An Introduction to Energy Balance Models

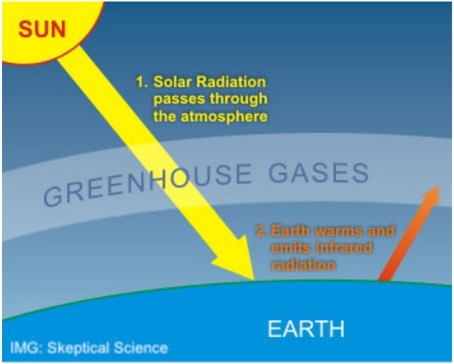
Alice Nadeau
(with a lot of slides from Dick McGehee)

University of Minnesota Mathematics of Climate Seminar

September 25, 2018

Conservation of energy

$$\text{temperature change} \sim \underbrace{\text{energy in}}_{\text{short wave radiation}} - \underbrace{\text{energy out}}_{\text{long wave radiation}}$$



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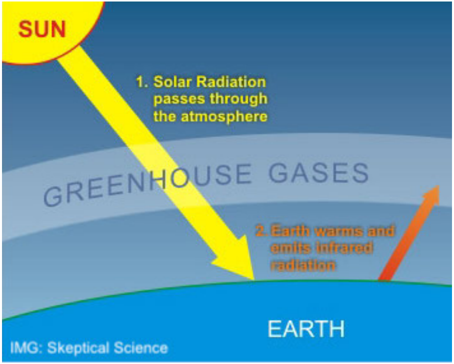
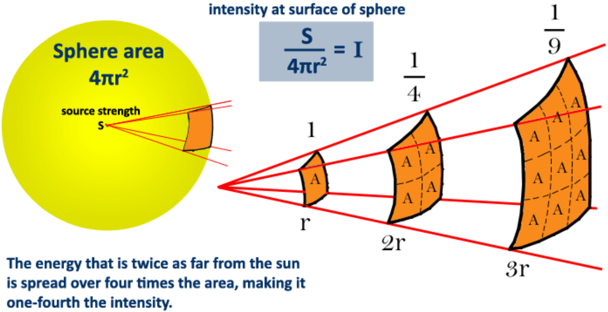


Everything else is detail!

Initial Thoughts

Annual radiation from the Sun $:= Q$

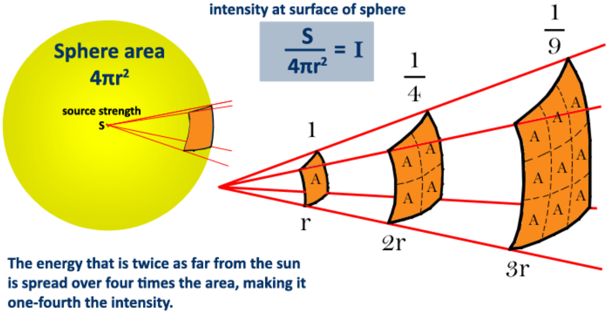
Finding Q



IDEAS, USBC Geography Dept.

$$I_{\text{Earth}} = \frac{\text{power flux} \cdot \text{surface area}}{4\pi r_{\text{Earth}}^2} = \frac{(\sigma T_{\text{Sun}})^4 (4\pi r_{\text{Sun}}^2)}{4\pi r_{\text{Earth}}^2} \approx 1368 \text{ W m}^{-2}$$

Finding Q



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$$Q = \frac{I_{\text{Earth}} \cdot \pi r_{\text{Earth}}^2}{4\pi r_{\text{Earth}}^2} \approx 342 \text{ W m}^{-2}$$

Initial Thoughts

Annual radiation from the Sun $:= Q$

Outgoing radiation $:= \sigma T^4$

$\rightarrow$ Stefan-Boltzmann Law

Dynamical Models

Perfect thermally conducting black body:

$$R\frac{dT}{dt} = Q - \sigma T^4$$

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Perfect thermally conducting black body plus albedo:

