An Introduction to Budyko's Energy Balance Model

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Mathematics of Climate Seminar
September 10, 2024

Mathematics and Climate Research Network

https://sites.google.com/view/math-climate

Budyko's Model

What determines the Earth's surface temperature?

temperature change ~ energy in – energy out

energy in from the Sun ↗ ↖ energy out from the Earth

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

energy in from the Sun: $Q = 342\ \mathrm{W/m^2}$

energy out from the Earth: $\sigma T^4\ \mathrm{W/m^2}$

$$T = (342/\sigma)^{1/4} = (342/5.67\times10^{-8})^{1/4} =$$
$$279\mathrm{K} = 6^\circ\mathrm{C} = 43^\circ\mathrm{F}$$

Earth's global mean temperature: 57°F *Not bad!*

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Budyko's Model

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Note that there is a differential equation lurking here.

temperature change ~ energy in – energy out

$$R\frac{dT}{dt} = Q - \sigma T^4$$

heat capacity → R

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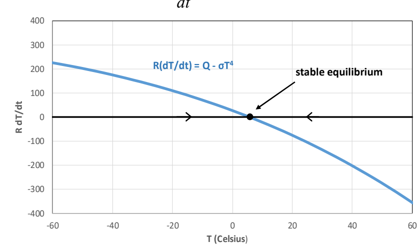
Budyko's Model

temperature change ~ energy in – energy out

heat capacity → $R\frac{dT}{dt} = Q - \sigma T^4$



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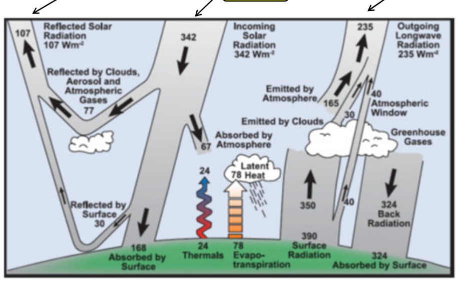
Budyko's Model

albedo insolation OLR



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

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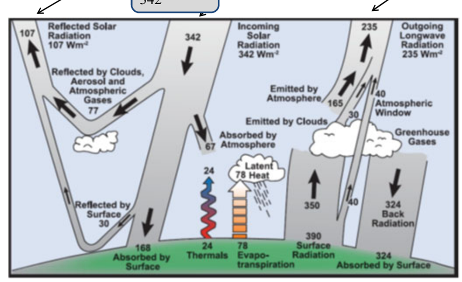
Budyko's Model

albedo $\frac{107}{342} \approx 0.313$ insolation OLR



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Budyko's Model

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α.

For Earth, $\alpha \approx 0.3$.

Budyko's Model

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Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = \left(0.7 \cdot Q/\sigma\right)^{1/4} = \left(0.7 \cdot 342 / 5.67 \times 10^{-8}\right)^{1/4}$$
$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

stable equilibrium

Budyko's Model

Albedo

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Way too cold!
What happened to 57°F?

Budyko's Model

Albedo

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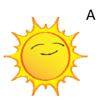
Way too cold!
What happened to 57°F?

255 K is actually the *photosphere* temperature.

Budyko's Model

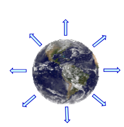
Photosphere



A photosphere is the deepest region of a luminous object, usually a star, that is transparent to photons of certain wavelengths.

https://en.wikipedia.org/wiki/Photosphere

For the Earth, the photosphere is where the long wave photons escape into space. It is high in the atmosphere where the temperature is 255 K.

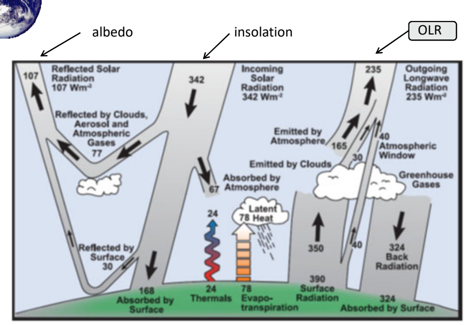


$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

T = photosphere temperature.

What about the surface temperature?

