Welander's Ocean Circulation Model

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Mathematics of Climate Seminar
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MCRN
Mathematics and Climate Research Network

https://sites.google.com/view/math-climate

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Welander's Model

20th Century Fox 2004

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Welander's Model

The Day After Tomorrow/Film synopsis

After climatologist Jack Hall (Dennis Quaid) is largely ignored by U.N. officials when presenting his environmental concerns, his research proves true when an enormous "superstorm" develops, setting off catastrophic natural disasters throughout the world. Trying to get to his son, Sam (Jake Gyllenhaal), who is trapped in New York with his friend Laura (Emmy Rossum) and others, Jack and his crew must travel by foot from Philadelphia, braving the elements, to get to Sam before it's too late.

Google Search

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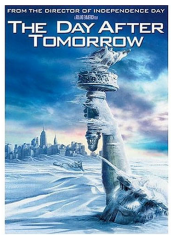
Welander's Model

The Day After Tomorrow/Science

The events depicted in the film are based on real science: that planetary warming can cause parts of the Northern Hemisphere to become colder.

Most of the events in the film are scientifically impossible or greatly exaggerated.



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Welander's Model

Dansgaard-Oeschger Events

"Global warming" can cause the Northern Hemisphere to cool.

Melting ice can lower the salinity of the North Atlantic, causing a decrease in the flow of the Atlantic Meridional Overturning Circulation (AMOC), slowing the heat transfer to the Northern Hemisphere.

This phenomenon is believed to have caused the ***Younger Dryas***.

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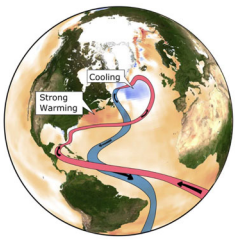
Welander's Model

The Washington Post

April 11, 2018

The oceans' circulation hasn't been this sluggish in 1,000 years. That's bad news.

The Atlantic Ocean circulation that carries warmth into the Northern Hemisphere's high latitudes is slowing down because of climate change, a team of scientists asserted Wednesday, suggesting one of the most feared consequences is already coming to pass.



Nature volume 556, pages191–196 (2018)

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Welander's Model

The Guardian US

August 25, 2025

Collapse of critical Atlantic current is no longer low-likelihood, study finds

Scientists say 'shocking' discovery shows rapid cuts in carbon emissions are needed to avoid catastrophic fallout

The Atlantic meridional overturning circulation is weakening and has collapsed in the distant past

https://www.theguardian.com/environment/2025/aug/28/collapse-critical-atlantic-current-amoc-no-longer-low-likelihood-study

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Welander's Model

INDEPENDENT

April 12, 2018

Gulf Stream current at 'record low' with potentially devastating consequences for weather, warn scientists

The Atlantic meridional overturning circulation (AMOC), the system of currents that transports warm water from the tropics via the Gulf Stream to the North Atlantic, plays a major role in regulating the world's climate.

A fictional depiction of AMOC's collapse was portrayed in ***The Day After Tomorrow***, and while the film's events were exaggerated, scientists say severe weather events are likely to result from the ongoing changes.

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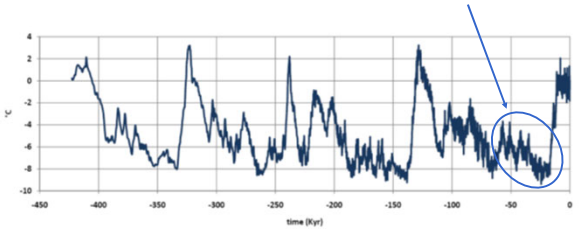
Welander's Model

Recent (last 400 Kyr) Temperature Cycles

Vostok Ice Core Data

What's with these oscillations?



J.R. Petit, *et al* (1999) Climate and atmospheric history of the past 420,000 years from the Vostok ice core, Antarctica, *Nature* 399, 429-436.