Albedo

Dynamical Models

Perfect thermally conducting black body:

$$R\frac{dT}{dt} = Q - \sigma T^4, \quad T^* = (Q/\sigma)^{1/4}$$

Perfect thermally conducting black body plus albedo:

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4, \quad T^* = ((1-\alpha)Q/\sigma)^{1/4}$$

Dynamical Models for *Surface Temperature*

Convert to surface temperature:

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT), \;\; T^* = ((1-\alpha)Q - A)/B$$

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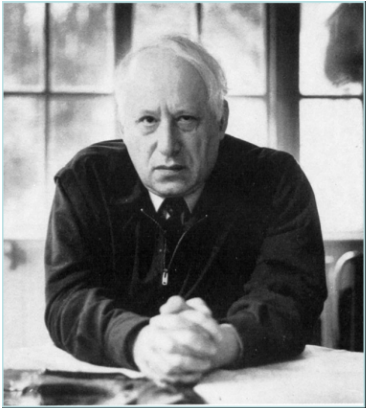
Include heat transport:

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t)) - C \cdot f(T), \;\; T^*(y) = \ldots$$

Budyko vs. Sellers

Mikhail I. Budyko

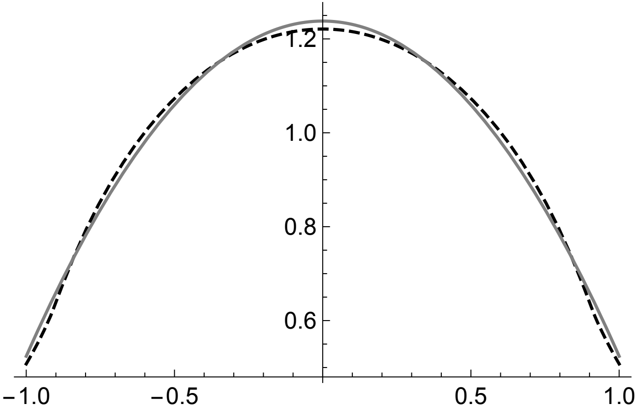


William D. Sellers

The Budyko Energy Balance Model

$$R\frac{\partial T}{\partial t} = \underbrace{Qs(y)(1-\alpha)}_{\text{incoming radiation}} - \underbrace{(A+BT(y,t))}_{\text{OLR}} - \underbrace{C\left(T(y,t) - \overbrace{\overline{T}(t)}^{\text{ann. avg. temp.}}\right)}_{\text{heat transport}}$$

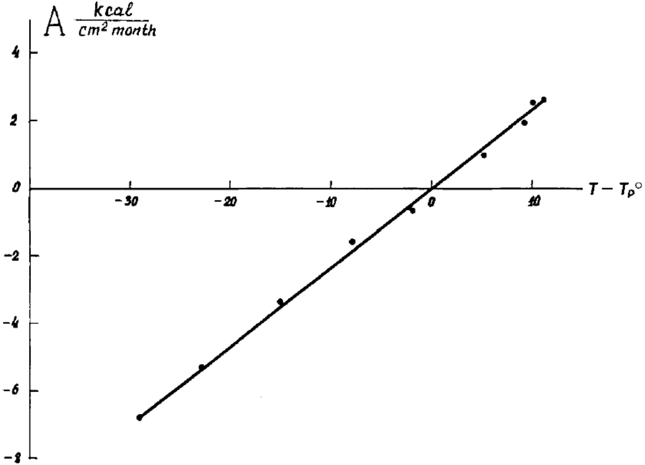
Incoming **Sol**ar Radi**ation** Distribution: $s(y)$



Dashed: from first principles, Solid: Quadratic approximation

Finding A, B and C

A, B, and C are empirical parameters



M. Budyko, "The effects of solar radiation variations on the climate of the Earth," *Tellus*, 21: 1969, 611–619.

$$R\frac{\partial T}{\partial t} = \underbrace{Qs(y)}_{\text{insolation}}\,(\underbrace{1-\alpha}_{\text{albedo}}) - \underbrace{(A + BT(y,t))}_{\text{OLR}} - \underbrace{C\left(T(y,t) - \overbrace{\overline{T}(t)}^{\int_0^1 T(y,t)dy} \right)}_{\text{heat transport}}$$