Budyko's Model



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Budyko's Model

OLR as a Function of Surface Temperature
(Outgoing Longwave Radiation)

$$\mathrm{OLR} \approx A + BT$$

A and B are determined from satellite observations.
T is surface temperature (*in Celsius*).

$$A = 202\ \mathrm{W/m^2}$$
$$B = 1.90\ \mathrm{W/m^2K}$$

Dynamics

Kelvin → $$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$ photosphere temperature

becomes

Celsius → $$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$ global mean surface temperature

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Math 5490

Energy Balance

OLR as a Function of Surface Temperature
(Outgoing Longwave Radiation)

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Math 5490 9/12/2023

Budyko's Model

OLR as a Function of Surface Temperature

$$\mathrm{OLR} \approx A + BT$$

Important:
A+BT is not a linear approximation to the Stefan-Boltzmann equation.

different

Dynamics

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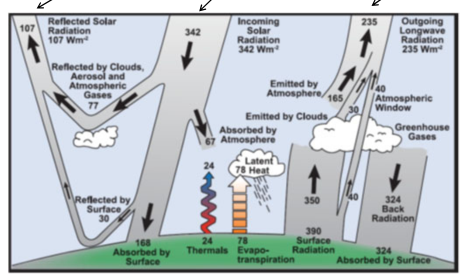
Budyko's Model

albedo   insolation   OLR



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Budyko's Model

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha) - A}{B}$$

Recall: Q = 342 W/m², A = 202 W/m², B = 1.9 W/m²K

Wikipedia: $0.30 \le \alpha \le 0.35$

$\alpha = 0.30$, T^* = 19.7°C = 67°F
$\alpha = 0.32$, T^* = 16.1°C = 61°F
$\alpha = 0.33$, T^* = 14.3°C = 58°F ← *Not bad!*
$\alpha = 0.35$, T^* = 10.7°C = 51°F

Earth's global mean temperature: 57°F

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Is it stable?

$$R\frac{dT}{dt} = \left(Q(1-\alpha) - A\right) - BT$$

Stable, since $B > 0$.

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What if Earth had more ice or less ice?
That would change the albedo α.

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Budyko's Model

Homogeneous Earth

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Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

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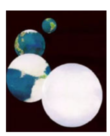
Current Earth: $\alpha = 0.33$, $T^* = 14.3$°C = 58°F

guesses → Ice-free Earth: $\alpha = 0.28$, $T^* = 23.3$°C = 74°F

guesses → Snowball Earth: $\alpha = 0.62$, $T^* = -38$ °C = -36°F

Why do we have ice caps?

If Earth was ever a snowball, how did we get out?



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Budyko's Model

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

What's missing?

Earth is not homogeneous. For example, it is warmer at the equator and colder at the poles. The temperature should depend on latitude.

Make T depend on y = sin(latitude)

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation distribution

$s(y) =$ distribution across latitudes $\left(\int_0^1 s(y)dy = 1\right)$

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Budyko's Model

Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Why do we use y = sine(latitude) instead of just latitude?

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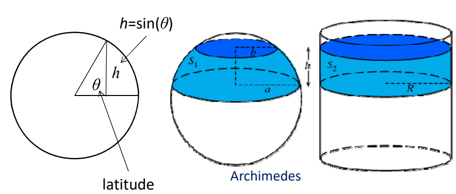
Budyko's Model

Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Why do we use y = sine(latitude) instead of just latitude?

Because y is directly proportional to surface area.



http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html

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Budyko's Model

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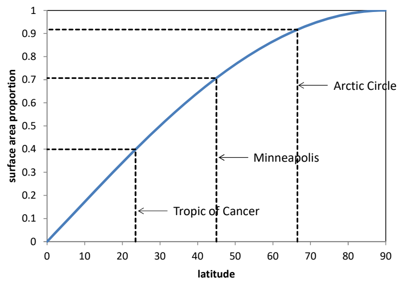


Budyko's Model

Latitiude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

$s(y) =$ distribution across latitudes $\left(\int_0^1 s(y)dy = 1\right)$

One can show that β = obliquity = 23.4°

$$s(y) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\theta - y\cos\beta\right)^2}\,d\theta$$

McGehee & Lehman, *SIAM J. Applied Dynamical Systems* 11 (2) (2012), 684–707.