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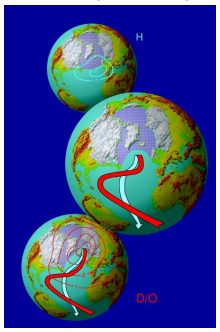
Welander's Model

Heinrich and Dansgaard-Oeschger events



http://www.pik-potsdam.de/~stefan/sampleimages.html

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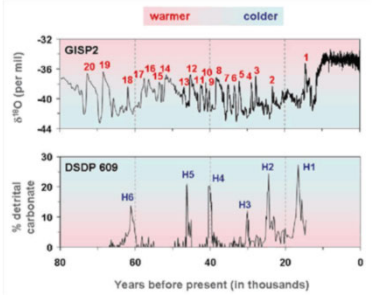
Welander's Model

Heinrich and Dansgaard-Oeschger events



http://www.ncdc.noaa.gov/paleo/abrupt/data3.html

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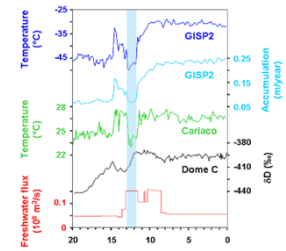
Welander's Model

The Younger Dryas



http://www.ncdc.noaa.gov/paleo/abrupt/data4.html

Mountain Avens
(*Dryas octopetala*)

https://www.ncdc.noaa.gov/abrupt-climate-change/The%20Younger%20Dryas

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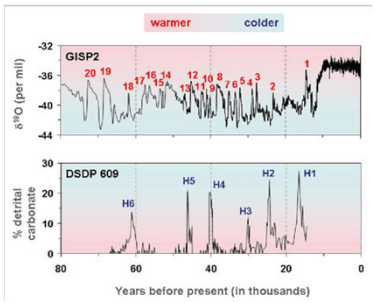
Welander's Model

Heinrich and Dansgaard-Oeschger events



What caused the oscillations?

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Welander's Model

What caused the D-O oscillations?

They could be self-oscillations in the natural dynamics of ocean circulation.

Pierre Welander, A simple heat-salt oscillator, *Dynamics of Atmospheres and Oceans* 6 (1982) 233-242.

R/V WeeLander is a 23-foot-long Beach Master work boat, informally named in honor of Professor Pierre Welander (1925—1996).

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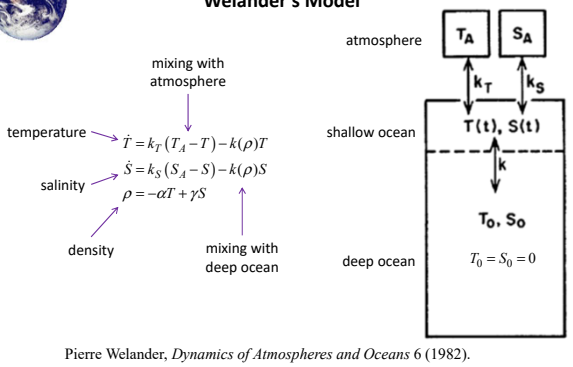
Welander's Model

Welander's Model



Pierre Welander, *Dynamics of Atmospheres and Oceans* 6 (1982).

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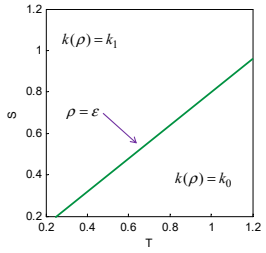
Welander's Model

Welander's Model

$$\dot{T} = k_T(T_A - T) - k(\rho)T$$
$$\dot{S} = k_S(S_A - S) - k(\rho)S$$
$$\rho = -\alpha T + \gamma S$$

The function k

$$k(\rho) = \begin{cases} k_0, & \rho < \varepsilon \\ k_1, & \rho > \varepsilon \end{cases}$$



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Welander's Model

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$$\dot{T} = k_T(T_A - T) - k(\rho)T$$
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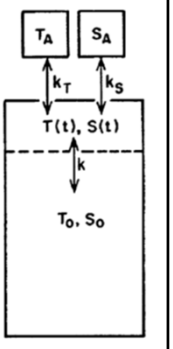
Welander chose scientifically reasonable values and dimensionless variables and constants