Incoming **So**lar Radi**ation Approximation:**

$$s(y) \approx 1 - 0.238(3y^2 - 1)$$

Symmetry assumption:

Equator $= 0 \leq y = \sin(\text{latitude}) \leq 1 =$ North Pole

Equilibrium Temperature Profile

$$0 = Qs(y)(1-\alpha) - (A + BT^*(y)) - C\left(T^*(y) - \overline{T^*}\right)$$

Equilibrium Temperature Profile

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Integrate to find $\overline{T^*}$:

$$\begin{aligned}
0 &= \int_0^1 \left[Qs(y)(1-\alpha) - (A + BT^*(y)) - C\left(T^*(y) - \overline{T^*}\right)\right] dy \\
&= Q \underbrace{\int_0^1 s(y)dy}_{1} - Q \underbrace{\int_0^1 s(y)\alpha dy}_{\overline{\alpha}} - A \underbrace{\int_0^1 dy}_{1} - B \underbrace{\int_0^1 T^*(y)dy}_{\overline{T^*}} \\
&\qquad - \underbrace{C\int_0^1 T^*(y)dy + C\int_0^1 \overline{T^*}dy}_{0} \\
&= Q(1-\overline{\alpha}) - (A + B\overline{T^*})
\end{aligned}$$

$\Rightarrow$**Equilibrium Global Mean Temperature:** $\overline{T^*} = \frac{1}{B}(Q(1-\overline{\alpha}) - A)$

Equilibrium Temperature Profile

$$0 = Qs(y)(1 - \alpha) - (A + BT^*(y)) - C\left(T^*(y) - \overline{T^*}\right)$$

$$\overline{T^*} = \frac{1}{B}(Q(1 - \overline{\alpha}) - A)$$

Equilibrium Temperature Profile

$$0 = Qs(y)(1-\alpha) - (A + BT^*(y)) - C\left(T^*(y) - \overline{T^*}\right)$$

$$\overline{T^*} = \frac{1}{B}(Q(1-\overline{\alpha}) - A)$$

Plug in $\overline{T^*}$ and solve for $T^*(y)$:

$$T^*(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha) - A + C\overline{T^*}\right))$$

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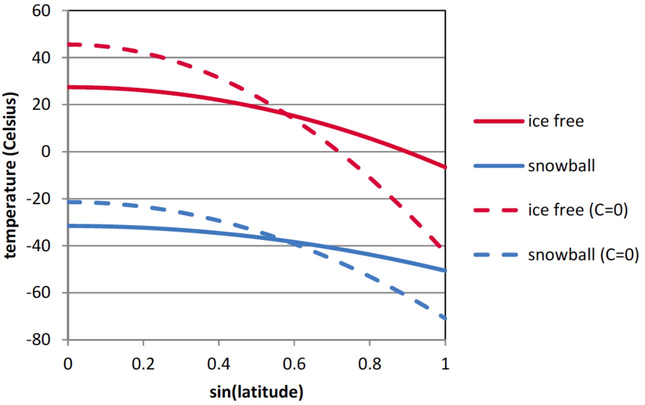
$\alpha = 0.32$

