An Introduction to Budyko's Energy Balance Model

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Mathematics of Climate Seminar
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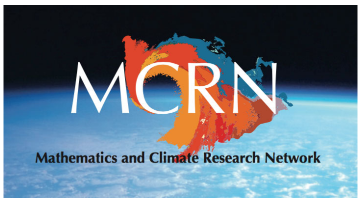

https://sites.google.com/view/math-climate

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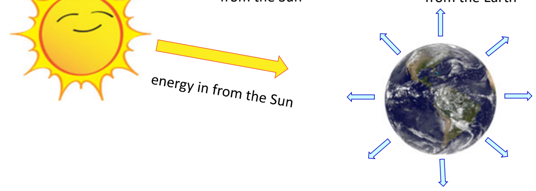
Budyko's Model

What determines the Earth's surface temperature?

temperature change ~ energy in – energy out

short wave energy from the Sun

long wave energy from the Earth



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Budyko's Model

Black-Body Radiation

Stefan-Boltzmann Law

$$F = \sigma T^4$$

power flux (W/m²) — watts per square meter

temperature (K) — kelvin

0 K = –273°C = "absolute zero"

Stefan-Boltzmann constant

$$\sigma \approx 5.67 \times 10^{-8}\ \mathrm{W/m^2}K^4$$

watt = joule per second = "power"

Reasonable approximation:
Every body in the solar system radiates energy according to this law.

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Budyko's Model

What determines the Earth's surface temperature?

temperature change ~ energy in – energy out

energy in from the Sun — energy out from the Earth

Simple Model

Assume that Earth is a perfectly thermally conducting black body in thermal equilibrium, so

energy in = energy out .

energy in from the Sun $Q = 342\ \mathrm{W/m^2}$

energy out from the Earth $\sigma T^4\ \mathrm{W/m^2}$

$$T = (342/\sigma)^{1/4} = (342/5.67\times10^{-8})^{1/4} =$$
$$279\mathrm{K} = 6^\circ\mathrm{C} = 43^\circ\mathrm{F}$$

Earth's global mean temperature: 57°F *Not bad!*

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Budyko's Model

What determines the Earth's surface temperature?

Simple Model

Assume that Earth is a perfectly thermally conducting black body.

energy in from the Sun $Q = 342\ \mathrm{W/m^2}$

energy out from the Earth $\sigma T^4\ \mathrm{W/m^2}$

$$T = (342/\sigma)^{1/4} = (342/5.67\times10^{-8})^{1/4} =$$
$$279\mathrm{K} = 6^\circ\mathrm{C} = 43^\circ\mathrm{F}$$

Note that there is a differential equation lurking here.

temperature change ~ energy in – energy out

heat capacity

$$R\frac{dT}{dt} = Q - \sigma T^4$$

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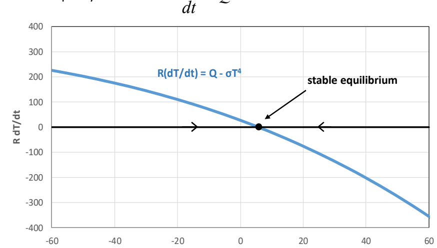
Budyko's Model

temperature change ~ energy in – energy out

heat capacity

$$R\frac{dT}{dt} = Q - \sigma T^4$$



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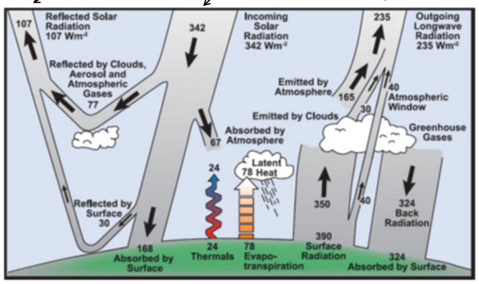
Budyko's Model

albedo insolation OLR



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

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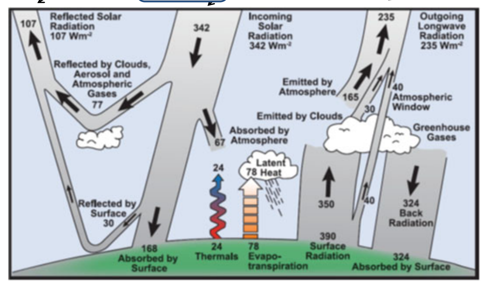
Budyko's Model

albedo $\frac{107}{342} \approx 0.313$ insolation OLR



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Budyko's Model

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α.
For Earth, $\alpha \approx 0.3$.