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1 - S) - k(\rho)S$$
$$\rho = -\alpha T + S$$

$$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$

$$\alpha = 0.8$$
$$\beta = 0.5$$



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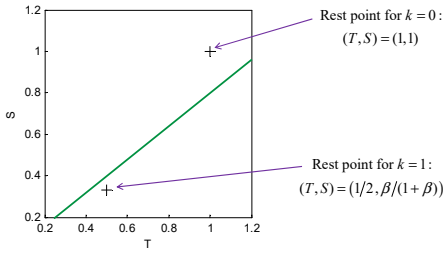

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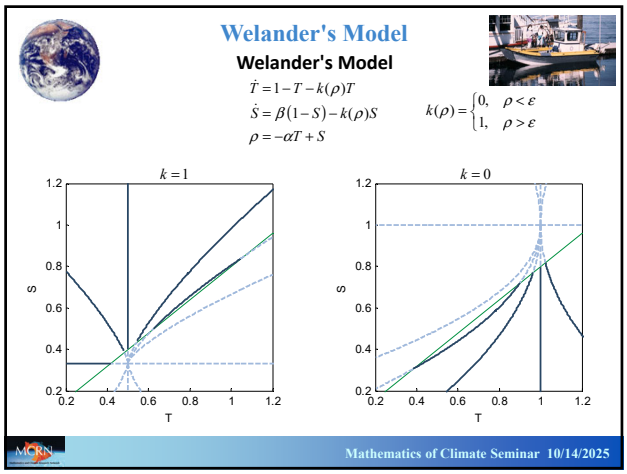

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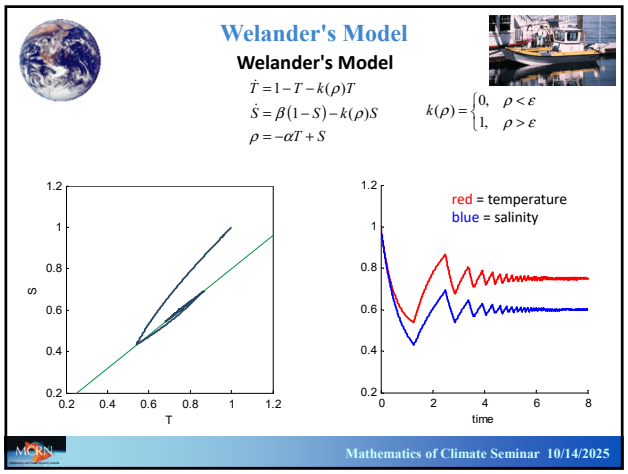

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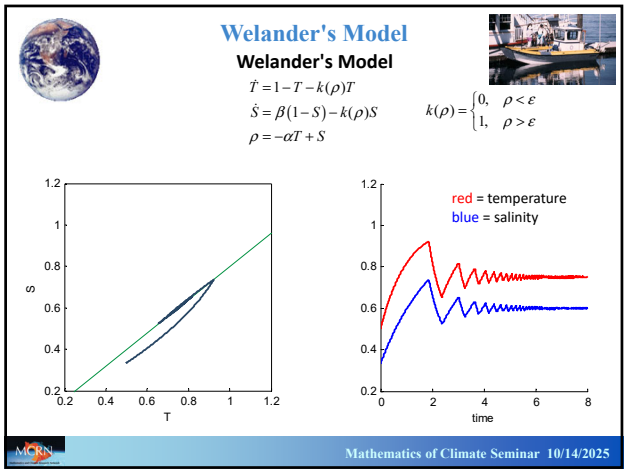

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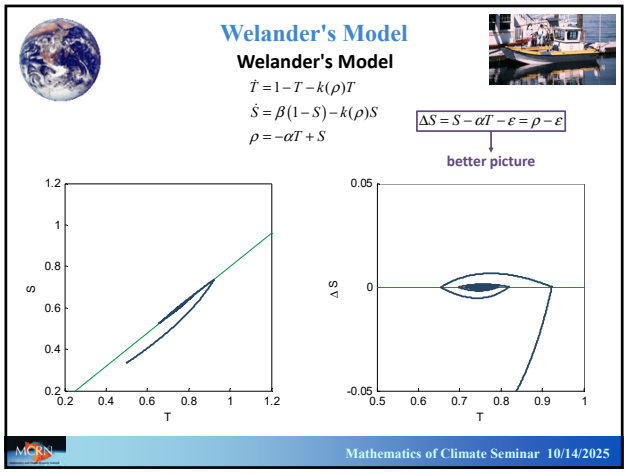