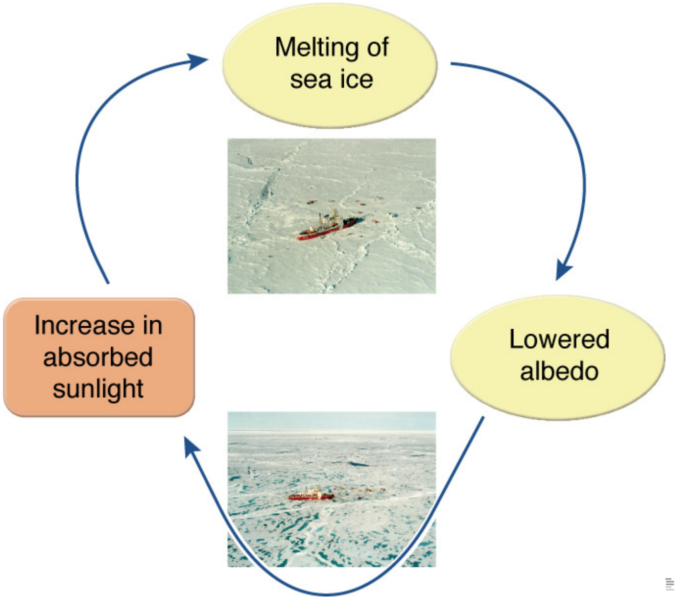
$\alpha = 0.62$

$C = 3.04$



From McGehee, Climate Seminar Sept. 19, 2017

Ice-Albedo Feedback



Non-uniform Albedo

$$R\frac{\partial T}{\partial t} = Qs(y)(1 - \underbrace{\alpha(y,\eta)}_{\text{albedo depends on latitude}}) - (A + BT(y,t)) - C\left(T - \overline{T^*}\right)$$

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Ice Line Assumption: There is one ice line, η, in the northern hemisphere north of which there is always ice.

$$\alpha(y,\eta) = \begin{cases} \alpha_1 & 0 \le y < \eta \\ \alpha_2 & \eta < y \le 1 \end{cases}, \qquad \alpha 1 < \alpha_2$$

Equilibrium Temperature Profile depends on the Ice Line

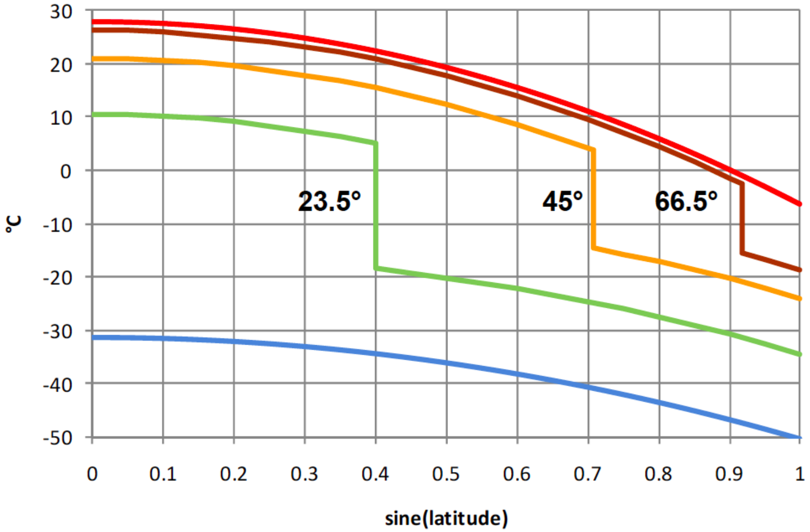
$$T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha) - A + C\overline{T^*}\right)\Big)$$

$$\overline{T_\eta^*} = \frac{1}{B}(Q(1-\overline{\alpha}(\eta)) - A)$$

where

$$\overline{\alpha}(\eta) = \int_0^1 s(y)\alpha(y,\eta)dy = \alpha_1\int_0^\eta s(y)dy + \alpha_2\int_\eta^1 s(y)dy$$

Equilibrium Temperature Profile depends on the Ice Line



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Dynamics of T

Experts only:

Theorem (Widiasih)

Let X be the space of functions where T lives and

$$L : X \to X; \qquad LT := C\overline{T} - (B + C)T.$$

If $f(y) = Qs(y)(1 - \alpha(y, \eta)) - A$, then Budyko's equation can be written as a linear vector field on X:

$$R\frac{dT}{dt} = f + LT.$$

Furthermore, the operator L has only point spectrum, with all eigenvalues negative. Therefore all solutions are stable.

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Everyone: For each fixed ice line η, there is a **globally stable** equilibrium solution for Budyko's equation.

Something seems wrong...