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241(3y^2 - 1)$$

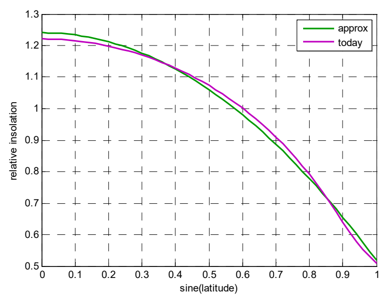
Chylek & Coakley, *J. Atmos. Sci.* 32 (1975), 675–679.

Budyko's Model

Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.5°



Budyko's Model

Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Note that an equilibrium temperature T is a function of y = sine(latitude), denoted $T^*(y)$

$$Qs(y)(1-\alpha) - (A + BT^*(y)) = 0$$

$$T^*(y) = \frac{Qs(y)(1-\alpha) - A}{B}$$

Budyko's Model

Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Equilbrium Solution

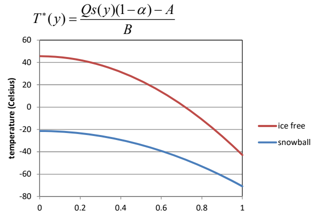
$$T^*(y) = \frac{Qs(y)(1-\alpha) - A}{B}$$

Tung*

α = 0.32: ice free
α = 0.62: snowball

Note that α = 0.32 is in the range of current Earth.

* K.K. Tung, *Topics in Mathematical Modeling*, Princeton U. Press, 2007



Budyko's Model

Latitude Dependence

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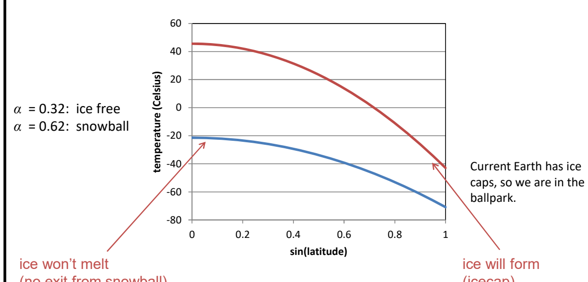

Current Earth has ice caps, so we are in the ballpark.

Budyko's Model

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

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Budyko's Model

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

The Second Law of Thermodynamics

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Budyko's Model

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

The Second Law of Thermodynamics

One simple statement of the law is that heat always moves from hotter objects to colder objects ...

https://en.wikipedia.org/wiki/Second_law_of_thermodynamics

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Budyko's Model

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

The Second Law of Thermodynamics

It's hotter at the equator than at the poles, so heat moves from the lower latitudes to the higher latitudes.

How?

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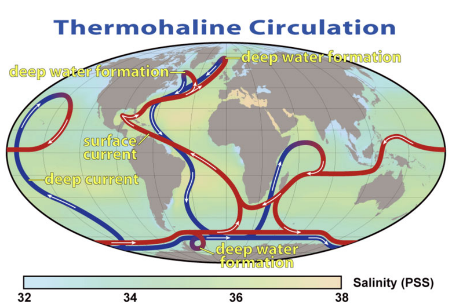
Budyko's Model

What's Missing?



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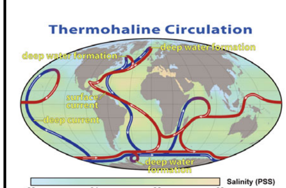
Budyko's Model

What's Missing?



Example

The Gulf Stream carries warm salty surface water from the Gulf of Mexico to the North Atlantic, where it cools, becomes more dense, and sinks, flowing south in the deep ocean.

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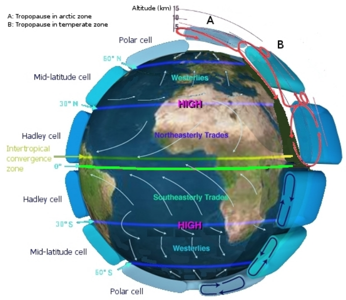
Budyko's Model

What's Missing?



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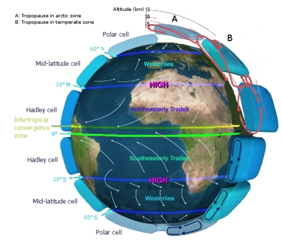
Budyko's Model

What's Missing?

Warm air from the surface along the equator rises and flows toward the poles in a series of cells moving heat from the equator to the poles.



