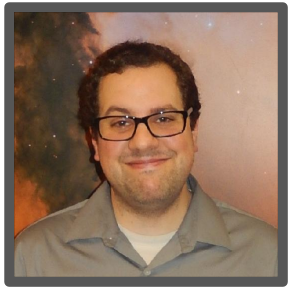
The Interplay of Rate and Noise Tipping and Applications

Katherine Slyman
University of North Carolina at Chapel Hill

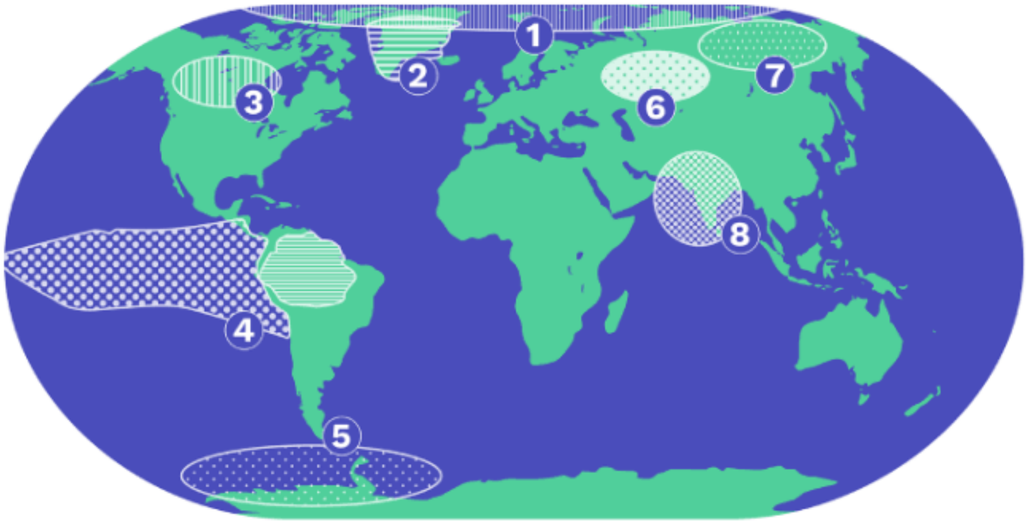
Chris Jones
UNC-CH



John Gemmer
Wake Forest

Motivation

Map of some Tipping Points



1. Artic Sea-Ice Loss
2. Atlantic Deep Water Formation
3. Boreal Forest Dieback
4. Change in ENSO Amplitude or Frequency
5. Instability of West Antartic Ice Sheet
6. Boreal Forest Dieback
7. Permafrost and Tundra Loss?
8. Indian Monsoon Chaotic Multistability

Source: Lenton, T. et al., "Tipping Elements in the Earth's climate system" PNAS, February 12th 2008

Rate-Induced Tipping

https://slideplayer.com/slide/6269988/

Rate-Induced Tipping

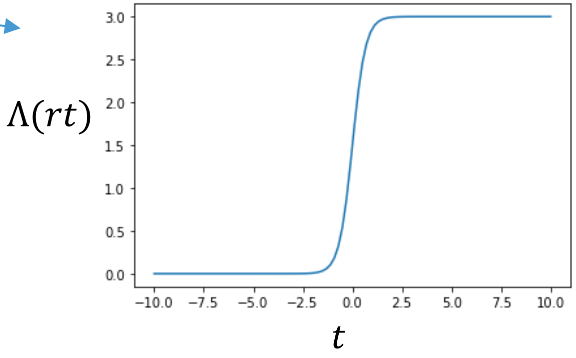
Start with the autonomous differential equation: $\dot{x} = f(x, \lambda)$

Ashwin, P., Wieczorek, S., Vitolo, R., & Cox, P. (2012).

Replace λ with $\Lambda(\mathrm{rt})$ and fixed $r > 0$.

Specifically assume:

1. $\lambda_- < \Lambda(rt) < \lambda_+ \ \forall t$.
2. $\Lambda(rt) \to \lambda_-$ as $t \to -\infty$ and $\Lambda(rt) \to \lambda_+$ as $t \to \infty$



This leads us to a nonautonomous system: $\dot{x} = f(x, \Lambda(\mathrm{rt}))$

Rate-Induced Tipping

We convert back into an autonomous system.

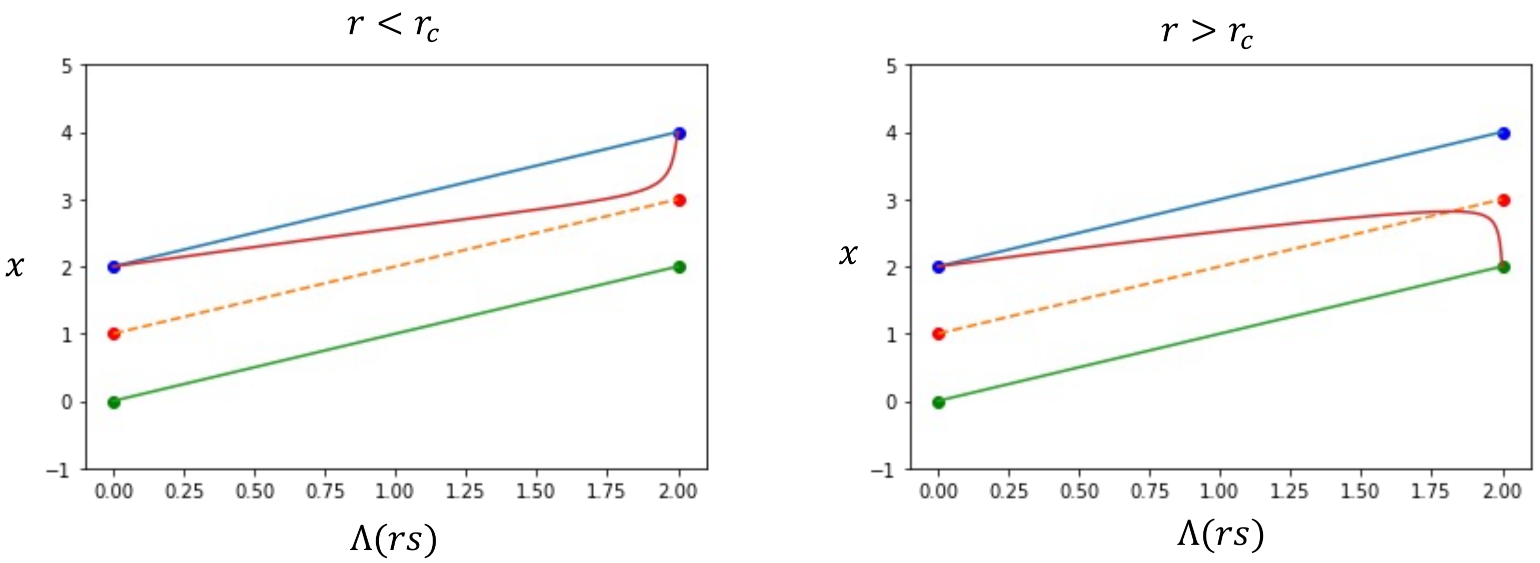
Approach:

If Λ is invertible, you can find an explicit expression for $\dot{\Lambda}$, in terms of Λ.

***Compactification**

Wieczorek, S., Xie, C., & Jones, C. K. (2021)

Rate-Induced Tipping Example 1



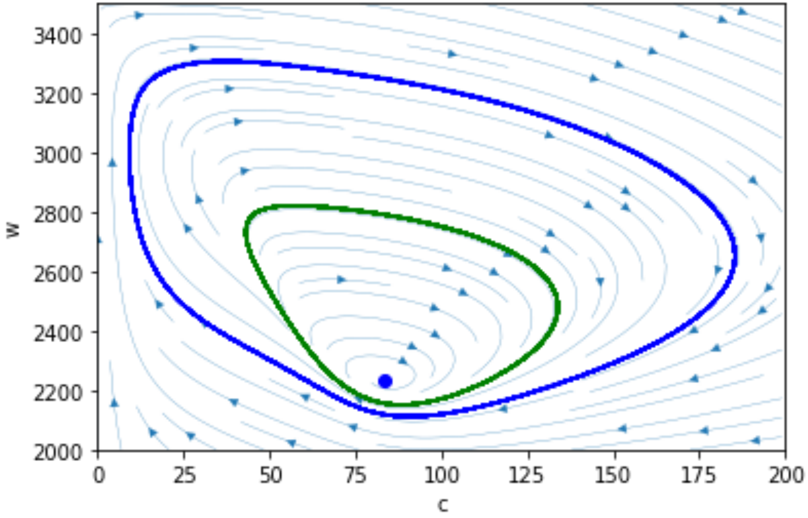
$$\dot{x} = -1(x - \Lambda(\mathrm{rs}))(\mathrm{x} - \Lambda(\mathrm{rs}) - 1)(\mathrm{x} - \Lambda(\mathrm{rs}) - 2)$$

$$\lambda_- = 0, \lambda_+ = 2$$

Rate-Induced Tipping Example 2

$$\dot{w} = \mu \left[1 - bs\left(c, c_p\right) + \theta \overline{s}\left(c, c_x\right) + \nu\right] - w + w_0,$$

