Filippov Systems and Multiflows

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Overview

Differential Equations

We often study autonomous differential equations:

$$\dot{x} = f(x)$$

A solution to this equation is a differentiable function

$$x : I \to X$$

that satisfies the equality

$$\frac{d}{dt}x(t) = f(x(t))$$

on some interval $I \in \mathbb{R}$.

Flows and Differential Equations

A **flow** is a continuous map $\varphi : \mathbb{R} \times X \to X$ satisfying the group properties

- $\varphi(0, x) = x$
- $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$

The flow relates to the differential equation

$$\dot{x} = f(x)$$

by letting $\varphi(t, x_0)$ correspond to the solution $x(t)$ with the initial condition $x(0) = x_0$.

Differential Inclusions

We want to study **differential inclusions**

$$\dot{x} \in F(x)$$

where F is a set-valued map.

A solution to this differential inclusion is an absolutely continuous function

$$x : I \to \mathbb{R}^n$$

that satisfies the inclusion

$$\frac{d}{dt}x(t) \in F(x(t))$$

almost everywhere on some interval $I \in \mathbb{R}$.

Filippov Systems Introduction

Filippov Domain:

- Start with open set $G \subset \mathbb{R}^n$
- G divided into open domains G_i
- Σ is set of boundary points of the G_i
- G is the union of all G_i and Σ

Filippov Convex Combination [4]:

- Continuous $f_i(x)$ defined in $\overline{G_i}$
- For $x \in G_i$, $F(x) = \{f_i(x)\}$
- For $x \in \Sigma$, $F(x)$ is the convex hull of all $f_i(x)$ such that x is a boundary point of G_i
- This defines a differential inclusion $\dot{x} \in F(x)$

Filippov Systems Introduction

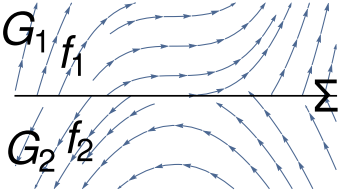


Figure: A planar Filippov system with $\mathbb{R}^2$ split into two regions.

$$\dot{x} \in F(x) = \begin{cases} f_1(x), & x \in G_1 \\ f_2(x), & x \in G_2 \\ \{\alpha f_2(x) + (1-\alpha) f_1(x) : \alpha \in [0,1]\} & x \in \Sigma \end{cases}$$

Behavior Near Splitting Boundary

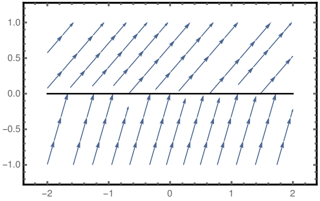


Figure: Crossing Region

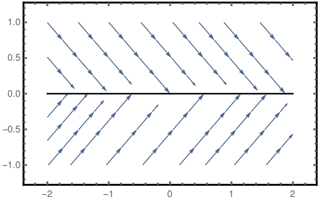


Figure: Attracting Region

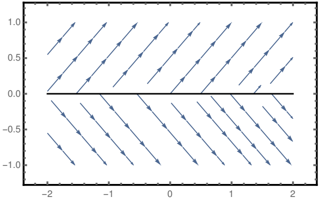


Figure: Repelling Region

Goal: Generalize Flows for Filippov Systems

Filippov systems have:

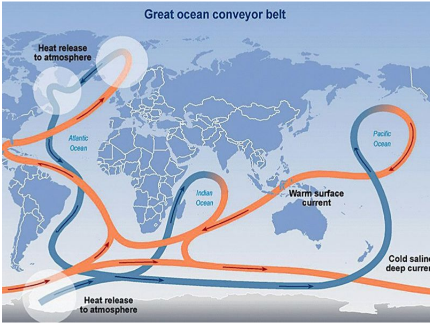
- Intersecting trajectories
- Non-unique solutions

This prevents Filippov systems from being flows:

- No group action
- Cannot be a map

Richard McGehee's Idea: **Multiflows**

Welander's Model: Atlantic Overturning Circulation



Atlantic meridional overturning circulation has changed convective strength in the past. Image: [16]

Welander's goal: Prove these changes could be internally driven, instead of relying on outside forcing [15].

Welander's Model

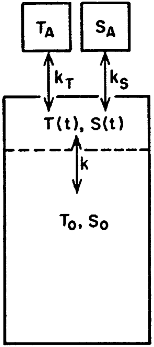


Figure: Deep Ocean and Shallow Ocean [15]

Ocean circulation box model: Planar system, salt (S) and temperature (T) are dynamic variables.