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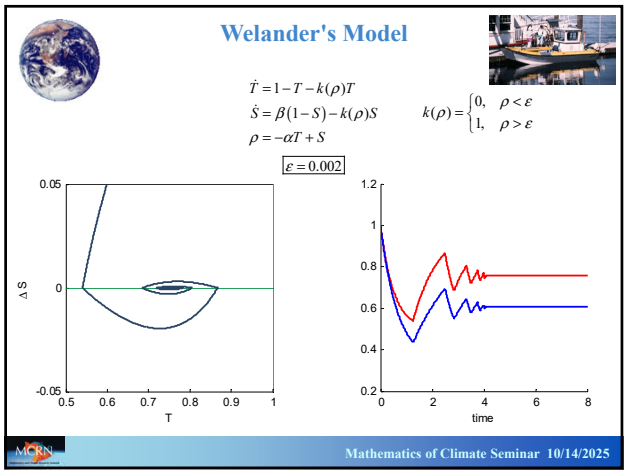

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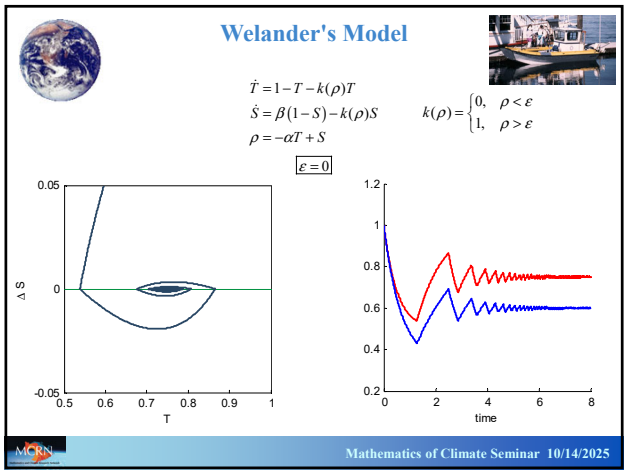

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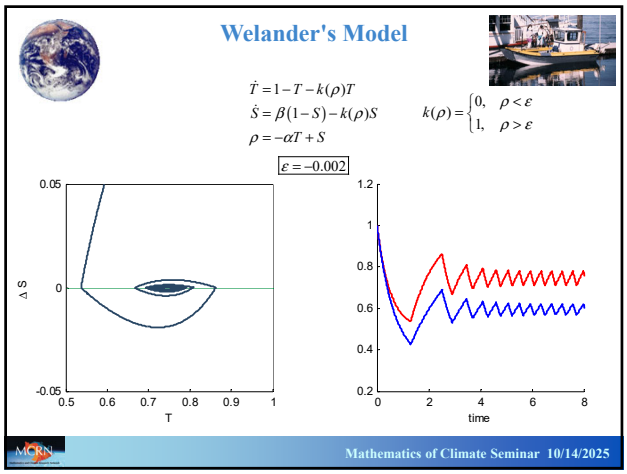

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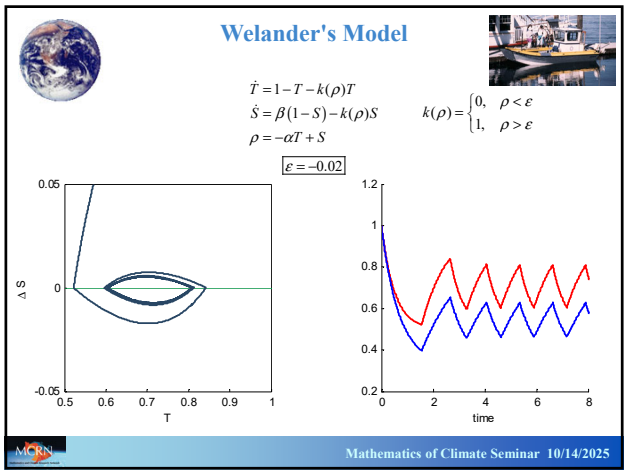

