Introduction to Multiflows

Richard McGehee
School of Mathematics
University of Minnesota

Mathematics of Climate Seminar
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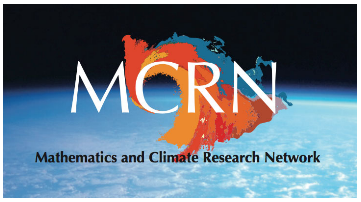

https://sites.google.com/view/math-climate

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Multiflows

Well-posed Problems

Jacques Hadamard

https://upload.wikimedia.org/wikipedia/commons/a/ae/Hadamard2_cropped.jpg

According to Wikipedia

The mathematical term ***well-posed problem*** stems from a definition given by 20th-century French mathematician Jacques Hadamard. He believed that mathematical models of physical phenomena should have the properties that:

1. a solution exists,
2. the solution is unique,
3. the solution's behaviour changes continuously with the initial conditions.

https://en.wikipedia.org/wiki/Well-posed_problem

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Well-posed Problems

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Metaphysical Preference?

https://en.wikipedia.org/wiki/Well-posed_problem

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Multiflows

Well-posed Problems

Basic Theorem from Dynamical Systems

If f is a smooth vector field in Euclidean space, then the initial value problem

$$\frac{dy}{dt} = f(y), \quad y(0) = x$$

satisfies the following conditions:

1. a solution exists,
2. the solution is unique,
3. the solution's behavior changes continuously with the initial conditions.

I.e., the initial value problem is **well-posed**.

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Multiflows

Well-posed Problems

Basic Theorem from Dynamical Systems

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1. a solution exists,
2. the solution is unique,
3. the solution's behavior changes continuously with the initial conditions.

What if f isn't smooth? What if is it only continuous? What happens?

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Multiflows

Example from Sophomore Calculus

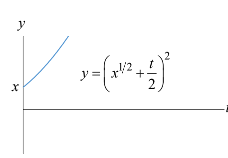
$$\frac{dy}{dt} = |y|^{1/2}, \quad y(0) = x$$

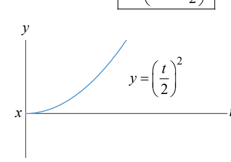
not smooth
not even Lipschitz continuous

Case: $y \ge 0$

$$\frac{dy}{dt} = y^{1/2}, \quad y^{-1/2}dy = dt, \quad \int y^{-1/2}dy = \int dt, \quad 2y^{1/2} = t + c$$

$$y(0) = x \;\Rightarrow\; 2x^{1/2} = c \;\Rightarrow\; 2y^{1/2} = t + 2x^{1/2} \;\Rightarrow\; \boxed{y = \left(x^{1/2} + \frac{t}{2}\right)^2}$$





Note that $y = 0$ is also a solution when $x = 0$.