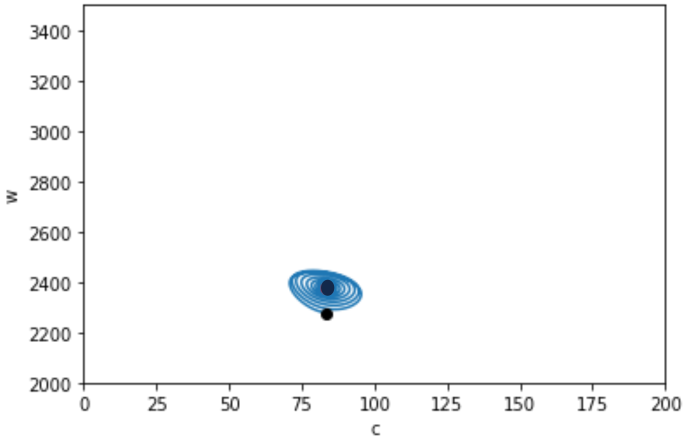
$$\dot{c}/f\left(c\right) = \mu \left[1 - bs\left(c, c_p\right) - \theta \overline{s}\left(c, c_x\right) - \nu\right] + w - w_0$$



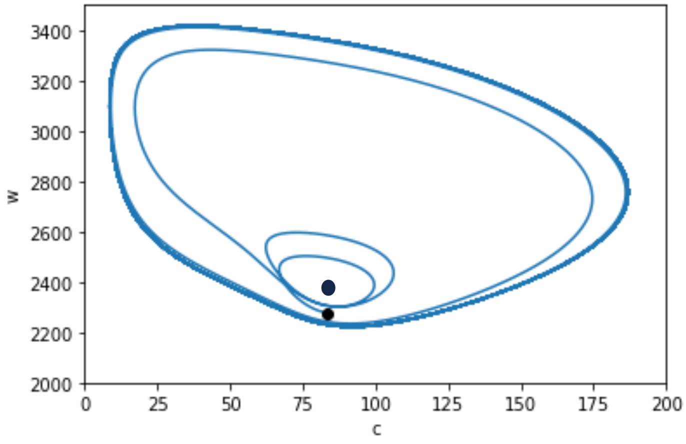
Rothman
Characteristic disruptions of an excitable carbon cycle (2019)

Rate-Induced Tipping Example 2

$r < r_c$



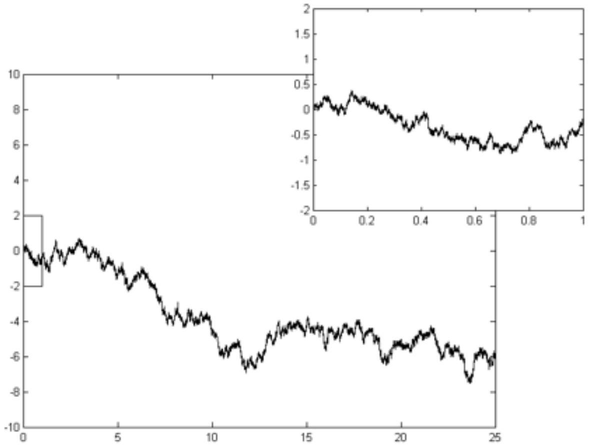
$r > r_c$



Noise-Induced Tipping

Modifying $\dot{x} = f(x, \lambda)$ to include the possibility of random effects, consider:

$$dx = f(x, \lambda)\,dt + g(x, t)dW$$



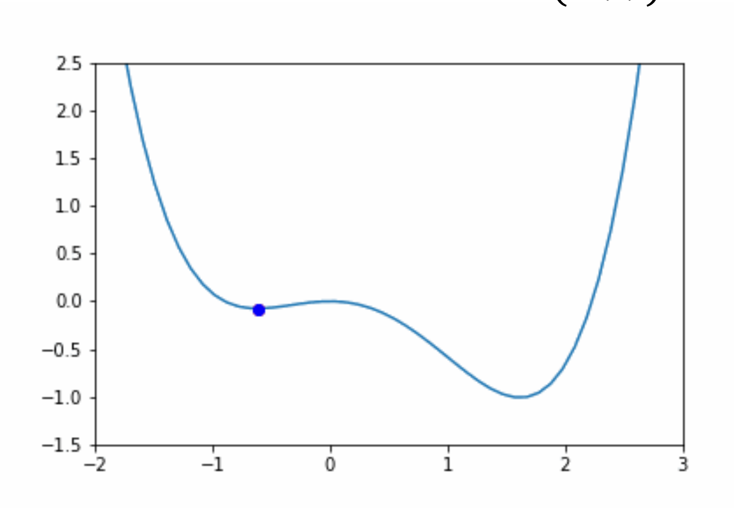
https://en.wikipedia.org/wiki/Wiener_process#/media/File:Wiener_process_zoom.png

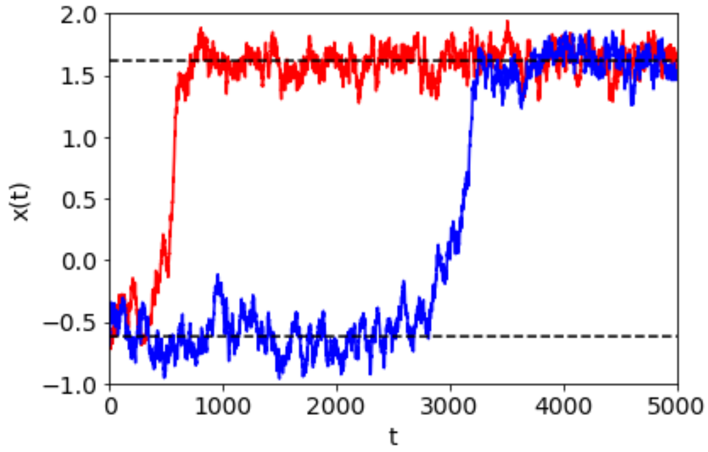
Noise-Induced Tipping

Modifying $\dot{x} = f(x, \lambda)$ to include the possibility of random effects, consider:

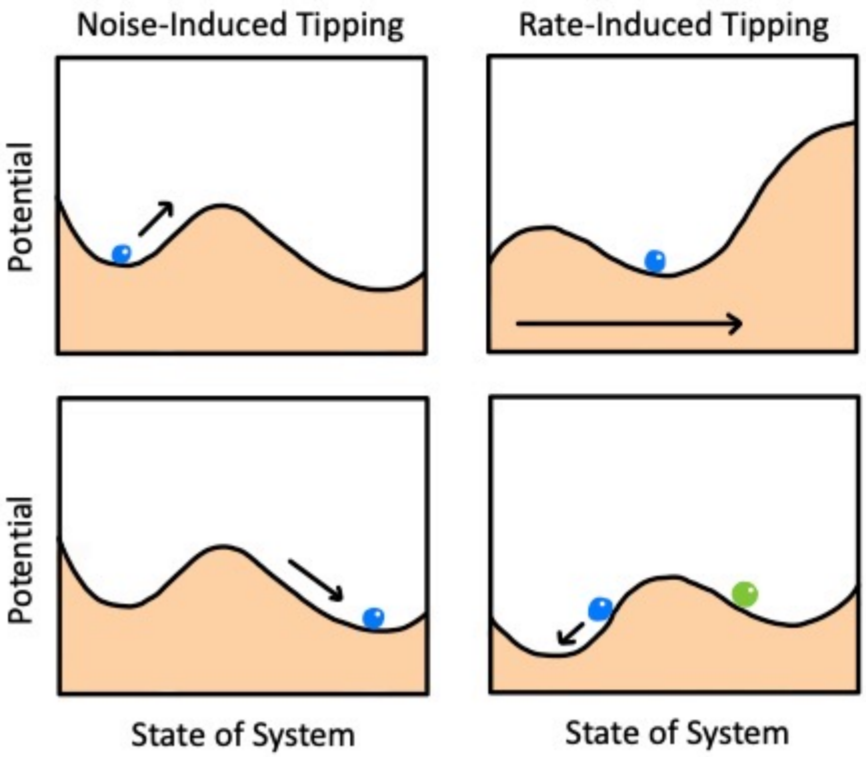
$$dx = f(x, \lambda)\, dt + g(x, t) dW$$

Gradient system: $dx = -\nabla V\big(x(t)\big) = (x - x^3 + x^2) dt + \sigma dW$





Noise vs. Rate Tipping



The Interplay

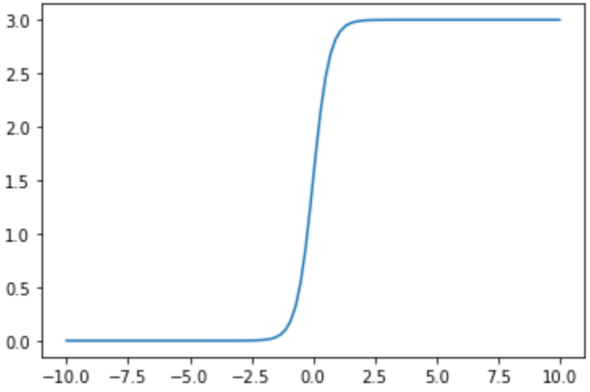
The interplay between noise and a ramp parameter results in tipping of the system before the critical rate is reached.

How does the noise strength affect the most probable path?