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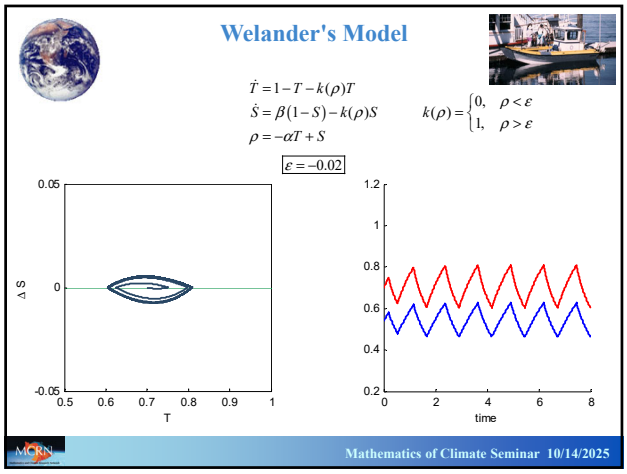

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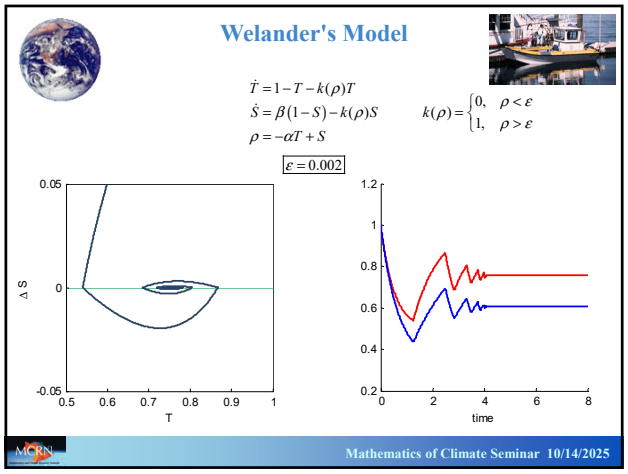

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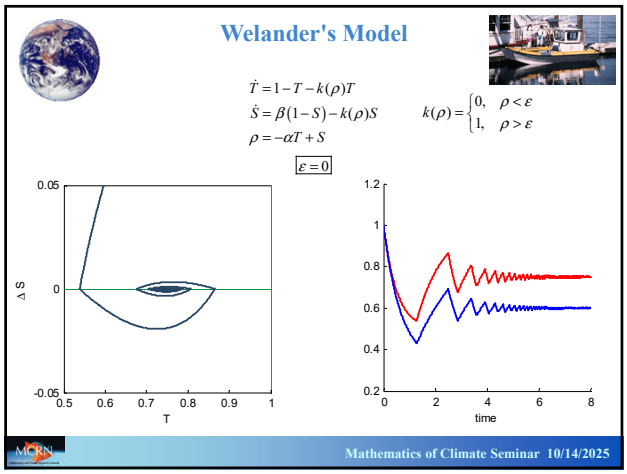

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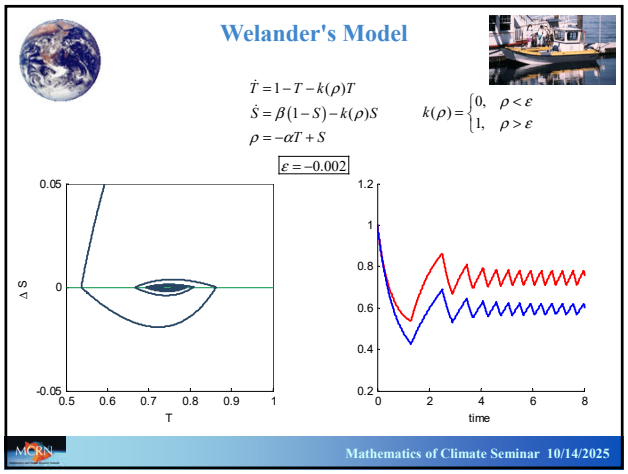