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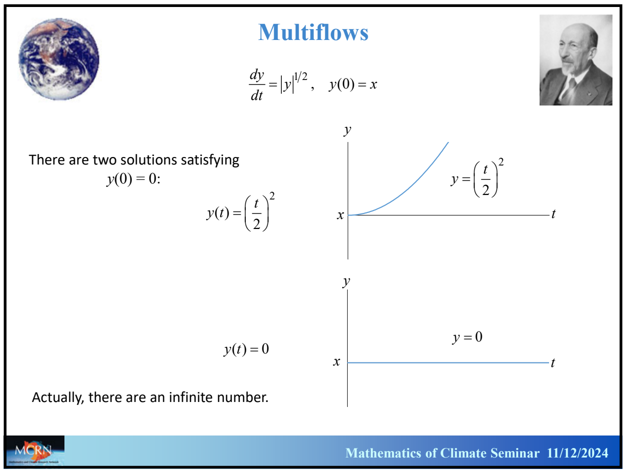
Multiflows
$\frac{dy}{dt} = |y|^{1/2}, \quad y(0) = x$
There are two solutions satisfying $y(0) = 0$:
$y(t) = \left(\frac{t}{2}\right)^2$
$y = \left(\frac{t}{2}\right)^2$
$y(t) = 0$
$y = 0$
Actually, there are an infinite number.
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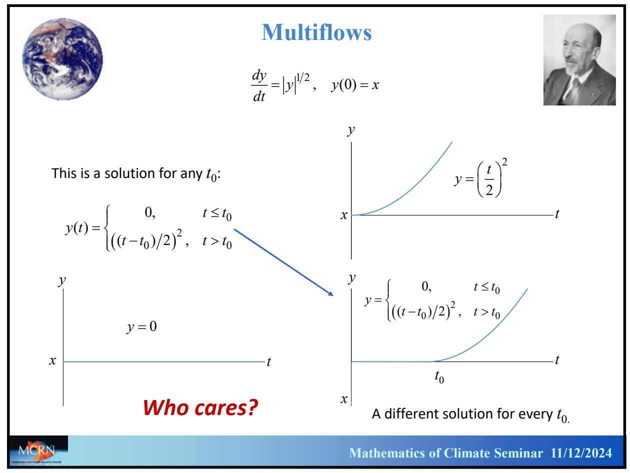
Multiflows
$\frac{dy}{dt} = |y|^{1/2}, \quad y(0) = x$
This is a solution for any t_0:
$y(t) = \begin{cases} 0, & t \le t_0 \\ ((t-t_0)/2)^2, & t > t_0 \end{cases}$
$y = \left(\frac{t}{2}\right)^2$
$y = 0$
$y = \begin{cases} 0, & t \le t_0 \\ ((t-t_0)/2)^2, & t > t_0 \end{cases}$
Who cares?
A different solution for every t_0.
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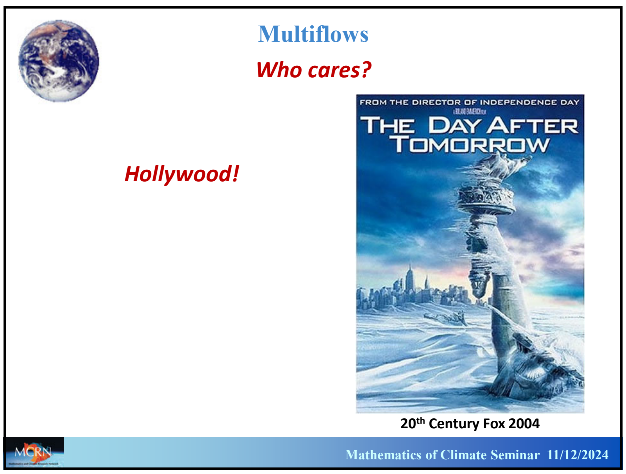
Multiflows
Who cares?
Hollywood!
FROM THE DIRECTOR OF INDEPENDENCE DAY
THE DAY AFTER TOMORROW
20th Century Fox 2004
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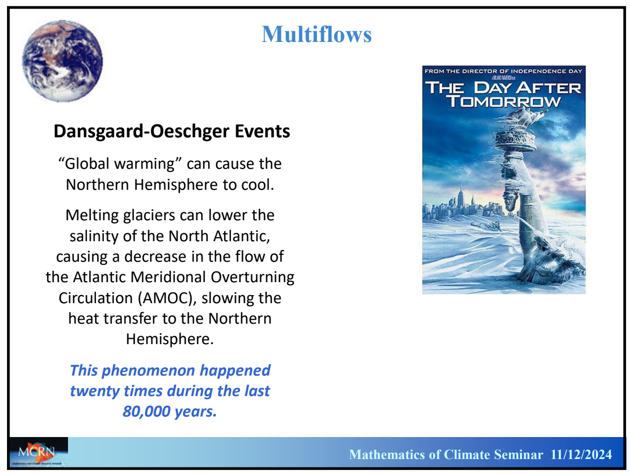
Multiflows
Dansgaard-Oeschger Events
"Global warming" can cause the Northern Hemisphere to cool.
Melting glaciers can lower the salinity of the North Atlantic, causing a decrease in the flow of the Atlantic Meridional Overturning Circulation (AMOC), slowing the heat transfer to the Northern Hemisphere.
This phenomenon happened twenty times during the last 80,000 years.
FROM THE DIRECTOR OF INDEPENDENCE DAY
THE DAY AFTER TOMORROW
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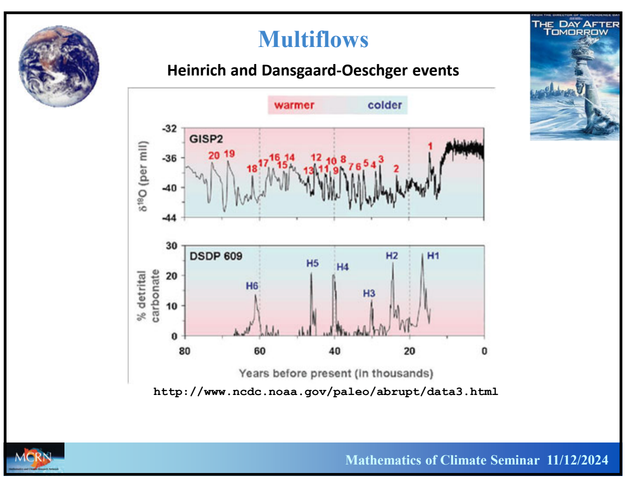
Multiflows
Heinrich and Dansgaard-Oeschger events
warmer
colder
GISP2
δ18O (per mil)
DSDP 609
% detrital carbonate
H6
H5
H4
H3
H2
H1
Years before present (in thousands)
http://www.ncdc.noaa.gov/paleo/abrupt/data3.html
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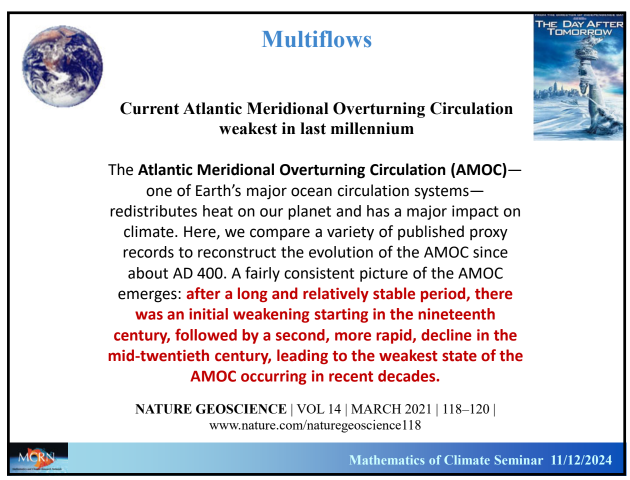
Multiflows
Current Atlantic Meridional Overturning Circulation weakest in last millennium
The Atlantic Meridional Overturning Circulation (AMOC)—one of Earth's major ocean circulation systems—redistributes heat on our planet and has a major impact on climate. Here, we compare a variety of published proxy records to reconstruct the evolution of the AMOC since about AD 400. A fairly consistent picture of the AMOC emerges: after a long and relatively stable period, there was an initial weakening starting in the nineteenth century, followed by a second, more rapid, decline in the mid-twentieth century, leading to the weakest state of the AMOC occurring in recent decades.
NATURE GEOSCIENCE | VOL 14 | MARCH 2021 | 118–120 | www.nature.com/naturegeoscience118
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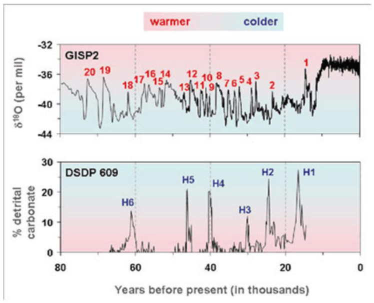
Multiflows