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Budyko's Model

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α.
For Earth, $\alpha \approx 0.3$.

Simple Model

Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = \left(0.7 \cdot Q/\sigma\right)^{1/4} = \left(0.7 \cdot 342/5.67\times10^{-8}\right)^{1/4}$$
$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

stable equilibrium

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Budyko's Model

Albedo

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

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Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

Way too cold!
What happened to 57°F?

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Budyko's Model

Albedo

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Dynamics

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Way too cold!
What happened to 57°F?

255 K is actually the *photosphere* temperature.

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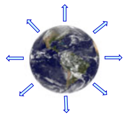
Budyko's Model

Photosphere

A photosphere is the deepest region of a luminous object, usually a star, that is transparent to photons of certain wavelengths.

https://en.wikipedia.org/wiki/Photosphere

For the Earth, the photosphere is where the long wave photons escape into space. It is high in the atmosphere where the temperature is 255 K.



$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

T = photosphere temperature.

What about the surface temperature?

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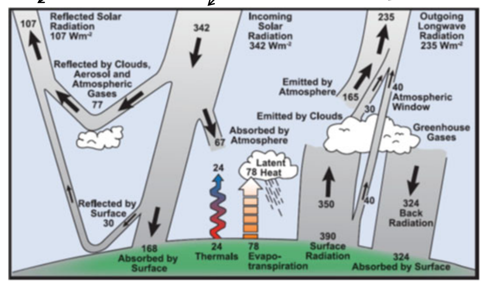
Budyko's Model

albedo

insolation

OLR



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Budyko's Model

OLR as a Function of Surface Temperature

(Outgoing Longwave Radiation)

$$\text{OLR} \approx A + BT$$

A and B are determined from satellite observations.

T is surface temperature (*in Celsius*).

$$A = 202 \text{ W/m}^2$$

$$B = 1.90 \text{ W/m}^2\text{K}$$

Kelvin

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

photosphere temperature

Celsius

becomes

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

global mean surface temperature

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Budyko's Model

OLR as a Function of Surface Temperature

$$\text{OLR} \approx A + BT$$

Important:

A+BT is not a linear approximation to the Stefan-Boltzmann equation.

different

Kelvin

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

$\boxed{\text{photosphere temperature}}$

Celsius

becomes

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

$\boxed{\text{global mean surface temperature}}$

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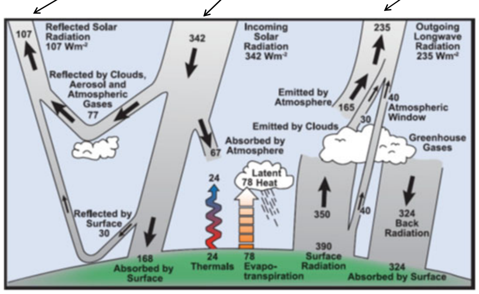
Budyko's Model

albedo

insolation

OLR



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Budyko's Model

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha) - A}{B}$$

Recall: Q = 342 W/m², A = 202 W/m², B = 1.9 W/m²K

Wikipedia: $0.30 \le \alpha \le 0.35$

$\alpha = 0.30$, T^* = 19.7°C = 67°F

$\alpha = 0.32$, T^* = 16.1°C = 61°F

$\alpha = 0.33$, T^* = 14.3°C = 58°F ← *Not bad!*

$\alpha = 0.35$, T^* = 10.7°C = 51°F

Earth's global mean temperature: 57°F

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Budyko's Model

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha) - A}{B}$$

Is it stable?

$$R\frac{dT}{dt} = \left(Q(1-\alpha) - A\right) - BT$$

Stable, since $B > 0$.

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Budyko's Model

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

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Is it stable?

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Stable, since $B > 0$.

What if Earth had more ice or less ice?
That would change the albedo α.

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Budyko's Model

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

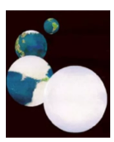
$$T^* = \frac{Q(1-\alpha) - A}{B}$$

Current Earth: $\alpha = 0.33$, $T^* = 14.3$°C = 58°F

guesses

Ice-free Earth: $\alpha = 0.28$, $T^* = 23.3$°C = 74°F

Snowball Earth: $\alpha = 0.62$, $T^* = -38$ °C = -36°F



Why do we have ice caps?

If Earth was ever a snowball, how did we get out?

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Budyko's Model

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

What's missing?

Earth is not homogeneous. For example, it is warmer at the equator and colder at the poles. The temperature should depend on latitude.