The Model Problem- Intro

From Ritchie and Sieber (2016): $$\frac{dx}{dt} = (x+y)^2 - 1$$

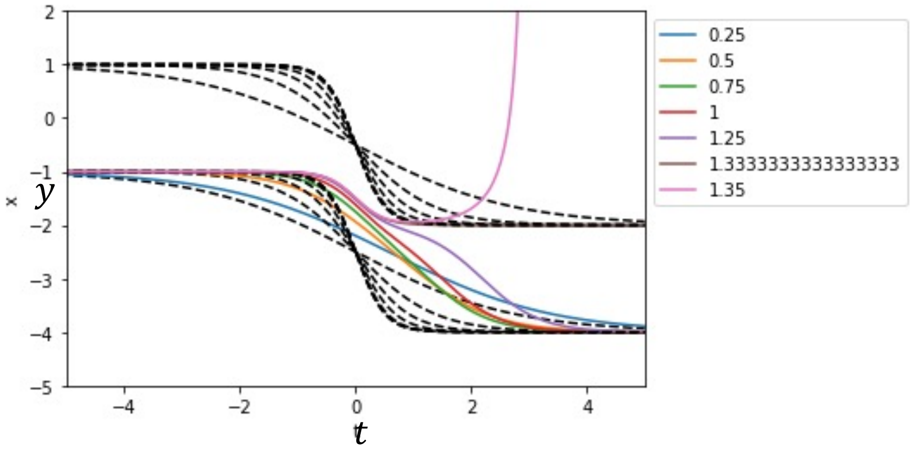
$$y(t) = \frac{3}{2}\left(1 + \tanh\left(\frac{3rt}{2}\right)\right)$$



The Model Problem- Intro

From Ritchie and Sieber (2016): $$\frac{dx}{dt} = (x+y)^2 - 1$$

$$y(t) = \frac{3}{2}\left(1 + \tanh\left(\frac{3rt}{2}\right)\right)$$



The Model Problem- Intro

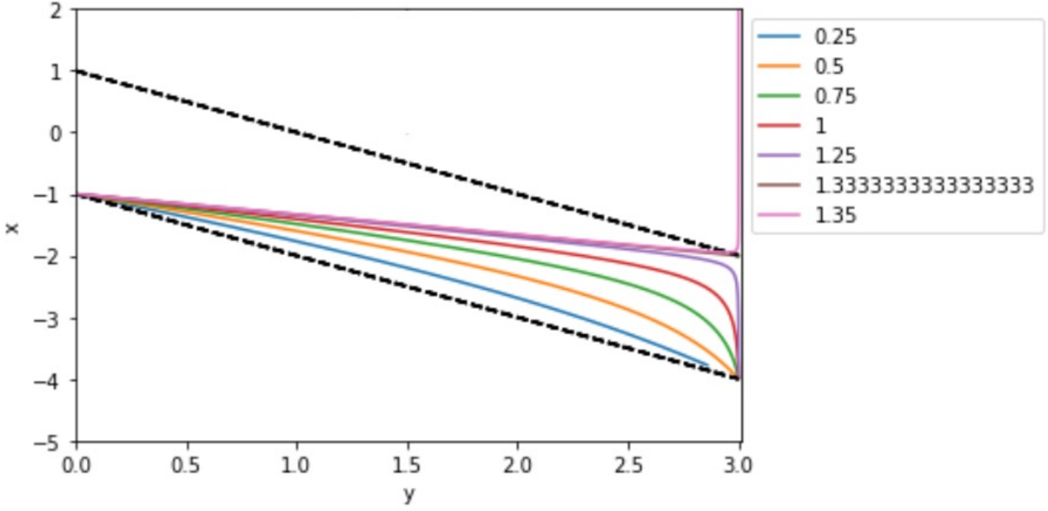
Now we are studying:

$$\frac{dx}{dt} = (x+y)^2 - 1$$

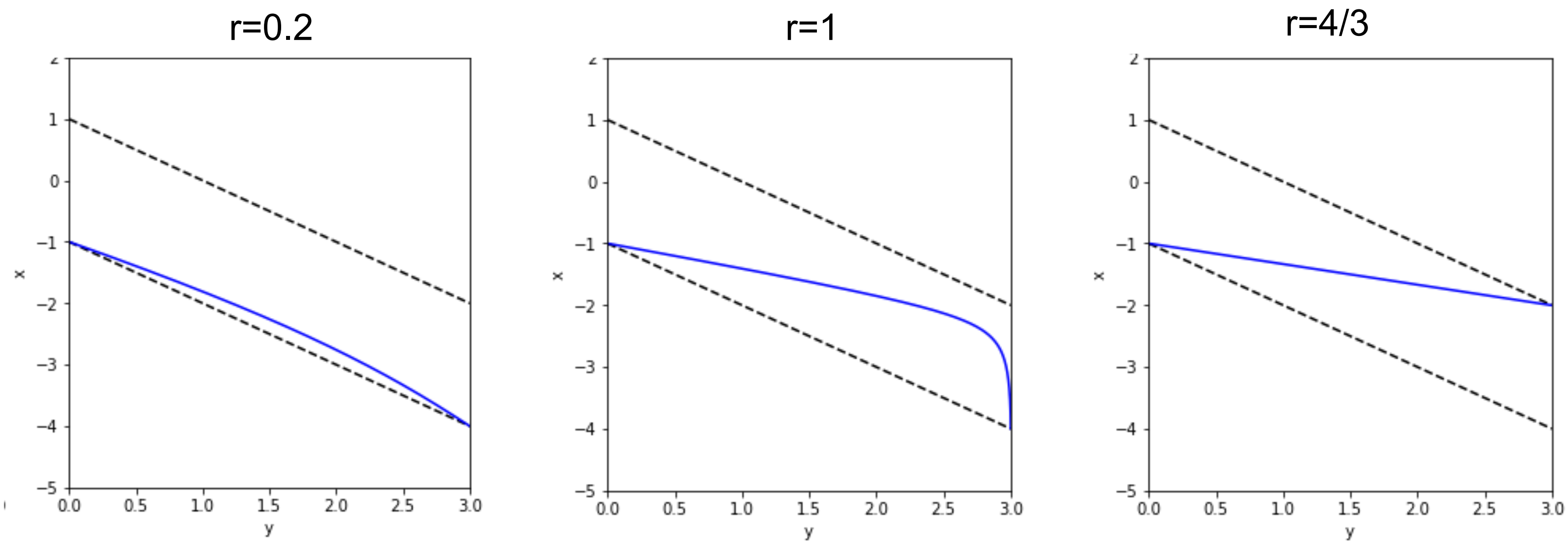
$$\frac{dy}{dt} = ry(3-y)$$

→

(-1,0), (-2,3) are saddle points.
(1,0) is a repeller.
(-4,3) is an attractor.



Sample Trajectories



At r_c, there is a heteroclinic connection between (−1,0) and (−2,3).

Perryman and Wieczorek (2015) found that r_c = 4/3 and that the connecting orbit is

$$x = -\frac{y}{3} - 1.$$

Freidlin-Wentzell Theory

Rewrite as stochastic: $dx = ((x+y)^2 - 1)dt + \sigma_1 dW_1 = f(x,y)dt + \sigma_1 dW_1$

$dy = (ry(3-y))dt + \sigma_2 dW_2 = g(y)dt + \sigma_2 dW_2$

The most probable path is a curve $\vec{r}(t)$ that minimizes the Freidlin-Wentzell functional:

$$I[r_1, r_2] = \int_{t_0}^{t_f} \frac{(\dot{r_1} - f)^2}{\sigma_1^2} + \frac{(\dot{r_2} - g)^2}{\sigma_2^2} dt$$

Most Probable Path Equations

Using calculus of variations results in the following Euler-Lagrange equations:

$$\ddot{r}_1 = f_y \dot{r}_2 + f f_x$$
$$\ddot{r}_2 = \frac{\sigma_2^2}{\sigma_1^2}\left(-\dot{r}_1 f_y + f f_y\right) + g g_y$$

We create a Hamiltonian system:

$$\dot{x} = f + \sigma_1^2 p$$
$$\dot{p} = -f_x p$$
$$\dot{y} = g$$
$$\dot{q} = -g_y q$$

$$H(x, p, y, q) = fp + gq + \frac{\sigma_1^2}{2} p^2$$

Most Probable Path Equations

$$\dot{x} = (x + y)^2 - 1 + p$$

$$\dot{p} = -2(x + y)p$$

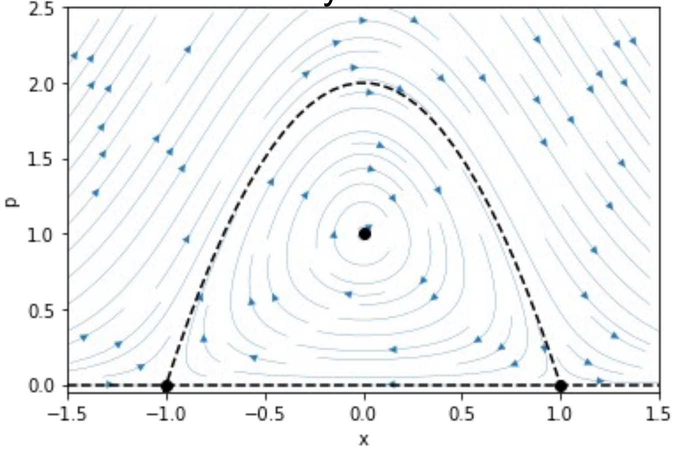
$$\dot{y} = ry(3 - y)$$

$$H(x, p, y) = ((x + y)^2 - 1)p + \frac{1}{2}p^2$$

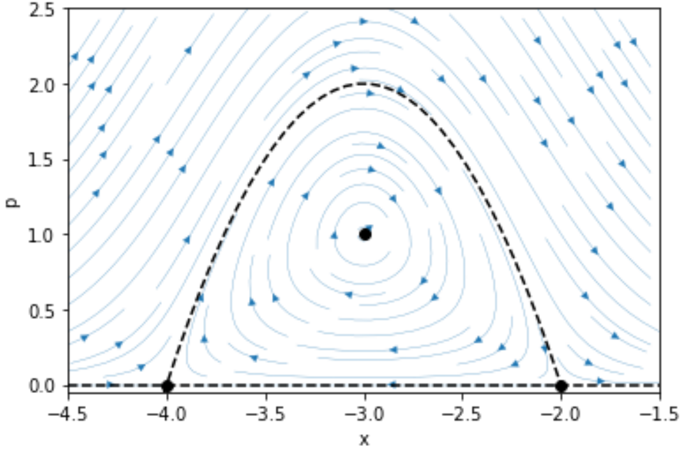
New Phase Space

$$J = \begin{pmatrix} 2(x+y) & 1 & 2(x+y) \\ -2p & -2(x+y) & -2p \\ 0 & 0 & 3r-2ry \end{pmatrix}$$