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Intuition:

- High temperature $\Rightarrow$ ice melts $\Rightarrow$ ice line moves north
- Low temperature $\Rightarrow$ ice forms $\Rightarrow$ ice line moves south

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How do we model our intuitions?

Dynamic Ice Line

Ice Formation Assumption: Permanent ice forms if the annual average temperature is below $T_c = -10\ ^\circ\mathrm{C}$ and melts if the annual average temperature is above T_c

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$$\frac{d\eta}{dt} = \epsilon(T^*_\eta(\eta) - T_c)$$

Dynamics of the Ice Line

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A + BT(y,t)) - C\left(T - \overline{T_\eta^*}\right), \qquad \frac{d\eta}{dt} = \epsilon(T_\eta^*(\eta) - T_c)$$

Experts only:

Theorem (Widiasih's Theorem)

For sufficiently small ϵ, the system has an attracting invariant curve given by the graph of a function $\Phi_\epsilon : [0,1] \to X$. On this curve, the dynamics are approximated by the equation

$$\frac{d\eta}{dt} = \epsilon(T_\eta^*(\eta) - T_c).$$

E. Widiasih, "Dynamics of the Budyko Energy Balance Model," *SIAM J. Appl. Dyn. Syst.*, 12(4), 2068–2092.

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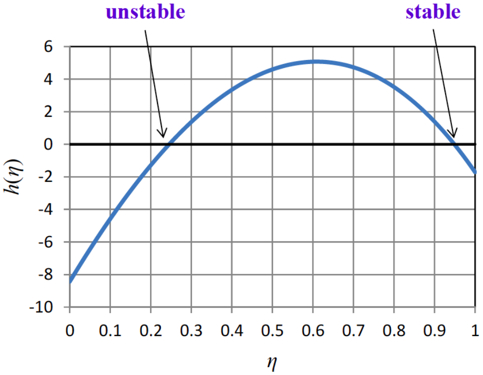
Everyone: $h(\eta) = T_\eta^*(\eta) - T_c$



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The Budyko–Widiasih Model

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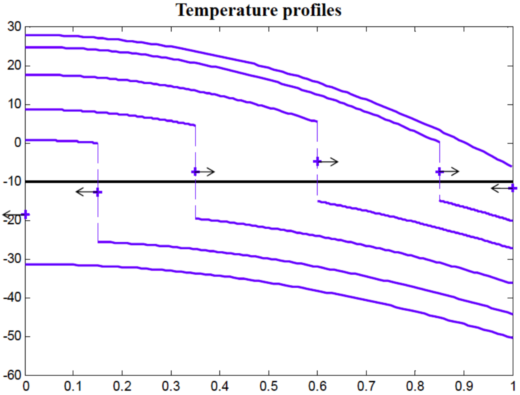

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Greenhouse Gasses in the Budyko–Widiasih Model

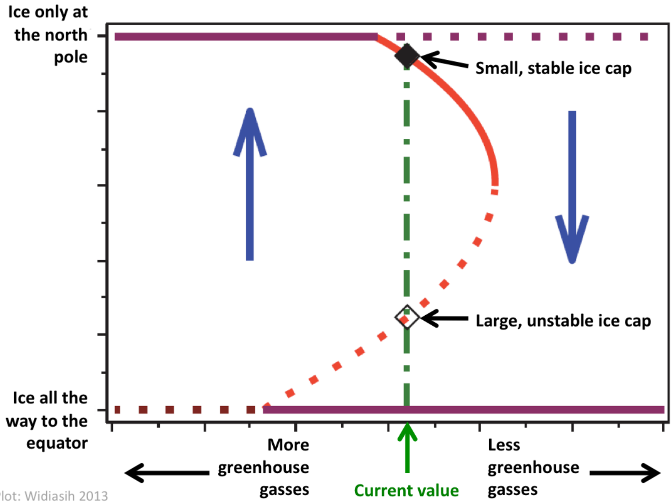
$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - \underbrace{(A + BT(y,t))}_{\text{outgoing long wave radiation}} - C\left(T - \overline{T_\eta^*}\right)$$

$$\frac{d\eta}{dt} = \epsilon h(\eta, A)$$

The parameter A is the **greenhouse gas parameter**.