Heinrich and Dansgaard-Oeschger events



http://www.ncdc.noaa.gov/paleo/abrupt/data3.html

What caused the oscillations?

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Multiflows

What caused the D-O oscillations?

They could be self-oscillations in the natural dynamics of ocean circulation.

Pierre Welander, A simple heat-salt oscillator, *Dynamics of Atmospheres and Oceans* **6** (1982) 233-242.

R/V WeeLander is a 23-foot-long Beach Master work boat, informally named in honor of Professor Pierre Welander (1925—1996).

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Multiflows

Welander Model

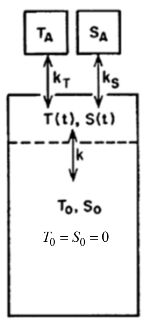
mixing with atmosphere

temperature → $\dot{T} = k_T(T_A - T) - k(\rho)T$

salinity → $\dot{S} = k_S(S_A - S) - k(\rho)S$

density → $\rho = -\alpha T + \gamma S$

mixing with deep ocean



Pierre Welander, *Dynamics of Atmospheres and Oceans* **6** (1982).

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Multiflows

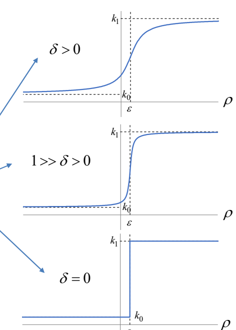
Welander Model

$$\dot{T} = k_T(T_A - T) - k(\rho)T$$
$$\dot{S} = k_S(S_A - S) - k(\rho)S$$
$$\rho = -\alpha T + \gamma S$$

The function k

$$k(\rho) = \frac{k_1 + k_0}{2} + \frac{k_1 - k_0}{\pi}\tan^{-1}\left(\frac{\rho - \varepsilon}{\delta}\right)$$

Limit as $\delta \to 0$ $\quad k(\rho) = \begin{cases} k_0, & \rho < \varepsilon \\ k_1, & \rho > \varepsilon \end{cases}$



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Multiflows

Welander Model

$$\dot{T} = k_T(T_A - T) - k(\rho)T$$
$$\dot{S} = k_S(S_A - S) - k(\rho)S$$
$$\rho = -\alpha T + \gamma S$$

Welander chose scientifically reasonable values and dimensionless variables and constants.

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1 - S) - k(\rho)S$$
$$\rho = -\alpha T + S$$

$$k(\rho) = \frac{1}{2} + \frac{2}{\pi}\tan^{-1}\left(\frac{\rho - \varepsilon}{\delta}\right)$$

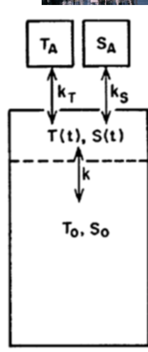
$\alpha = 0.8$

$\beta = 0.5$

ε: small parameter

Limit as $\delta \to 0$ $\quad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$

Welander simulated the case $\delta = 0$, finding a periodic orbit, and concluded the same would hold for $\delta > 0$.