y=0



y=3



(-1,0,0), (1,0,0) -- saddles, (0,1,0) -- center

↓

Eigenvalues: -2, 2, 3r

(-4,0,3), (-2,0,3) -- saddles, (-3,1,3) -- center

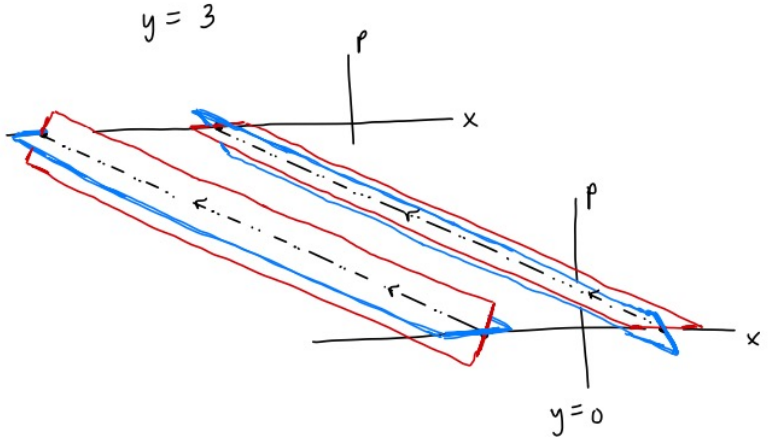
↓

Eigenvalues: 2, -2, -3r

Theorem

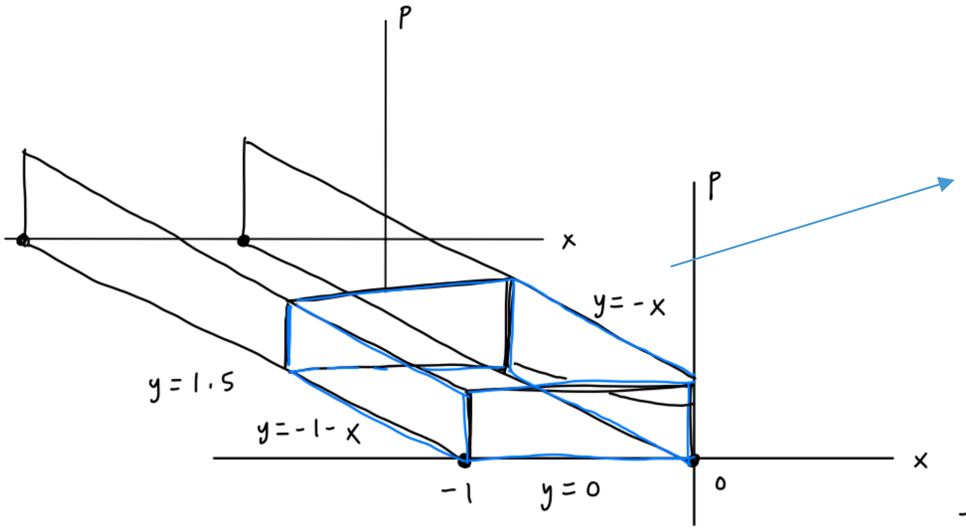
There exists a heteroclinic connection between the saddle points (−1,0,0) and (−2,0,3) that goes through the plane $y = -x$ *at* $y = \frac{3}{2}$ *for* $r \leq r_c$.

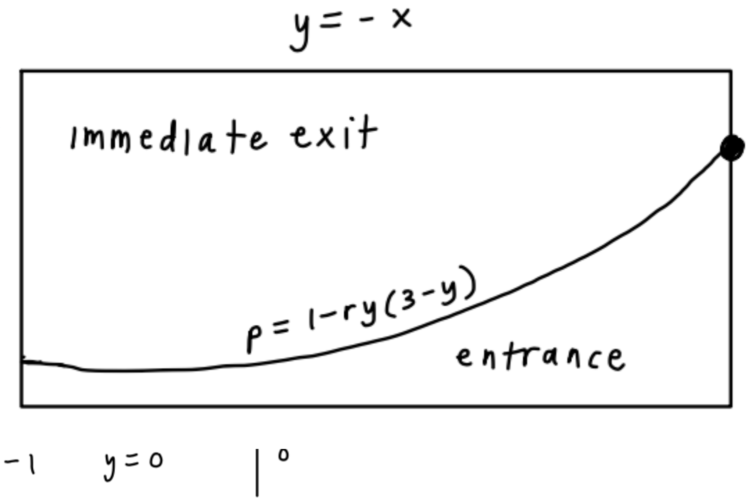
**We will show this is the most probable path!*



Wazewski

Wazewski Principle:
Let W^- be the immediate exit set of W and let W^0 be the eventual exit set of W. If W^- is closed relative to W^0, then W is a Wazewski set and the map K: $W^0 \to W^-$, that takes each point to the first where it exits W is continuous. Note that $W^- \subset W^0$.

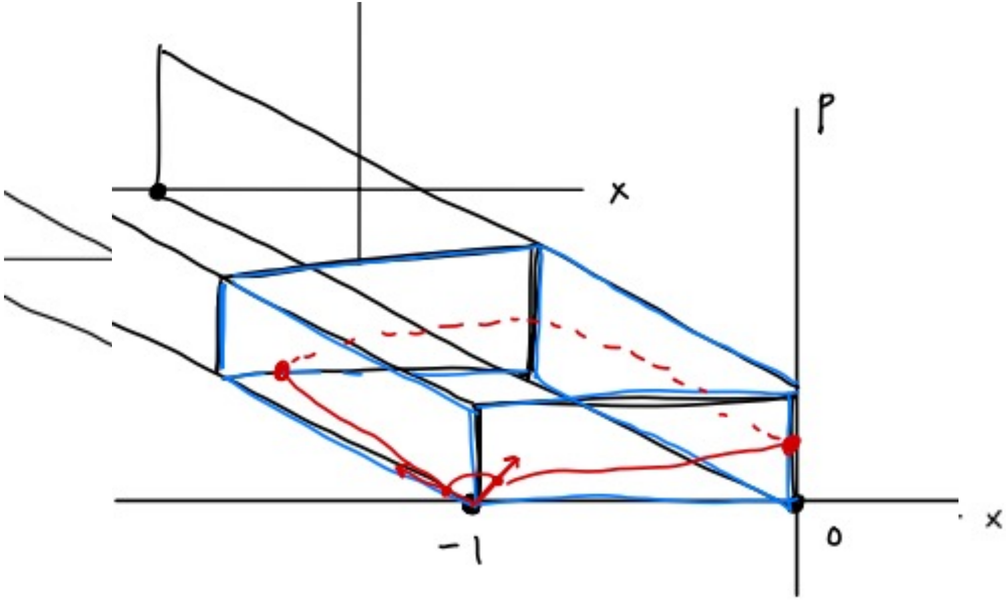




Wazewski

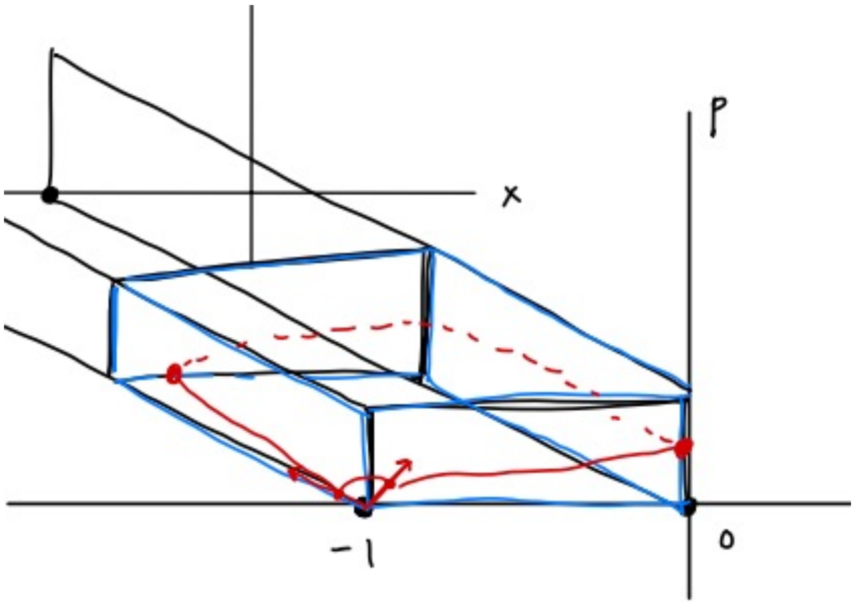
Shooting Argument!

Take a sphere of small radius around (-1,0,0) intersected with the unstable manifold of (-1,0,0) that lies in W.



Wazewski

Shooting Argument!



Symmetry

Make the change of variables $\tau = -t$ to get the time reversed system.