Make T depend on y = sin(latitude)

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation distribution

$$s(y) = \text{distribution across latitudes} \quad \left(\int_0^1 s(y)dy = 1\right)$$

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Budyko's Model

Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Why do we use y = sine(latitude) instead of just latitude?

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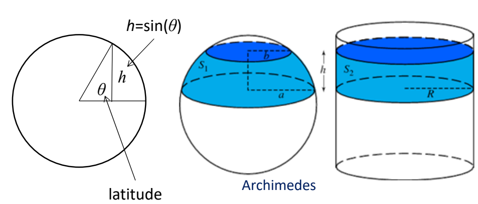
Budyko's Model

Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Why do we use y = sine(latitude) instead of just latitude?

Because y is directly proportional to surface area.



http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html

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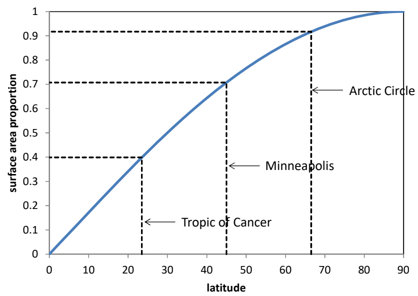
Budyko's Model

Why *y* ?



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Budyko's Model

Latitiude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha)-(A+BT(y,t))$$

$s(y)=$ distribution across latitudes $\left(\int_0^1 s(y)dy=1\right)$

One can show that β = obliquity = 23.4°

$$s(y)=\frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\theta-y\cos\beta\right)^2}\,d\theta$$

McGehee & Lehman, *SIAM J. Applied Dynamical Systems* 11 (2) (2012), 684–707.

Chylek and Coakley's quadratic approximation:

$$s(y)\approx 1-0.241\left(3y^2-1\right)$$

Chylek & Coakley, *J. Atmos. Sci.* 32 (1975), 675–679.

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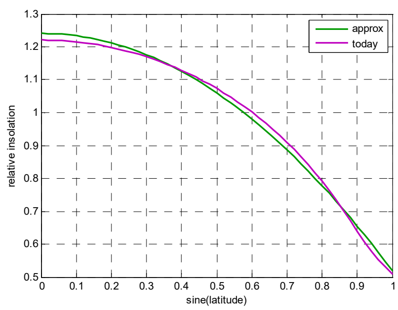
Budyko's Model

Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.5°



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Budyko's Model

Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha)-(A+BT(y,t))$$

Note that an equilibrium temperature T is a function of y = sine(latitude), denoted $T^*(y)$

$$Qs(y)(1-\alpha)-(A+BT^*(y))=0$$

$$\boxed{T^*(y)=\frac{Qs(y)(1-\alpha)-A}{B}}$$

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Budyko's Model

Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha)-(A+BT(y,t))$$

Equilbrium Solution

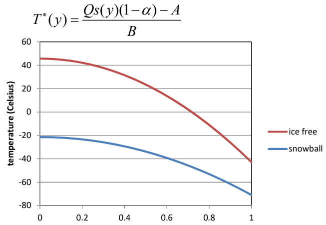
$$T^*(y)=\frac{Qs(y)(1-\alpha)-A}{B}$$

Tung*

α = 0.32: ice free
α = 0.62: snowball

Note that α = 0.32 is in the range of current Earth.

* K.K. Tung, *Topics in Mathematical Modeling*, Princeton U. Press, 2007



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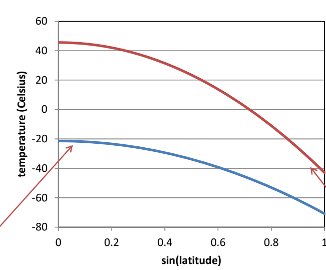
Budyko's Model

Latitude Dependence

α = 0.32: ice free
α = 0.62: snowball



Current Earth has ice caps, so we are in the ballpark.

ice won't melt (no exit from snowball)

ice will form (icecap)

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Budyko's Model

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

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Budyko's Model

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

The Second Law of Thermodynamics

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Budyko's Model

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

The Second Law of Thermodynamics

One simple statement of the law is that heat always moves from hotter objects to colder objects ...

`https://en.wikipedia.org/wiki/Second_law_of_thermodynamics`

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Budyko's Model

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

The Second Law of Thermodynamics

It's hotter at the equator than at the poles, so heat moves from the lower latitudes to the higher latitudes.

How?