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Multiflows

Welander Model

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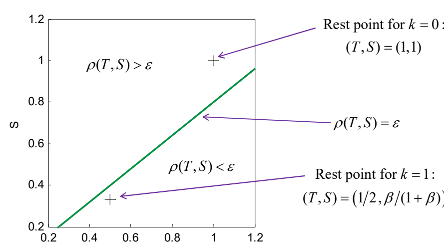

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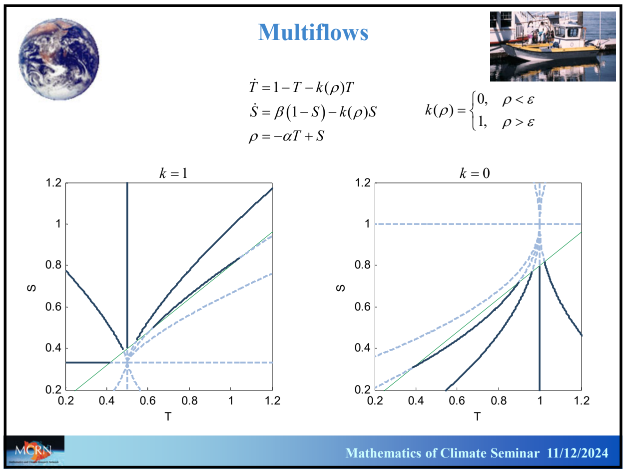

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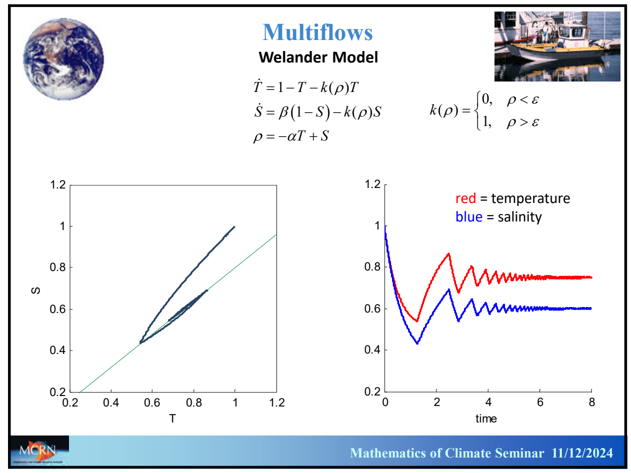

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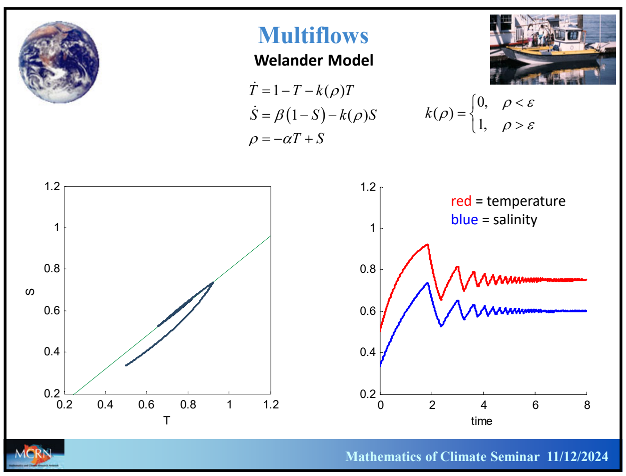

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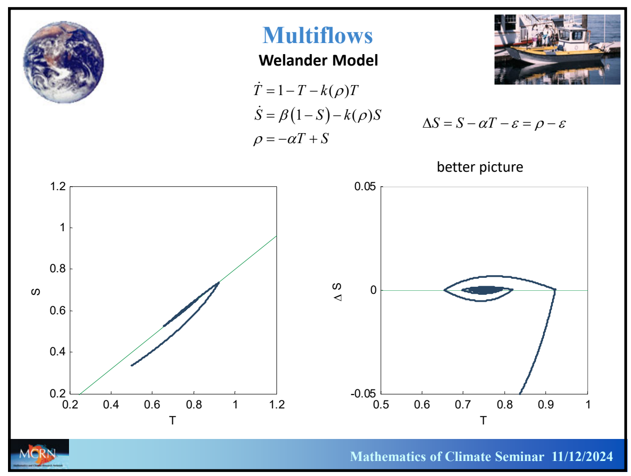

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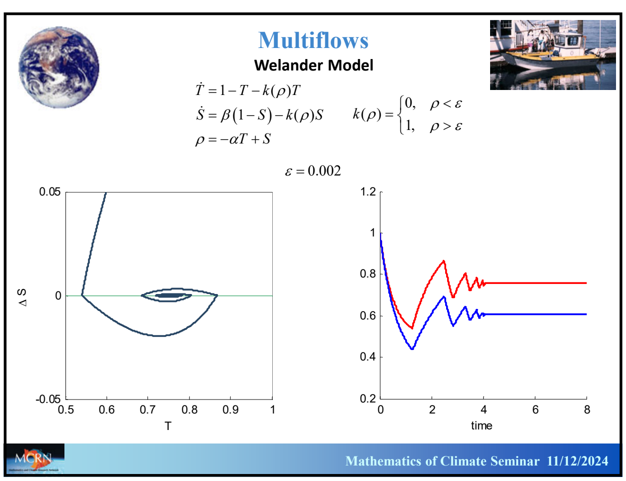