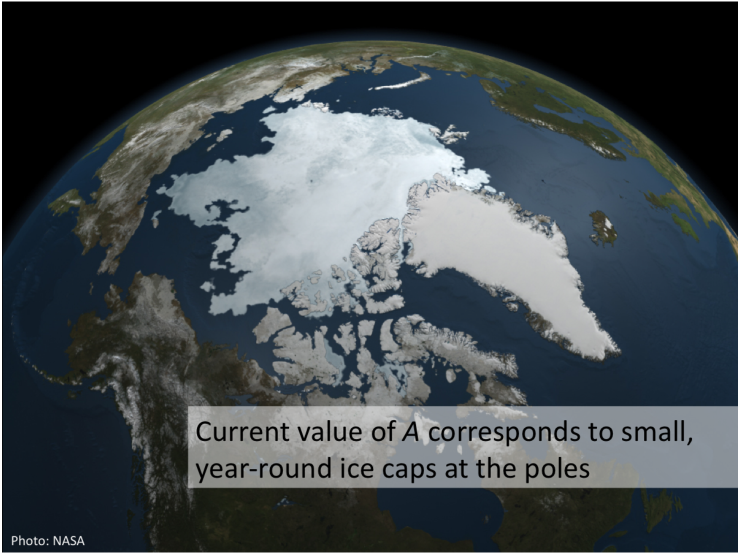
Bifurcation Diagram for A

$$\frac{d\eta}{dt} = \epsilon h(\eta, A)$$



Plot: Widiasih 2013

Current Earth



Bifurcation Diagram for A

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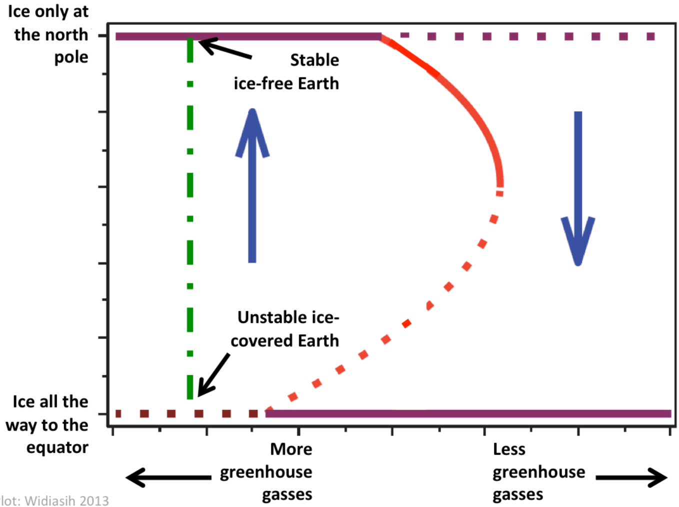
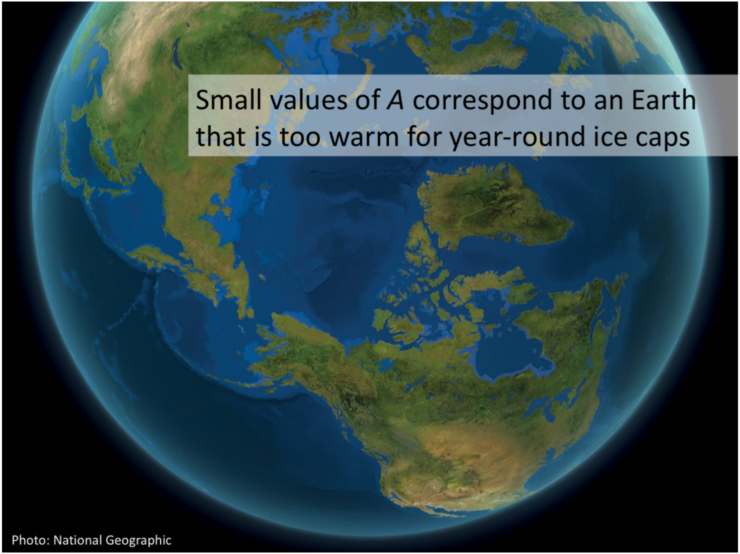


Plot: Widiasih 2013

Future Earth?