Welander's goal: Show internally driven ocean convection strength oscillations, instead of relying on outside forcing.

Welander's Model

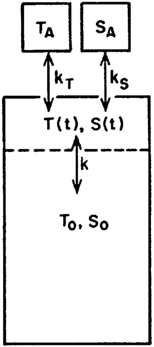


Figure: Deep Ocean and Shallow Ocean [15]

$$\dot{T} = k_T(T_A - T) - k(\rho)T$$
$$\dot{S} = k_S(S_A - S) - k(\rho)S$$
$$\rho = -\alpha T + \gamma S$$

Smooth Version:

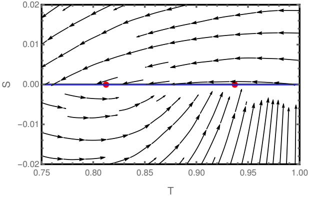
$$k(\rho) = \frac{1}{\pi}\tan^{-1}\left(\frac{\rho - \epsilon}{a}\right) + \frac{1}{2}$$

Nonsmooth Version:

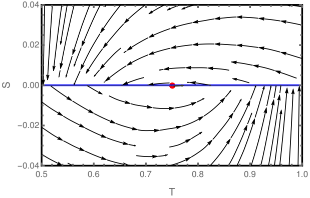
$$k(\rho) = \begin{cases} k_1 & \rho > \epsilon \\ 0, & \rho < \epsilon \end{cases}$$

Σ: Line $\rho = \epsilon$

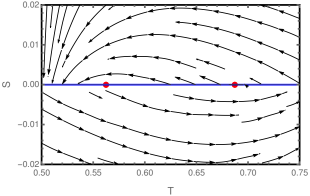
Welander's Model: Fused Focus Bifurcation



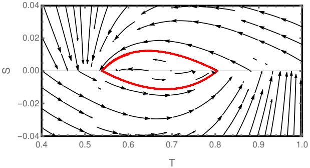
(a) $\epsilon > 0$



(b) $\epsilon = 0$



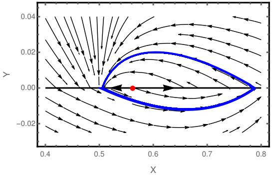
(c) $\epsilon < 0$



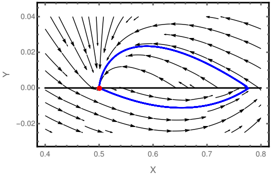
(d) Periodic Orbit, $\epsilon < 0$

Figures and Analysis: Julie Leifeld [7]

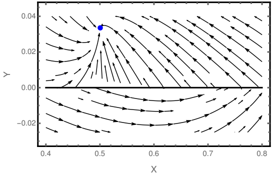
Welander's Model: Border Collision Bifurcation



(e) $0 > \epsilon > \epsilon_0$



(f) $\epsilon = \epsilon_0$



(g) $\epsilon < \epsilon_0 := -\frac{1}{15}$

A prominent paper [6] claimed to classify all planar bifurcations in Filippov systems, but missed this one [7].

Behaviour of Filippov Systems

We want to understand some of the strange behaviour of Filippov systems.

Our goal is to see what features of a *flow* must be changed in order to fit Filippov systems.

Almost Everywhere Condition

Solutions typically lose differentiabililty when they reach the splitting boundary Σ. For this reason, we only demand that $\dot{x} \in F(x)$ almost everywhere.

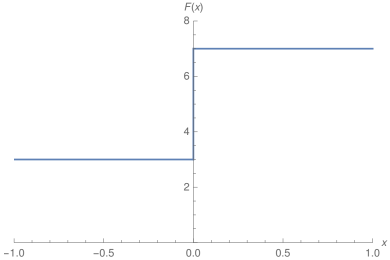


Figure: $F(x) = \begin{cases} 3, & x < 0 \\ 7, & x > 0 \\ [3,7] & x = 0 \end{cases}$