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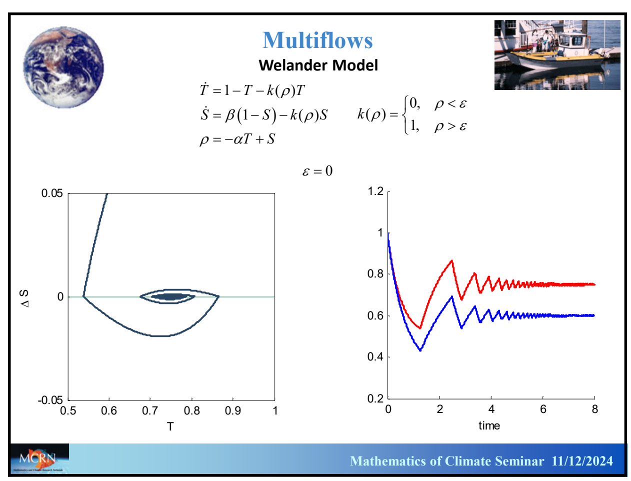

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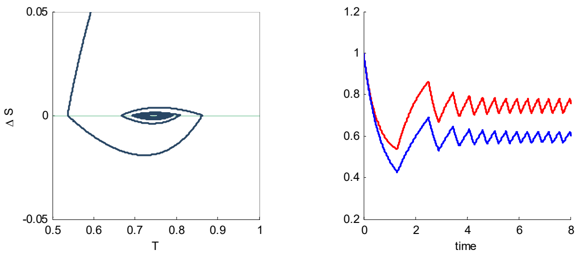
Multiflows

Welander Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$
$$\rho = -\alpha T + S$$

$\varepsilon = -0.002$



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Multiflows

Welander Model

$$\dot{T} = 1 - T - k(\rho)T$$
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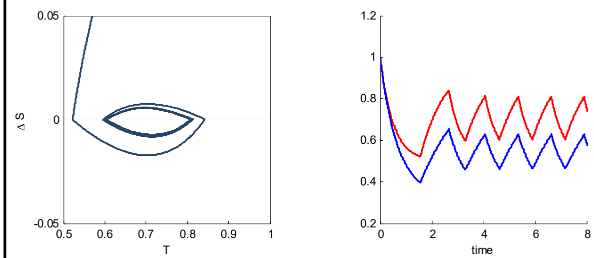
$$\rho = -\alpha T + S$$

$\varepsilon = -0.02$



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Multiflows

Welander Model

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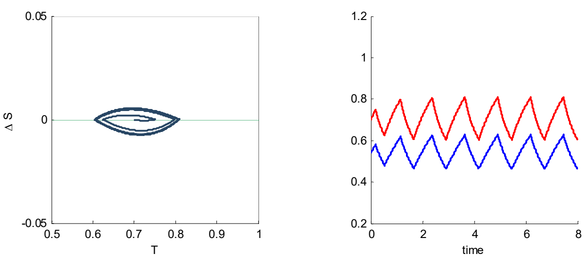

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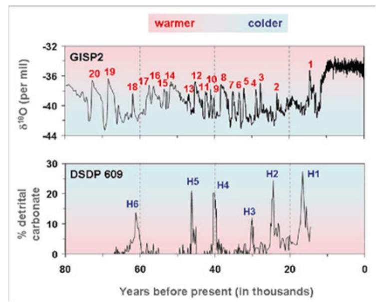
Multiflows

Heinrich and Dansgaard-Oeschger events



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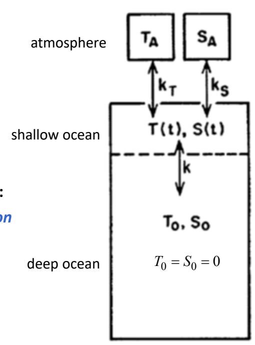
Multiflows

Welander found the discontinuous limit easier to analyze. He assumed that the self-oscillations he found in the discontinuous limit held for the smooth system with a small parameter.

Juliann Leifeld, 2016:

Welander's assumption was correct.



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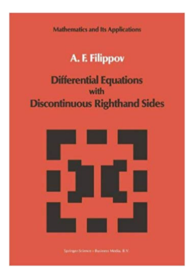
Multiflows

Filippov Systems



Kluwer Academic Publishers

Roughly

- The Euclidean space is partitioned by a finite number of sets.
- The boundaries are codimension 1.
- The vector field can be thought of as a finite number of vector fields, each defined and smooth on a partition set, including the boundary.
- The individual vector fields take different values on the boundaries.

