 342 W/m^2
 $s(y)$ = distribution across latitudes ($\int_0^1 s(y) dy = 1$)

One can show that

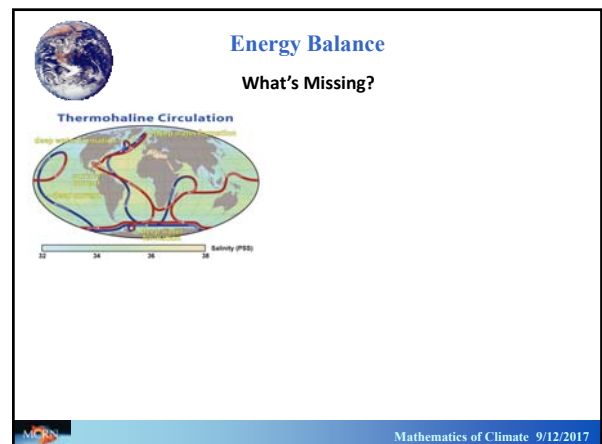
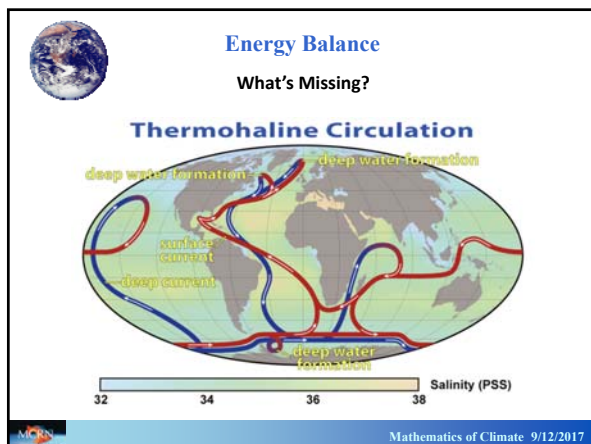
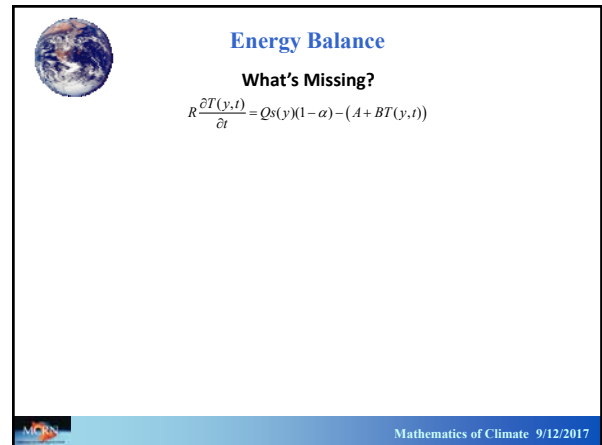
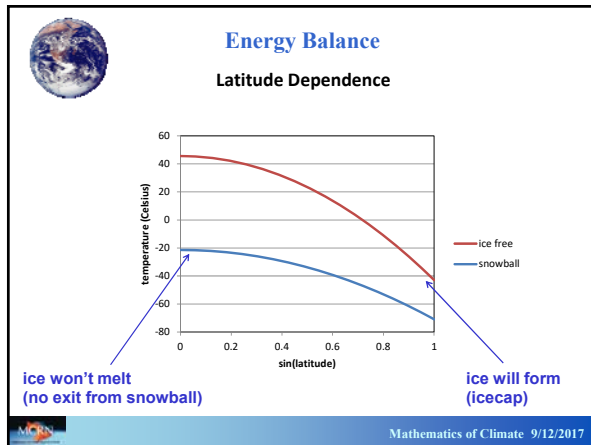
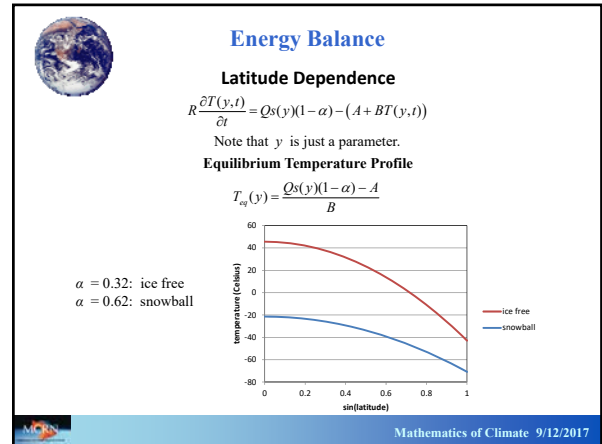
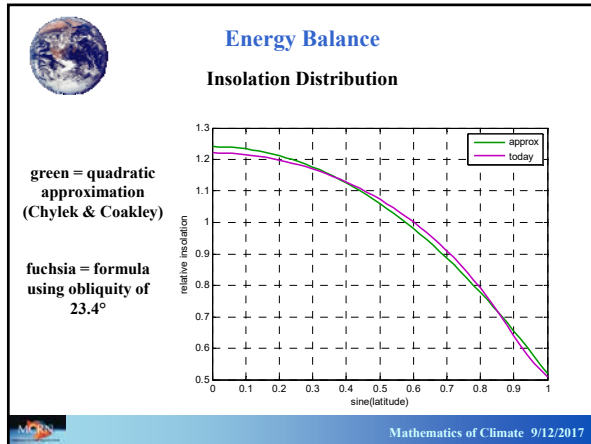
$$s(y) = \frac{2}{\pi} \int_0^{2\pi} \sqrt{1 - (1 - y^2 \sin^2 \beta \cos \theta - y \cos \beta)^2} d\theta$$