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Budyko's Model

What's Missing?

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Budyko's Model

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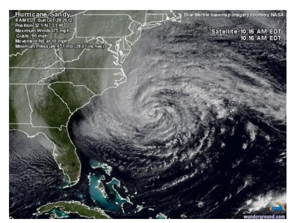

Atlantic hurricanes move heat from the equatorial Atlantic up the coast of North America.

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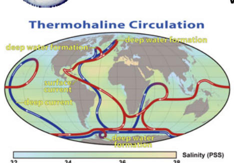
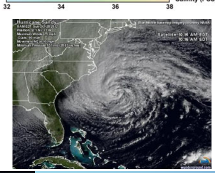
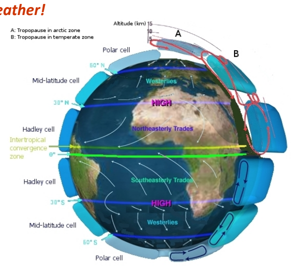
Budyko's Model

What's Missing?





Weather!



The second law of thermodynamics

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Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right) + C\left(\overline{T}(t) - T(y,t)\right)$$

2nd law

global mean temperature $\overline{T}(t) = \int_0^1 T(y,t)dt$

$\left(\overline{T}(t) - T(y,t)\right)$ interpretation

Each point on Earth's surface is trying to assume the global mean temperature. If the temperature at a point is below the the global mean, then it heats up. If the temperature at that point is above the mean, then it cools off.