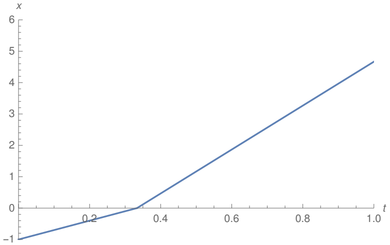


Figure: Solution $x(t)$ to $\dot{x} \in F(x)$

Intersecting Trajectories

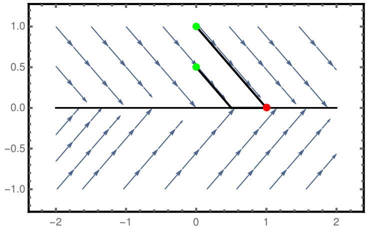


Figure: Intersecting Trajectories in a simple Filippov System

Cannot obey group properties:

$$\phi_t(\phi_{-t}(x)) = \phi_{t-t}(x) \neq \phi_0(x) = x$$

Intersecting Trajectories

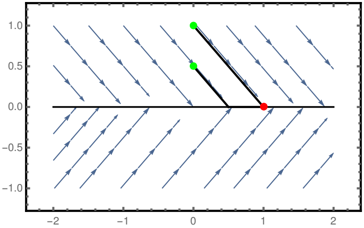


Figure: Intersecting Trajectories in a Filippov System

Solution: Monoid Action (Semiflow)

Multiple Solutions

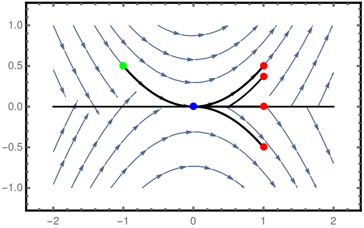


Figure: Four different solutions of a Filippov system

$$\dot{(x, y)} \in H(x,y) := \begin{cases} \{(1, x)\}, & y > 0 \\ \{(1, \beta) : \beta \in [-x, x]\} & y = 0 \\ \{(1, -x)\}, & y < 0 \end{cases}$$

Dealing with Nonuniqueness

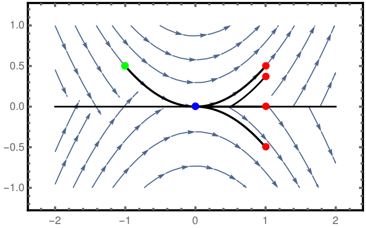


Figure: Four different solutions of a Filippov system

Can we ignore nonuniquness?

"Repelling sliding motion cannot be reached by following the system flow forward in time." [2]

The example to the left (as well as Welander's model) indicate that this method is not robust.

Dealing with Nonuniqueness

We can follow a unique vector during sliding motion (the vector that stays on the splitting boundary).

This approach is followed by Kuznetsov et. al.[6]

The phase portraits are different in forward and backwards time.

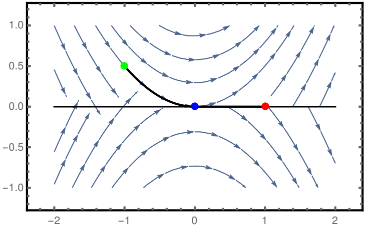


Figure: A unique sliding solution is chosen

Nearby Smooth Systems

We often want to use nonsmooth systems to understand nearby smooth systems [5].

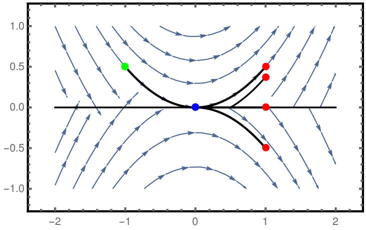


Figure: Four different solutions of a Filippov system

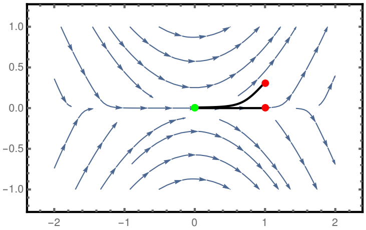


Figure: $(\dot{x}, \dot{y}) = (1, \tanh(\gamma y)x)$

As $\gamma \to \infty$, this system limits to the Filippov system on the left.

Theorems about Solutions

For a Filippov system $\dot{x} \in F(x)$ on an open domain G, the following results hold [4]:

- For each initial condition, solutions exist on some interval $(-\delta, \delta)$.
- $|F(x)|$ is bounded in a compact domain.
- Solutions lying in a compact domain are equicontinuous.
- Solutions are continued up to the boundary of any compact domain.
- The limit of a uniformly convergent sequence of solutions is a solution.

Multiflows

A multiflow is an object that is intended to generalize the concept of flows to Filippov systems.