β = obliquity = 23.4°

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241(3y^2 - 1)$$


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Energy Balance

What's Missing?

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Energy Balance

What's Missing?

Thermohaline Circulation

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What's Missing?

<https://www.nytimes.com/2017/09/04/us/hurricane-irma.html>

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Energy Balance

What's Missing?

Weather!

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Energy Balance

Budyko's Model

$$R \frac{\partial T}{\partial t} = Qs(y)(1 - \alpha) - (A + BT) + C(T - T)$$

global mean temperature $\bar{T}(t) = \int_0^{\pi} T(y, t) dy$ *Weather*

Second Law of Thermodynamics:
Energy travels from hot places to cold places.

Budyko's equation as a dynamical system:
 T lives in a function space (temperature as a function of latitude).

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Budyko's Model


Why y ?

$$R \frac{\partial T(y, t)}{\partial t} = Qs(y)(1 - \alpha) - (A + BT(y, t)) + C(\bar{T}(t) - T(y, t))$$

global mean temperature: $\bar{T}(t) = \int_0^{\pi} T(y, t) dt$

Why do we use $y = \text{sine}(\text{latitude})$ instead of just latitude?

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Budyko's Model

Why y ?


$$R \frac{\partial T(y,t)}{\partial t} = Qs(y)(1 - \alpha) - (A + BT(y,t)) + C(\bar{T}(t) - T(y,t))$$

global mean temperature $\bar{T}(t) = \int_0^1 T(y,t) dy$

Why do we use $y = \text{sine}(\text{latitude})$ instead of just latitude?

Because y is directly proportional to surface area.

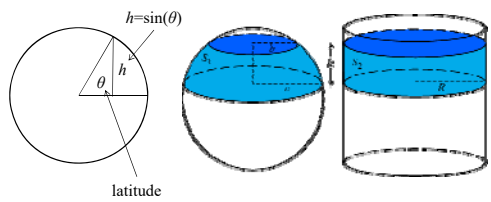
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Budyko's Model

Why $y = \text{sine}(\text{latitude})$?

Archimedes




$h = \sin(\theta)$

latitude

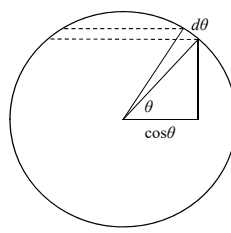
<http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html>

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Budyko's Model

Why $y = \text{sine}(\text{latitude})$?



surface area of a unit sphere $\int_{-\pi/2}^{\pi/2} 2\pi \cos \theta d\theta = 2\pi \sin \theta \Big|_{-\pi/2}^{\pi/2} = 4\pi$

average over the sphere of a function of latitude $f(\theta)$

$$\frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} f(\theta) 2\pi \cos \theta d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} f(\theta) \cos \theta d\theta$$


(substitute $y = \sin(\theta)$) $= \frac{1}{2} \int_{-1}^1 f(\arcsin y) dy$

average over the sphere of a function $T(y)$

$$\bar{T} = \frac{1}{2} \int_{-1}^1 T(y) dy$$

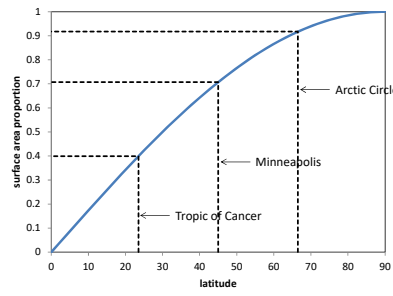
if T is symmetric across the equator: $\bar{T} = \int_0^1 T(y) dy$

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Budyko's Model

Why $y = \text{sine}(\text{latitude})$?



surface area proportion


latitude

Arctic Circle

Minneapolis

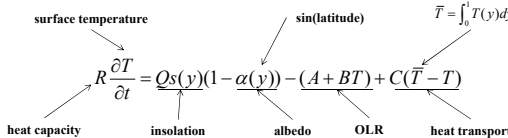
Tropic of Cancer

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Budyko's Model

Budyko's Equation



$$R \frac{\partial T}{\partial t} = Qs(y)(1 - \alpha(y)) - (A + BT) + C(\bar{T} - T)$$

heat capacity insolation albedo OLR heat transport

Symmetry assumption: $0 \leq y = \text{sine}(\text{latitude}) \leq 1$

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241(3y^2 - 1)$$

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