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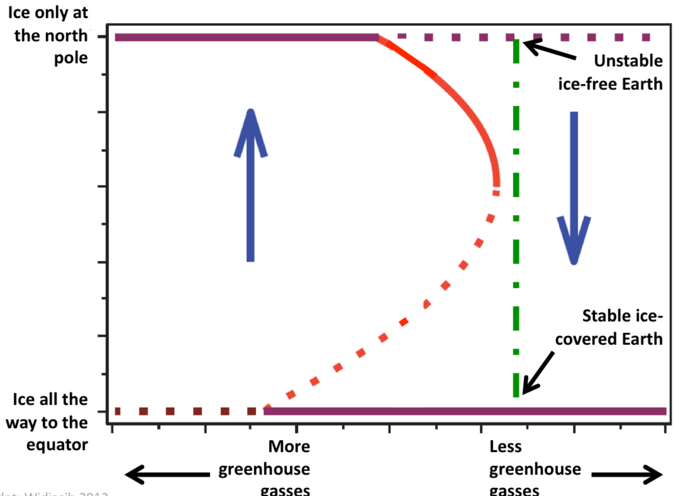
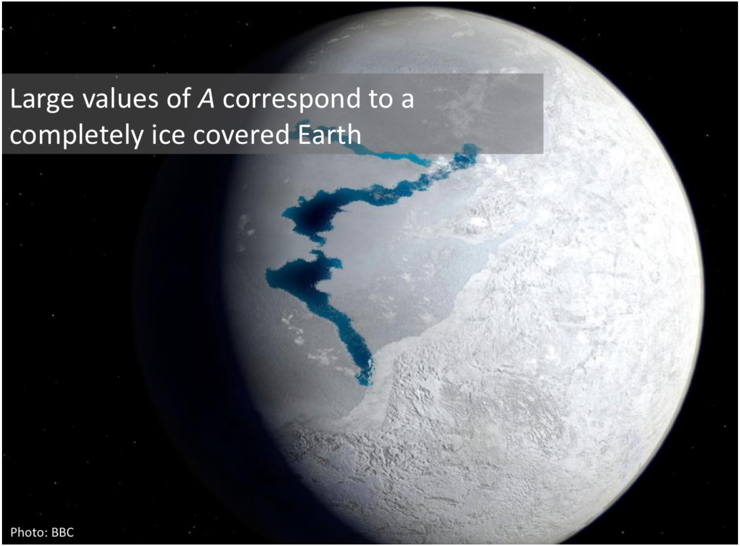


Plot: Widiasih 2013

Past Earth?

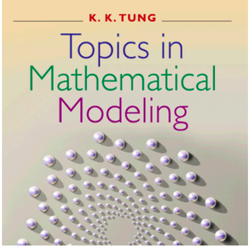

Large values of A correspond to a completely ice covered Earth
Photo: BBC

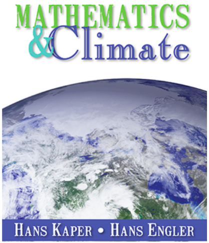
Evidence for Snowball Earth

Hoffman & Schrag, Snowball Earth, Scientific American, January 2000, 68-75

Further Reading

Everyone:





Experts:

- Barry, McGehee, Widiasih. (2017) "Nonsmooth Frameworks for and Extended Budyko Model."
- McGehee and Lehman. (2012) "A paleoclimate model of ice-albedo feedback forced by variations in Earth's orbit."
- MeGehee and Widiasih. (2014) "A quadratic approximation to Budyko's ice-albedo feedback bodel with ice line dynamics."
- Walsh (2016) "Periodic orbits for a discontinuous vector field arising from a conceptual model of glacial cycles."
- Widiasih. (2013) "Dynamics of the Budyko Energy Balance Model."

Thank you!