A.F. Filippov*

*https://alchetron.com/Aleksei-Fedorovich-Filippov

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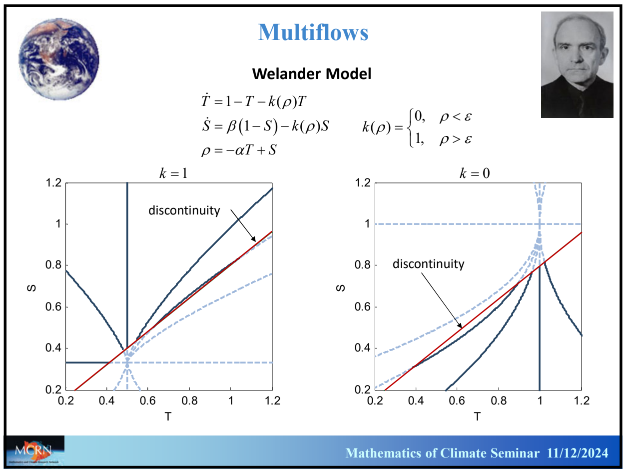

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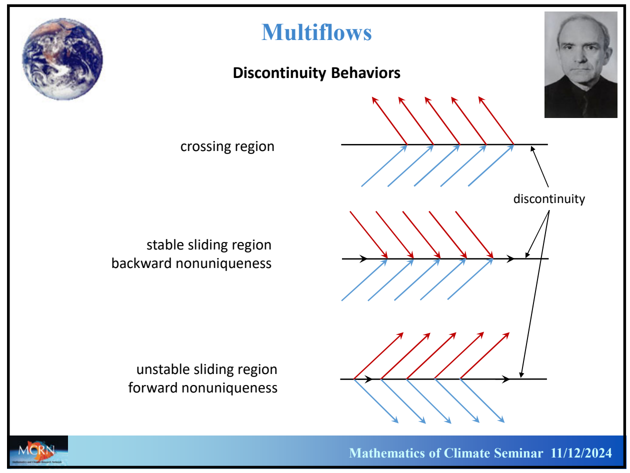

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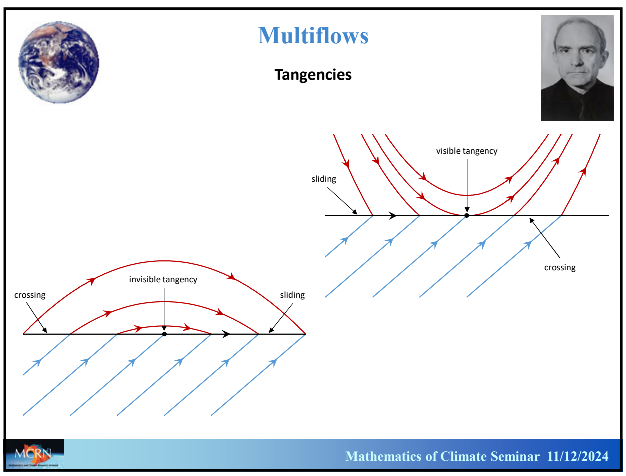

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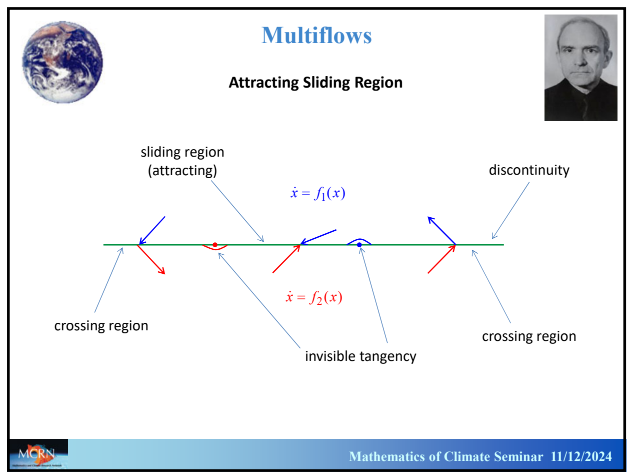

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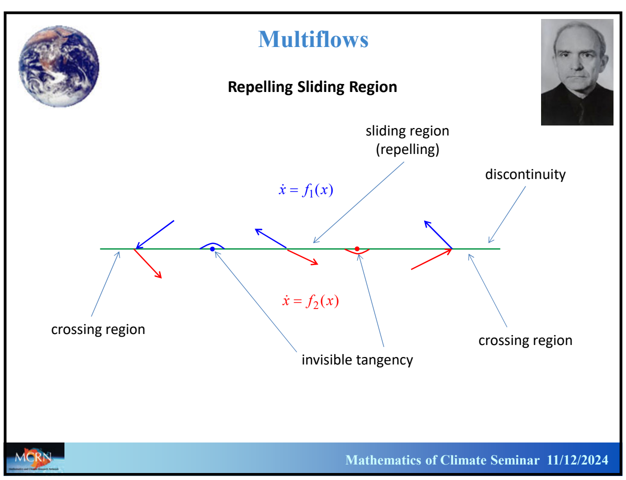