Before we define multiflows, we need some background.

Relations

A **relation** on a topological space X is a subset of $X \times X$.

If F and G are both relations on X, then we can define the composition:

$$F \circ G = \{(x, z) \in X \times X : \exists y \in X \text{ s.t. } (x, y) \in G, (y, z) \in F\}$$

The Closed Graph Theorem

Let X be a topological space and let Y be a Hausdorff space.

$f : X \to Y$ is continuous

$\downarrow$

The graph of f is closed

The Closed Graph Theorem

Let X be a topological space and let Y be a compact Hausdorff space.

$$f : X \to Y \text{ is continuous}$$

$$\updownarrow$$

$$\text{The graph of } f \text{ is closed}$$

Graph of a Flow

The graph of a **flow** ϕ on a compact set X is a closed subset of $\mathbb{R} \times X \times X$ such that for each $t \in \mathbb{R}$, ϕ^t contains exactly one pair $(x, y) \in X \times X$ for each $x \in X$ and the group properties hold:

- $\phi^0 = \{(x, x) : x \in X\}$
- $\phi^{t+s} = \phi^t \circ \phi^s$

Where $\phi^t := \{(x, y) \in X \times X : (t, x, y) \in \phi\}$

Can we modify flows to fit Filippov Systems?

~~The graph of a **flow**~~ ϕ on a compact set X is a closed subset of $\mathbb{R}^+ \times X \times X$ such that ~~for each $t \in \mathbb{R}$, ϕ^t contains exactly one pair $(x, y) \in X \times X$ for each $x \in X$ and~~ the ~~group~~ *monoid* properties hold:

- $\phi^0 = \{(x, x) : x \in X\}$
- $\phi^{t+s} = \phi^t \circ \phi^s$

Where $\phi^t := \{(x, y) \in X \times X : (t, x, y) \in \phi\}$

Multiflows

A **multiflow** Φ on a compact space X is a closed subset of $\mathbb{R}^+ \times X \times X$ satisfying the monoid properties:

- $\Phi^0 = \{(x, x) : x \in X\}$
- $\Phi^{t+s} = \phi^t \circ \phi^s$

Where $\Phi^t := \{(x, y) \in X \times X : (t, x, y) \in \Phi\}$

Filippov Systems give rise to Multiflows

Theorem: Let $\dot{x} \in F(x)$ be a Filippov system on an open domain $G \subset \mathbb{R}^n$, and let $K \subset G$ be compact. Let Φ be the set of all points

$$\{(T, a, b) \in \mathbb{R}^+ \times K \times K\}$$

such that there exists a solution $x : [0, T] \to K$ satisfying $x(0) = a$ and $x(T) = b$.

Then the set Φ is a multiflow over K.

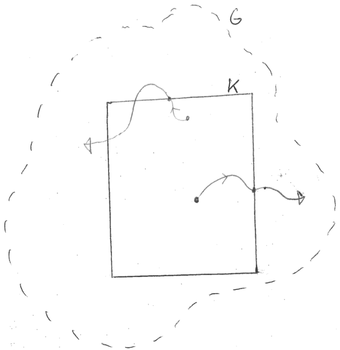


Figure: Once solutions leave K, they are no longer included in Φ.

Concepts Related to Multiflows

Several other researchers have attempted to generalize the concept of a flow to systems with nonuniqueness [11][14][3][1].

The **set-valued dynamical system** described by Oyama [13] is particularly close to multiflows.

The key distinction between multiflows and these other objects is that multiflows do not demand that solutions exist for all time.

Concepts Related to Multiflows

A correspondence $\Phi : [0, \infty) \times X \to X$ on a compact subset $X \subset \mathbb{R}^n$ is a **set-valued dynamical system** [13] if it meets the following conditions:

1. $\Phi_t(x)$ is nonempty for all t, x
2. $\Phi_0(x) = x$
3. $\Phi_t(\Phi_s(x)) = \Phi_{t+s}(x)$
4. Φ is compact valued and upper-semicontinouous.

Filippov systems cannot be described by this object because their solutions do not (in general) remain in a compact set for all time.

Future Work

Rewrite Filippov's Proofs

Generalize some topological concepts from flows to multiflows:

- ω-limit sets [12]
- Chain Recurrence
- Attractors and Attractor Blocks [12]

Conley Index Theory

Semicontinuity of Multiflows

References I

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