Transform the x, p, and y by: $\hat{x} = -x - 3$

$\hat{p} = p$

$\hat{y} = 3 - y$

Once substituted: $\hat{x}' = (\hat{x} + \hat{y})^2 - 1 + \hat{p}$

$\hat{p}' = -2(\hat{x} + \hat{y})\hat{p}$

$\hat{y}' = r\hat{y}(3 - \hat{y})$

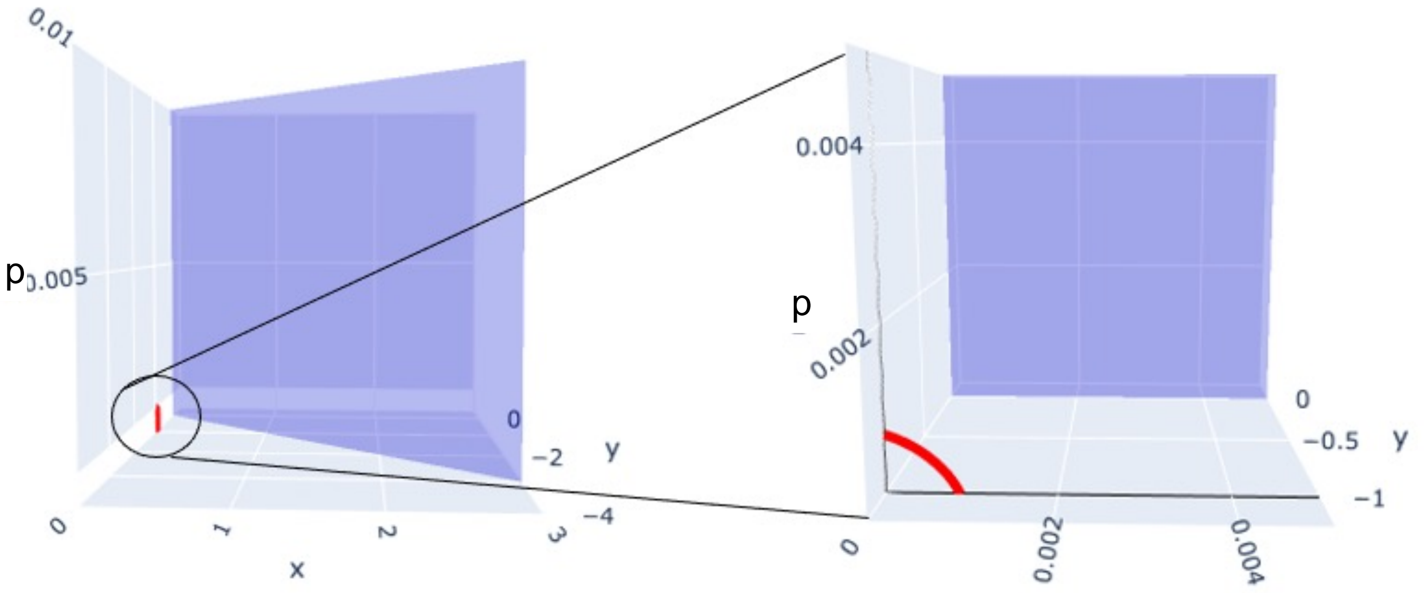
The Model Problem- Conclusion

The intersection of the unstable manifold of (−1,0,0) and the plane $y = -x$ was continuous for $y \leq \frac{3}{2}$ for $r \leq r_c$.

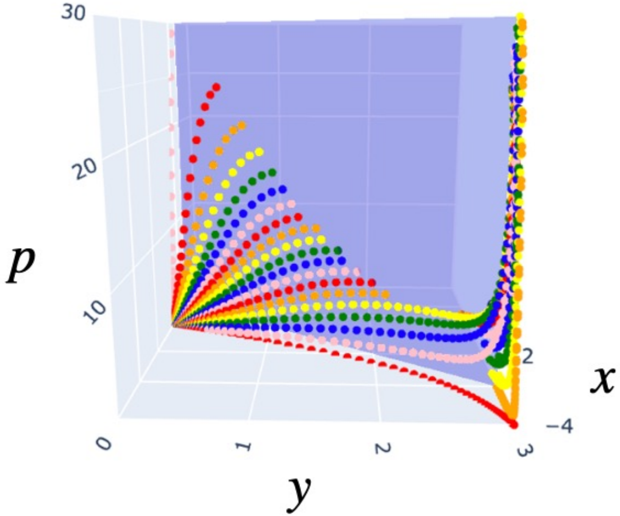
The intersection of the stable manifold of (−2,0,3) and the plane $y = -x$ is continuous for $y \geq \frac{3}{2}$ for $r \leq r_c$.

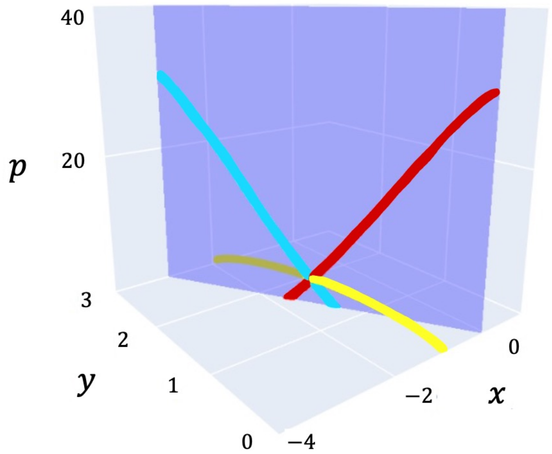
The unstable and stable manifolds will always intersect in the plane $y = -x$ at $y = \frac{3}{2}$, implying a heteroclinic connection between the saddles for $r \leq r_c$.

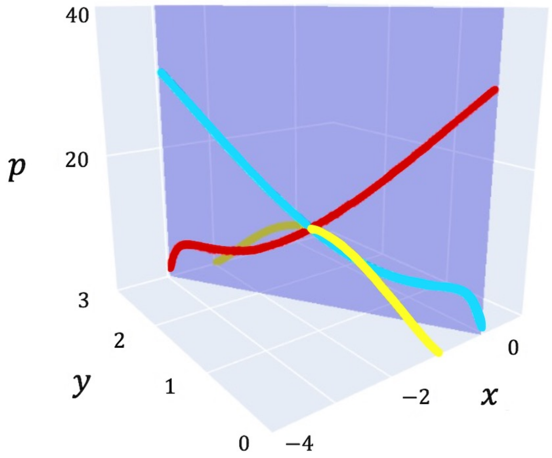
Simulations



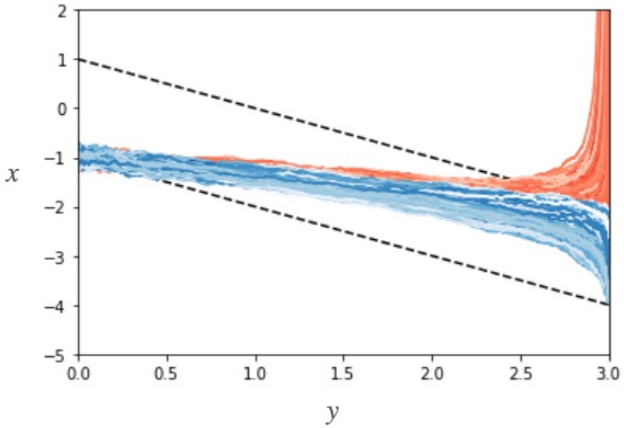
Simulations



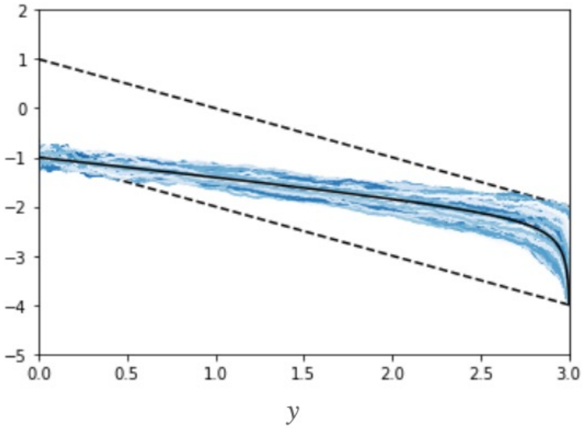




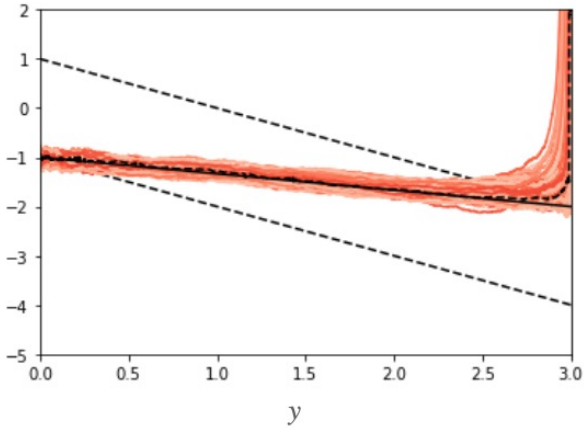
Simulations



1000 realizations



923 do not tip



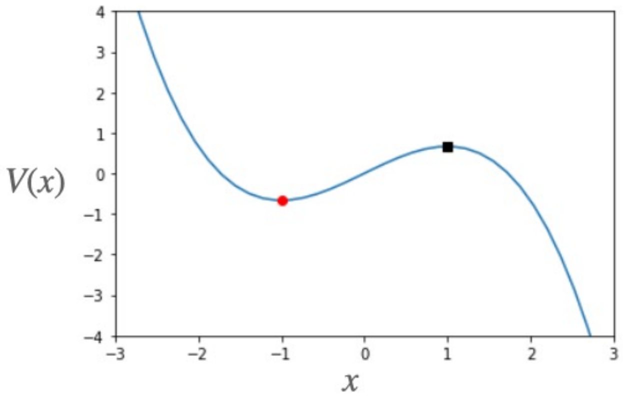
77 tip

$r = 1, \sigma = .15$

Too Much Noisy Influence?