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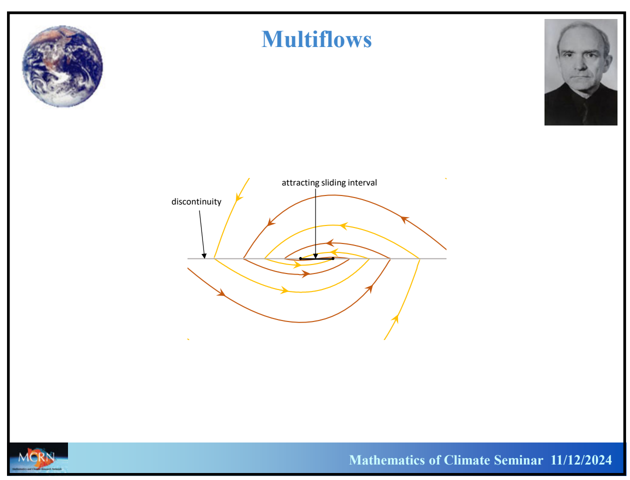

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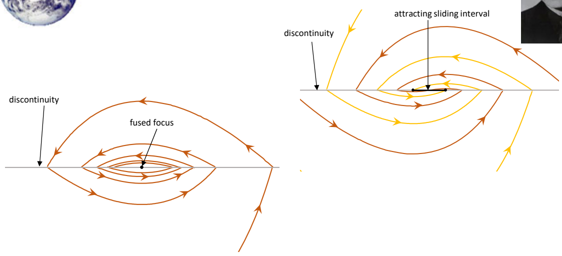

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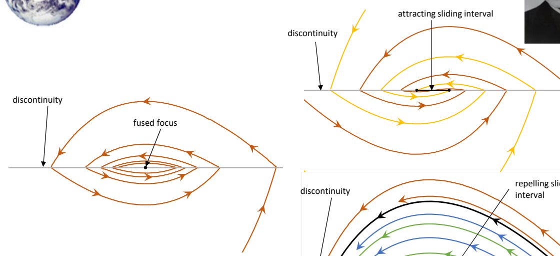
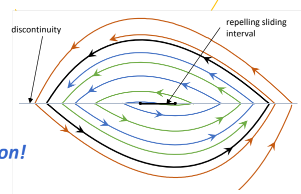
Multiflows





Looks a lot like a Hopf bifurcation!

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Multiflows

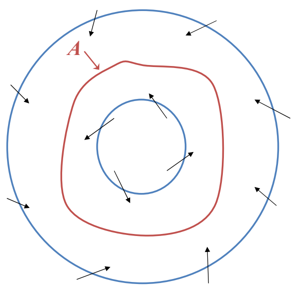
Conley Index Theory

Inside the annulus, there is an attracting set that "looks like" a circle.

Question

Is there a Conley theory for Filippov systems?

If so, what good would it be?



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Multiflows

Relations

Definitions

A *relation* on a set X is a subset of $X \times X$.

Let f and g be relations on a set X. The *composition* of f and g is the relation
$f \circ g = \{(x,z) \in X \times X :$ there exists $y \in X$ with $(x,y) \in g$ and $(y,z) \in f\}$.

Theorem

If f and g are closed relations on a compact metric space X, then $f \circ g$ is closed.

Note

The space of closed relations on a compact metric space X, together with composition as the binary operator, is a semigroup with identity
$\mathrm{id} = \{(x,y) \in X \times X : x = y\}$.

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Multiflows

Definition

Let X be a compact metric space.
A *multiflow* on X is a subset Φ of $[0,\infty) \times X \times X$ satisifying
(1) Φ is closed,
(2) $\Phi^0 = \mathrm{id}$,
(3) $\Phi^{t+s} = \Phi^t \circ \Phi^s$.

where

$$\Phi^t = \{(x,y) \in X \times X : (t,x,y) \in \Phi\}$$

Note

A multiflow Φ determines a semigroup homomorphism
from $[0,\infty)$ to the space of closed relations: $t \mapsto \Phi^t$.

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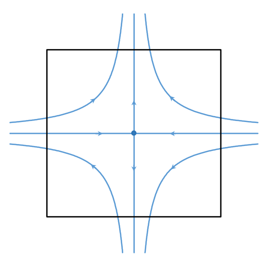
Multiflows

Theorem: (*Cameron Thieme 2020*) A flow restricted to a compact set is a multiflow.



McGehee & Sander, A new proof of the stable manifold theorem. *Z. angew. Math. Phys.* **47** (1996), 497-513.

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Multiflows

Multiflow Examples

Theorem: (*McGehee 2018*) A continuous vector field generates a multiflow.