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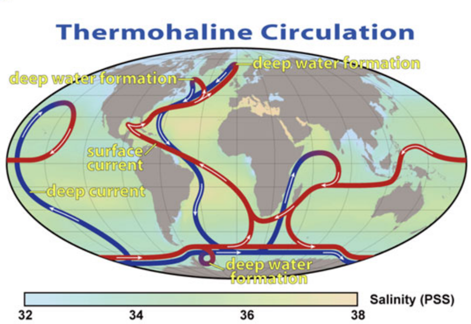
Budyko's Model

What's Missing?



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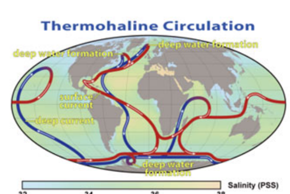
Budyko's Model

What's Missing?



Example

The Gulf Stream carries warm salty surface water from the Gulf of Mexico to the North Atlantic, where it cools, becomes denser, and sinks, flowing south in the deep ocean.

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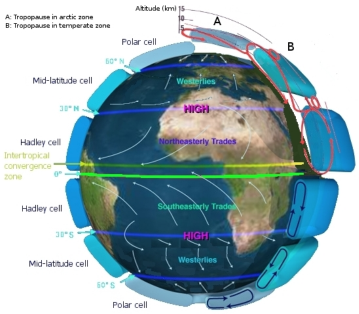
Budyko's Model

What's Missing?



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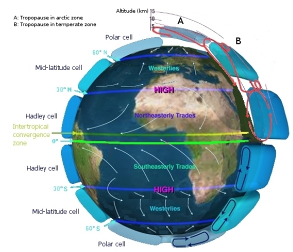
Budyko's Model

What's Missing?

Warm air from the surface along the equator rises and flows toward the poles in a series of cells moving heat from the equator to the poles.



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Budyko's Model

What's Missing?

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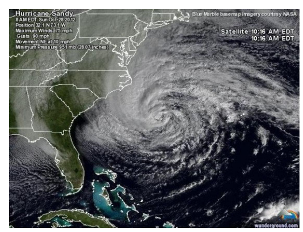
Budyko's Model

What's Missing?



Atlantic hurricanes move heat from the equatorial Atlantic up the coast of North America.

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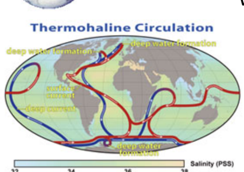
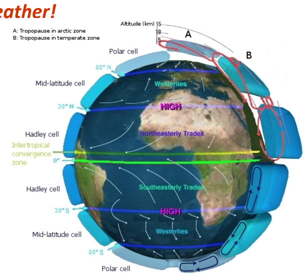
Budyko's Model

What's Missing?



Weather!



The second law of thermodynamics

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Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right) + \boxed{C\left(\bar{T}(t) - T(y,t)\right)}$$

2nd law

global mean temperature $\bar{T}(t) = \int_0^1 T(y,t)dt$

$\left(\bar{T}(t) - T(y,t)\right)$ interpretation

Each point on Earth's surface is trying to assume the global mean temperature. If the temperature at a latitude is below the global mean, then it heats up. If the temperature at that latitude is above the mean, then it cools off.

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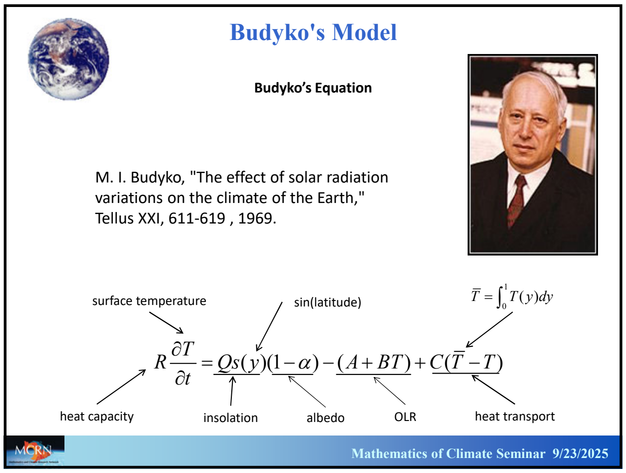

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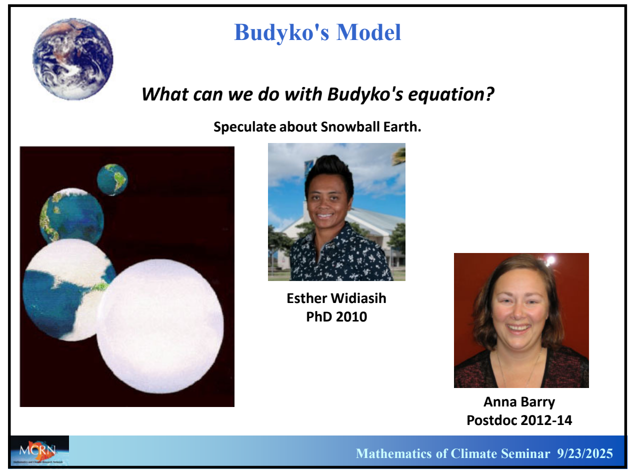