$$dx = -\nabla V + \sigma dW = (x^2 - 1)dt + \sigma dW$$

$$V = -\frac{1}{3}x^3 + x$$



$$\mathbb{E}[\tau] \approx e^{\frac{2\Delta V}{\sigma^2}} > 10^{12}$$

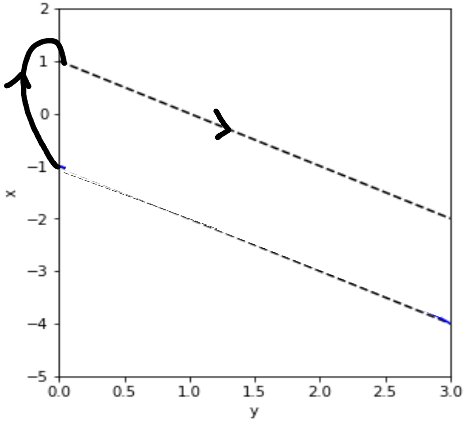
Without the ramp, tipping will be extremely rare!

r	σ_1 range	MC time to tip
.75	.15 – .3	13 – 14
.85	.1 – .3	11.5 – 12.5
1	.08 – .25	9.7 – 10.5
1.1	.08 – .25	8.6 – 9.5

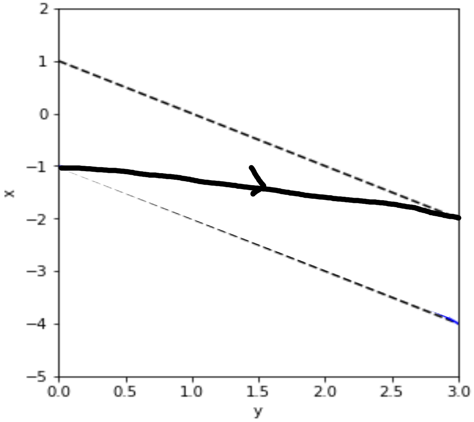
Ramp only – no tipping
Noise only – tipping extremely rare
Interplay – facilitates tipping on finite timescale

Considering the Action Value

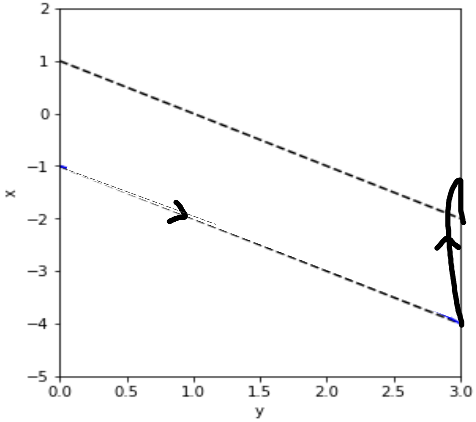
3 ways to tip:



Action=5.33



r= 1.1, Action= .023
r= 1 , Action= .054
r= .75, Action= .226
r= .5 , Action= .684
r= .01, Action= 5.30*

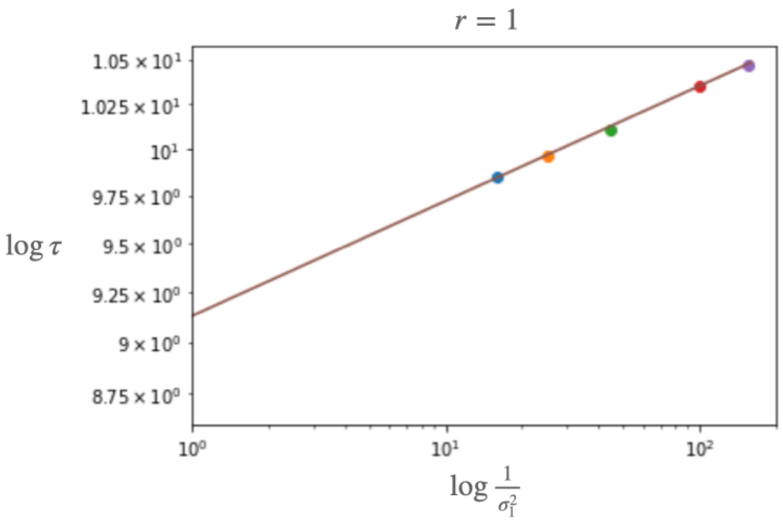


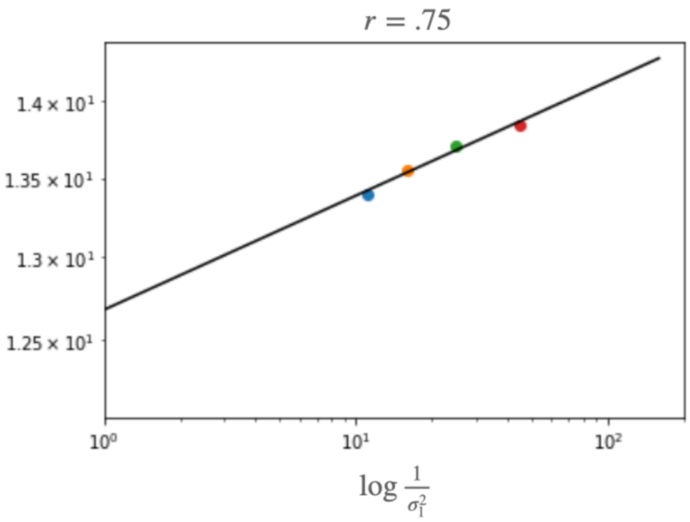
Action=5.33

Expected Time to Tip?

Scaling law emerges! *Found using converged Monte Carlo simulations

Linear relationship in log-log space holds true for multiple r, σ





Impacts of Tropical Cyclones

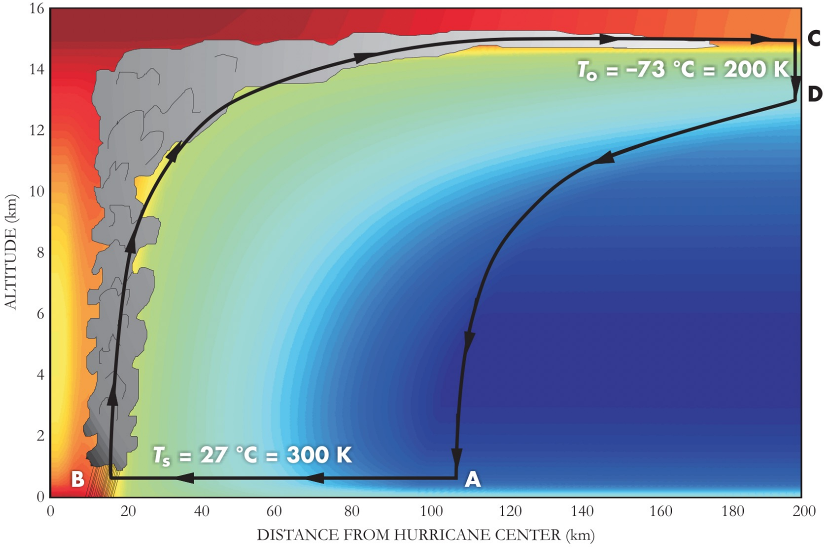
- Some of the most costly natural disasters
  - Property damages and lives lost

- Hurricane Dorian
  - ~ $7 billion, 400+ missing, reef damage
  - tourism and fishing industries

- Further understanding needed
  - Formation, intensification, tracking, dissipation
  - Risk and damage prediction

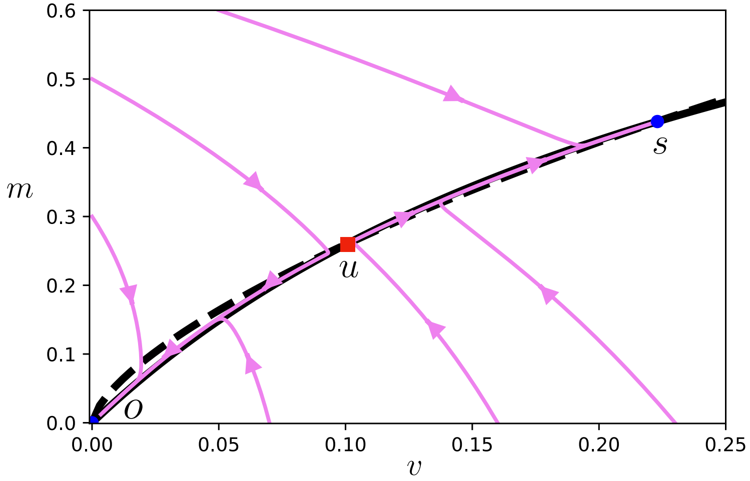
https://www.theatlantic.com/photo/2019/09/hurricane-dorian-damage-bahamas-photos/597463/