Lischitz continuous vector fields generate flows.
Continuous vector fields might not.
Why?
Uniqueness of solutions disappears.
Example:

$$\frac{dy}{dt}=|y|^{1/2},\quad y(0)=x$$

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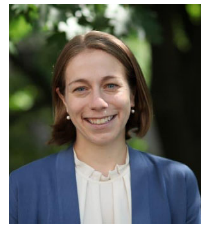
Multiflows

Multiflow Examples

Theorem: (*Cameron Thieme 2020*)
Filippov Systems generate multiflows.



Theorem: (*Kate Meyer 2019*) Certain control systems generate multiflows. Specifically, the system

$$\frac{dx}{dt}=f(x)+g(t),\quad x\in\mathbb{R}^n,\ f \text{ is smooth},\ \|g\|_\infty \le r,$$

generates a multiflow.

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Multiflows

So what?

I.e., so we show that something generates a multiflow. ***What good is that?***

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Multiflows

So what?

I.e., so we show that something generates a multiflow. ***What good is that?***

What is "good"?

Modeling Dogma
A ***good*** model is well-posed.

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Multiflows

So what?

I.e., so we show that something generates a multiflow. ***What good is that?***

What is "good"?

Modeling Dogma
A ***good*** model is well-posed.

Heresy Avoidance
A bad model is ***pretty good*** if it says something nice about a "nearby" good model.

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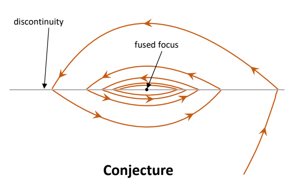
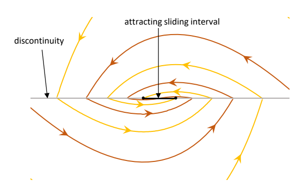
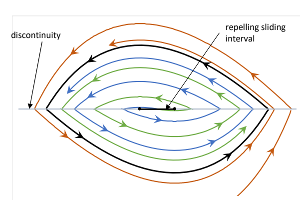
Multiflows







Conjecture
If a Filippov system is the continuous limit of a smooth system (in the "right" toplogy on multiflows), then the smooth system will have a nearby attractor limiting to the periodic orbit.
Sort of a Conley Hopf bifurcation.

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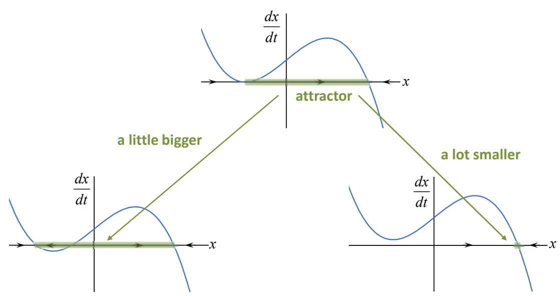
Multiflows
Semicontinuity of Attractors

As parameters change, attractors can get a little bigger or a little smaller or a lot smaller, but they can't get a lot bigger.



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Multiflows

Multiflows

Cameron Thieme, PhD Thesis, 2020

Thieme gave a definition of semicontinuity in a one-parameter family of multiflows.

Thieme gave conditions concluding that Conley Theory is valid for these one-parameter families.

Welander's model satisfies the Thieme's definition and conditions.

Welander's "nearby" smooth model has an attractor with the topology of (homotopic to) a circle.

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Multiflows

Open Questions

Can Thieme's definition and conditions be generalized further?

Are there other interesting examples?
Artic sea ice?
Controlling epidemics?
Adaptive chemotherapy?

McGehee's Pipedream
Can singular perturbation problems (aka fast-slow systems) be formulated as multiflow systems?

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Multiflows

Well-posed Problems

According to Wikipedia

The mathematical term ***well-posed problem*** stems from a definition given by 20th-century French mathematician Jacques Hadamard. He believed that mathematical models of physical phenomena should have the properties that:
1. a solution exists,
2. the solution is unique,
3. the solution's behaviour changes continuously with the initial conditions.

https://en.wikipedia.org/wiki/Well-posed_problem

Jacques Hadamard

https://upload.wikimedia.org/wikipedia/commons/a/ae/Hadamard2_cropped.jpg

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Multiflows

Not-Too-Badly Posed Problem

The mathematical term ***well-posed problem*** stems from a definition given by 20th-century French mathematician Jacques Hadamard. He believed that mathematical models of physical phenomena should have the properties that:
1. a solution exists,
2. the solution is unique,
3. the solution's behaviour changes continuously with the initial conditions.

Avoiding Heresy:
A mathematical model is ***not-too-badly-posed*** if it can be described as a multiflow. I.e., uniqueness can be abandoned and continuity with initial conditions can be replaced by semicontinuity with nearby systems.

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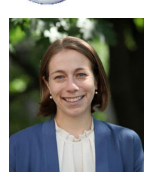
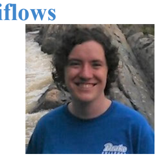
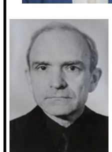
Multiflows






Thank-you!



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