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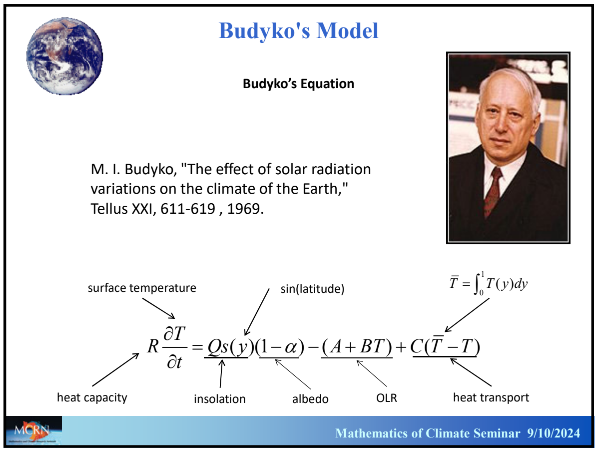
Budyko's Model
Budyko's Equation
M. I. Budyko, "The effect of solar radiation variations on the climate of the Earth," Tellus XXI, 611-619 , 1969.
surface temperature
sin(latitude)
$\bar{T} = \int_0^1 T(y)dy$
$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha)-(A+BT)+C(\bar{T}-T)$
heat capacity
insolation
albedo
OLR
heat transport
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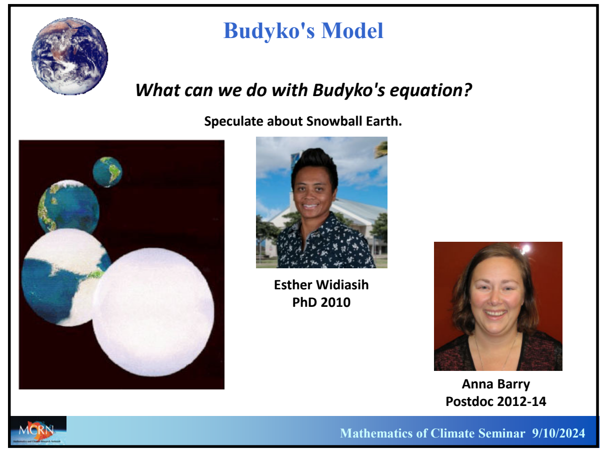
Budyko's Model
What can we do with Budyko's equation?
Speculate about Snowball Earth.
Esther Widiasih
PhD 2010
Anna Barry
Postdoc 2012-14
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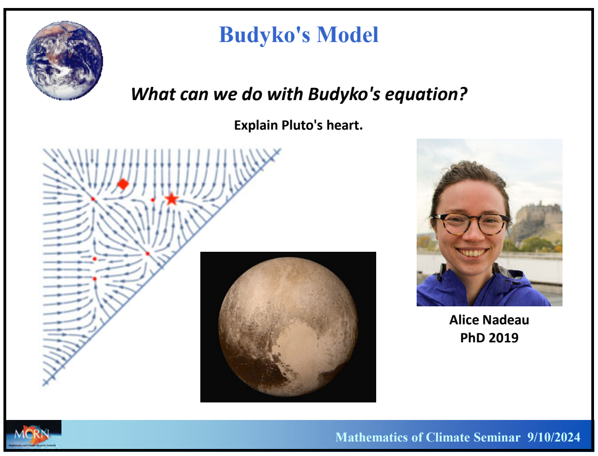
Budyko's Model
What can we do with Budyko's equation?
Explain Pluto's heart.
Alice Nadeau
PhD 2019
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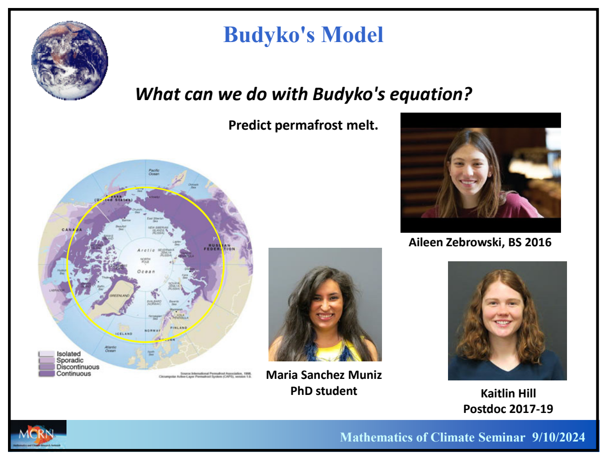
Budyko's Model
What can we do with Budyko's equation?
Predict permafrost melt.
Aileen Zebrowski, BS 2016
Maria Sanchez Muniz
PhD student
Kaitlin Hill
Postdoc 2017-19
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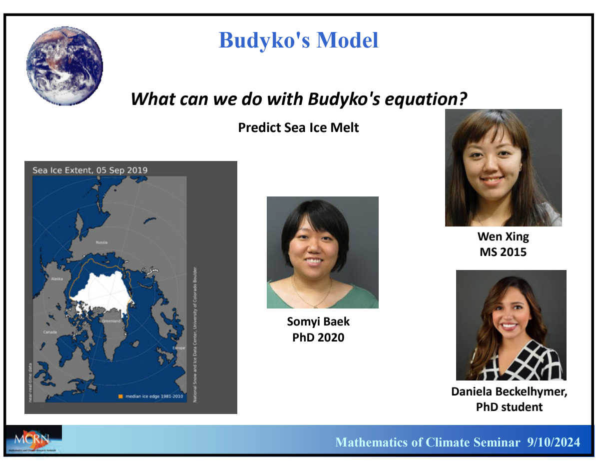
Budyko's Model
What can we do with Budyko's equation?
Predict Sea Ice Melt
Sea Ice Extent, 05 Sep 2019
Wen Xing
MS 2015
Somyi Baek
PhD 2020
Daniela Beckelhymer,
PhD student
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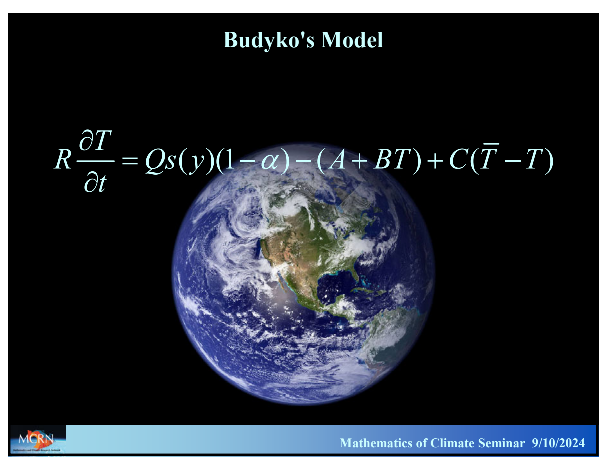
Budyko's Model
$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha)-(A+BT)+C(\bar{T}-T)$
Mathematics of Climate Seminar 9/10/2024