Hurricane Problem



$$\frac{dv}{d\tau} = (1-\gamma)\left(\frac{V_p}{V_p^-}\right)^2 m^3 - (1-\gamma m^3)v^2,$$

$$\frac{dm}{d\tau} = (1-m)v - cm,$$



v – wind speed

m – inner core moisture

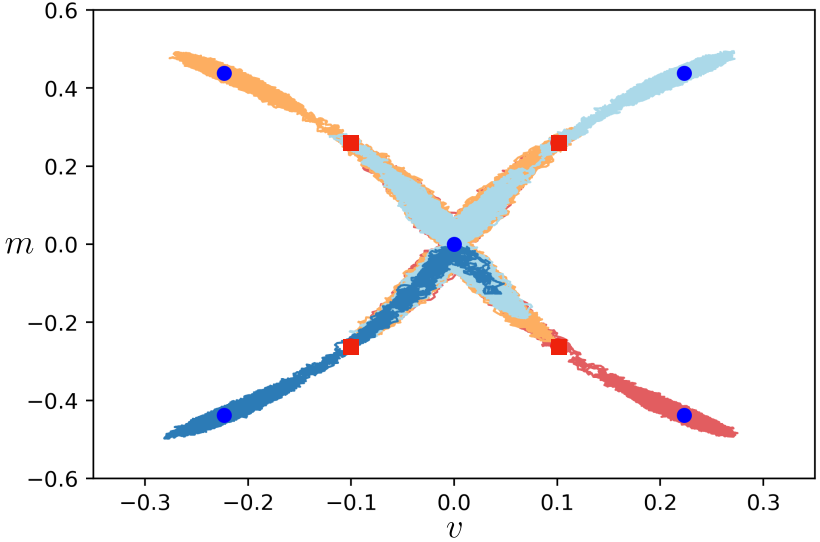
Vp – full potential intensity

c – wind shear

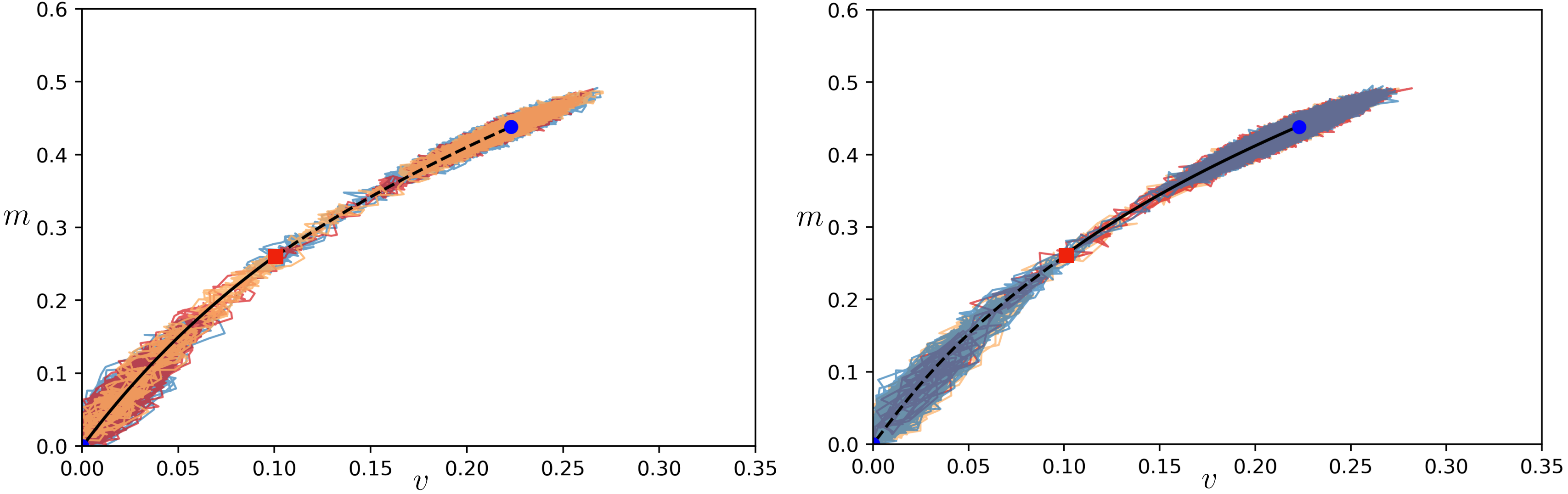
Noise-Induced Tipping

$$\tilde{f}(v,m) = \begin{cases} f(v,m) & \text{if } v,m \geq 0 \\ -f(-v,m) & \text{if } v < 0, m < 0 \\ -f(-v,-m) & \text{if } v < 0, m < 0 \\ f(v,-m) & \text{if } v > 0, m < 0 \end{cases},$$

$$\tilde{g}(v,m) = \begin{cases} g(v,m) & \text{if } v,m \geq 0 \\ -g(-v,m) & \text{if } v < 0, m < 0 \\ -g(-v,-m) & \text{if } v < 0, m < 0 \\ g(v,-m) & \text{if } v > 0, m < 0 \end{cases}.$$



Noise-Induced Tipping



Can form or kill a storm!

Rate-Induced Tipping

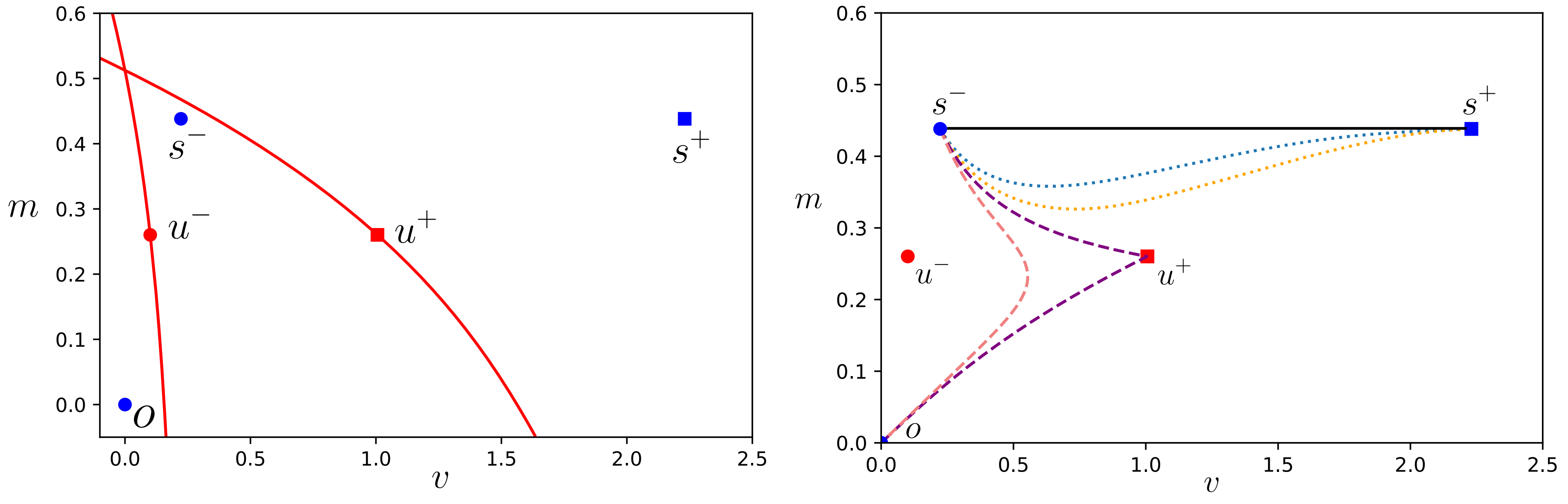
$$\frac{dv}{d\tau} = \frac{(1-\gamma)V_p(\Lambda(s))^2}{V_p^{-2}}m^3 - (1-\gamma m^3)v^2,$$