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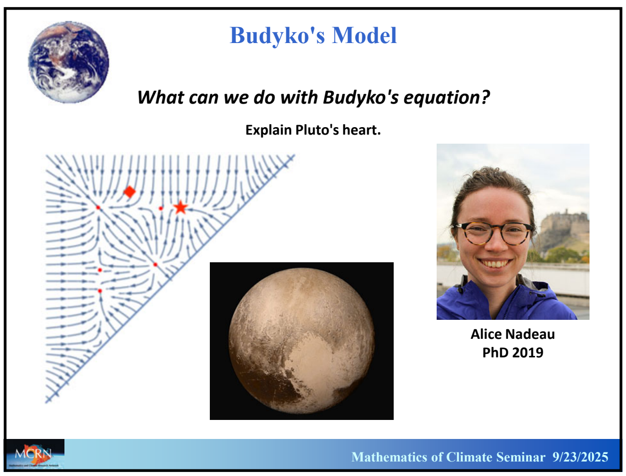

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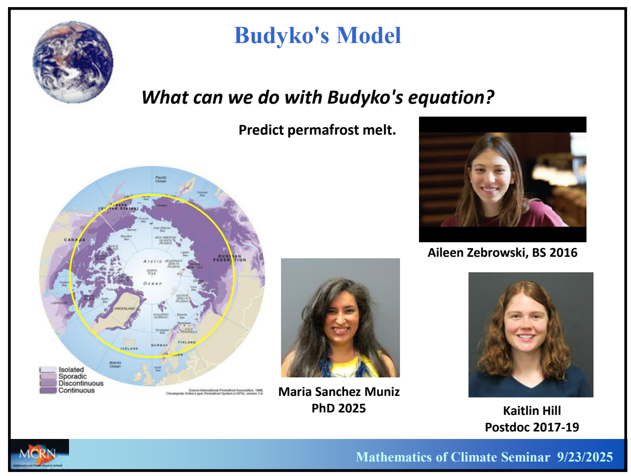

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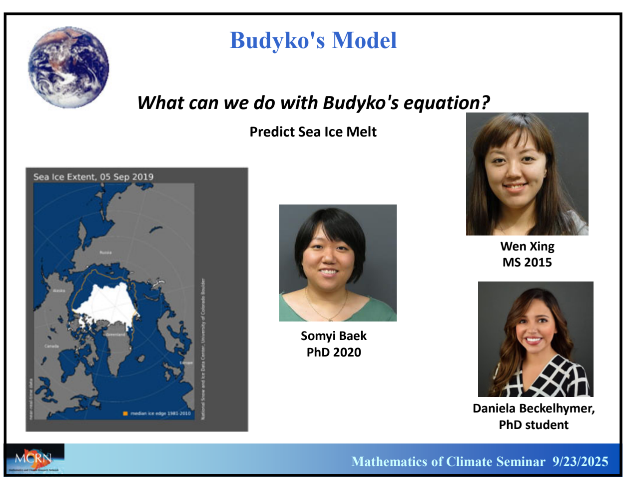

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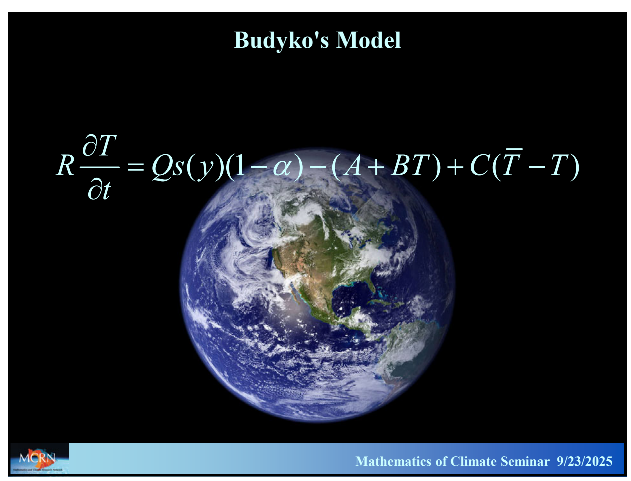

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