$$\frac{dm}{d\tau} = (1-m)v - c(\Lambda(s))m,$$

$$\frac{ds}{d\tau} = r.$$

Now assume ramping on the wind shear and max potential velocity

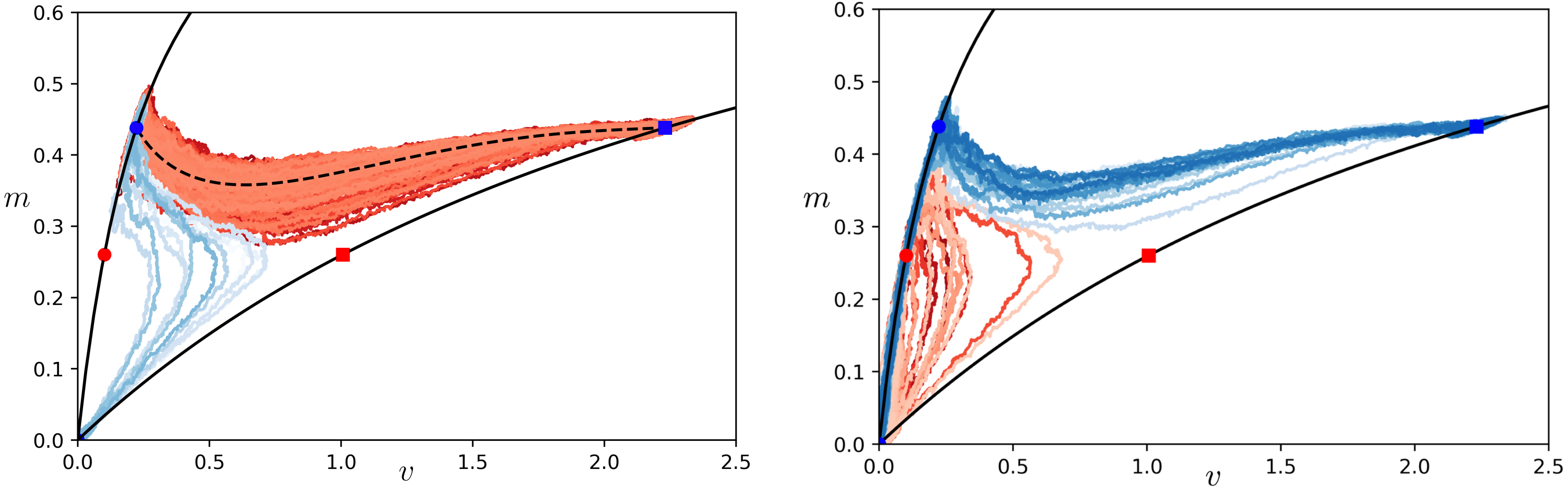
Rate-Induced Tipping



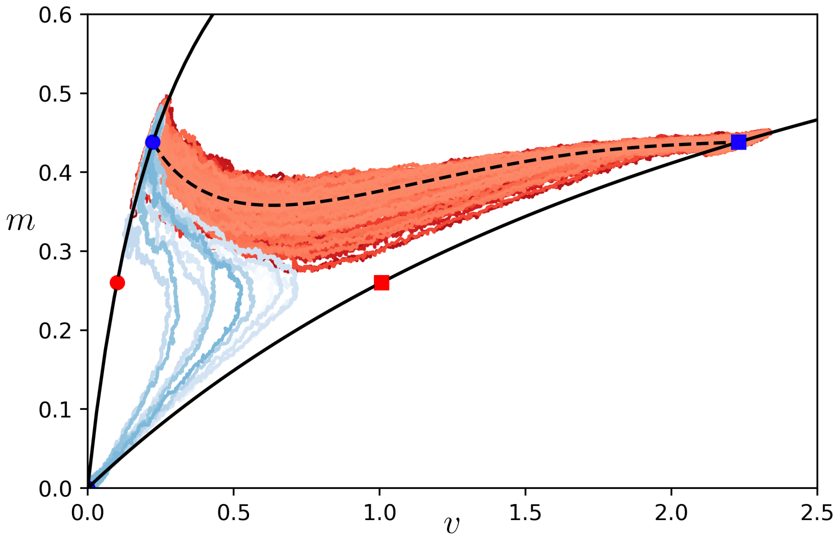
Both wind shear and max potential velocity both increasing to kill a storm!

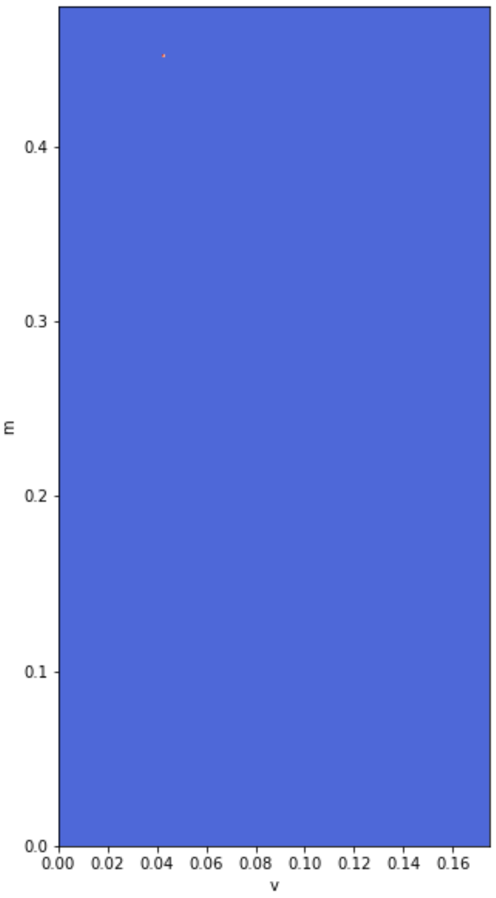
Can kill a storm but cannot form a storm!

Rate and Noise Together

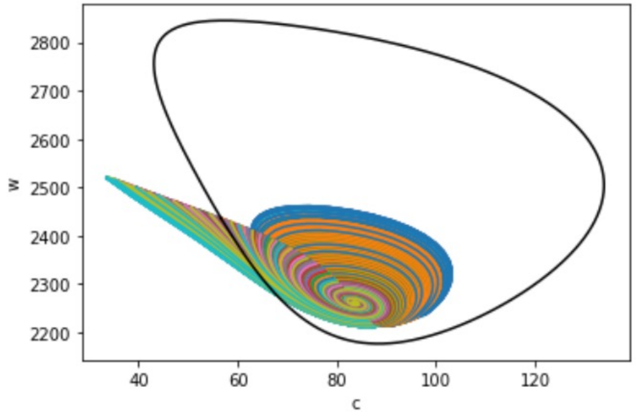


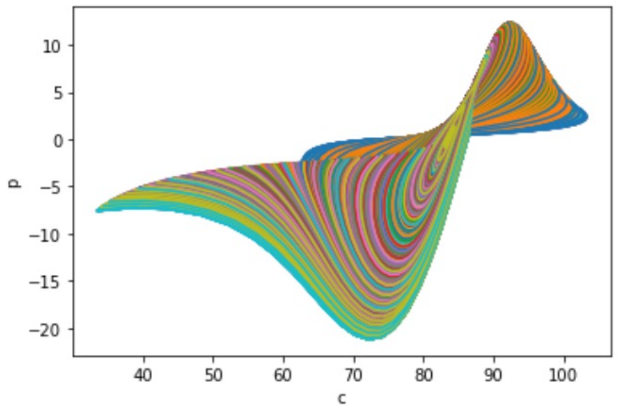
Rate and Noise Together

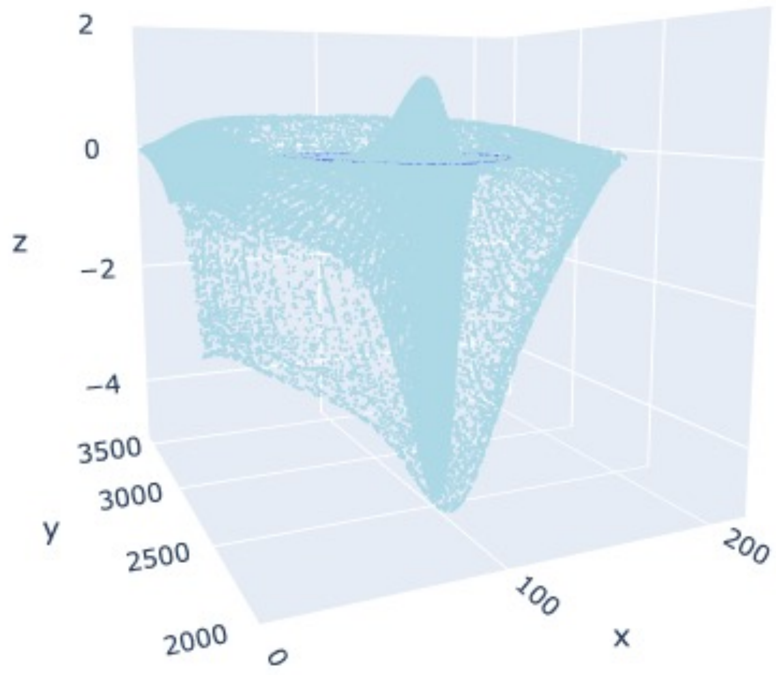




Back to the Carbon Cycle







Back to the Carbon Cycle

$$83.5809 < c < 83.5829 \text{ and } 4.6411\times10^{-56}\left(-2.51072\times10^{21}\sqrt{-7.04742\times10^{67}\,c^2+1.17807\times10^{70}\,c-4.92328\times10^{71}} - 2.29898\times10^{51}\,c+1.92153\times10^{53}\right) < p < \text{[illegible]}.6411\times10^{-56}\left(2.51072\times10^{21}\sqrt{-7.04742\times10^{67}\,\text{[illegible]}\times10^{70}\,c-4.92328\times10^{71}} - 2.29898\times10^{51}\,c+1.921\text{[illegible]}\right) \text{[illegible]nd } q =$$

$$-6.39788\times10^{-38}\sqrt{\left(-6\text{[illegible]}\,c^2-1.50389\times10^{64}\,c\,p+1.1273\times10^{70}\,c - 7.04742\times1\text{[illegible]} + 1.25698\times10^{66}\,p-4.71109\times10^{71}\right)} - 0.000525393\,c-0.212229\,p+0.0439134 \text{ and } w =$$

$$1.21773\times10^{-34}\sqrt{\left(-6.74368\times10^{67}\,c^2-1.50389\times10^{64}\,c\,p+1.1273\times10^{70}\,c - 7.04742\times10^{67}\,p^2+1.25698\times10^{66}\,p-4.71109\times10^{71}\right)} - 2.76038\times10^{-7}\,c-0.000111503\,p+2260.29$$

GROSS

Thank you!

Rate and Noise-Induced Tipping Working in Concert

Katherine Slyman[1] and Christopher K. Jones[2]

[1)]*Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, North Carolina 27517*

[2)]*Renaissance Computing Institute, University of North Carolina at Chapel Hill, Chapel Hill, North Carolina 27517*

(*Electronic mail: kslyman@ad.unc.edu.)

(Dated: 4 October 2022)

Rate-induced tipping occurs when a ramp parameter changes rapidly enough to cause the system to tip between co-existing, attracting states. We show that the addition of noise to the system can cause it to tip well below the critical rate at which rate-induced tipping would occur. Moreover it does so with significantly increased probability over the noise acting alone. We achieve this by finding a global minimizer in a canonical problem of the Freidlin-Wentzell action functional of large deviation theory that represents the most probable path for tipping. This is realized as a heteroclinic connection for the Euler-Lagrange system associated with the Freidlin-Wentzell action and we find it exists for all rates less than or equal to the critical rate. Its role as most probable path is corroborated by direct Monte Carlo simulations.

Office of Naval Research Science & Technology USN ONR

renci

NSF

MCRN