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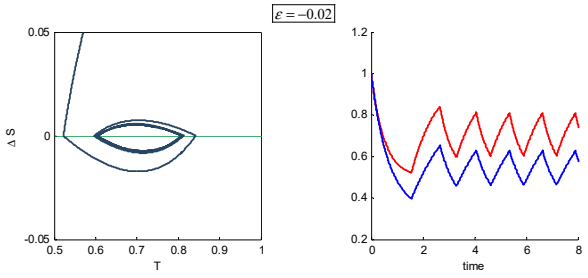
Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S$$
$$\rho = -\alpha T + S$$
$$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$

$\varepsilon = -0.02$



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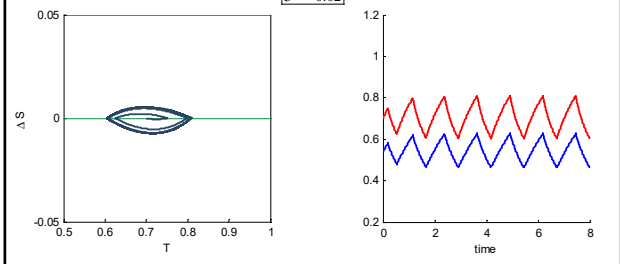

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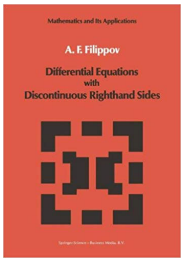
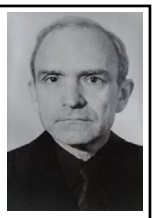
Welander's Model

Filippov Approach



Kluwer Academic Publishers



A.F. Filippov*

Roughly

- The Euclidean space is partitioned by a finite number of sets.
- The boundaries are codimension 1.
- The vector field can be thought of as a finite number of vector fields, each defined and smooth on a partition set, including the boundary.
- The individual vector fields take different values on the boundaries.

*https://alchetron.com/Aleksei-Fedorovich-Filippov

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The function k

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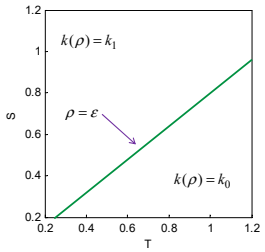
Welander's model is a Filippov system.



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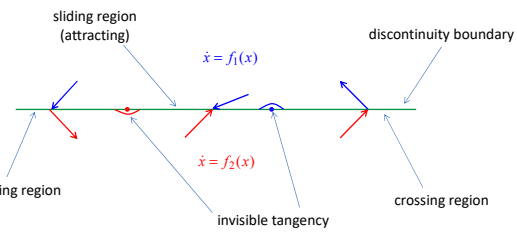
Welander's Model



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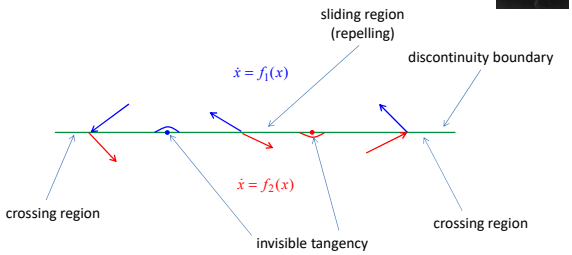
Welander's Model



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$$\dot{T} = 1 - T - k(\rho)T$$
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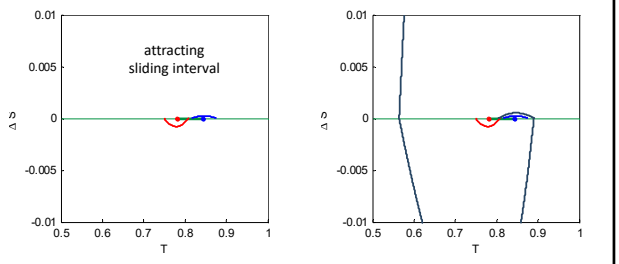
$$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$

$\varepsilon = 0.025$



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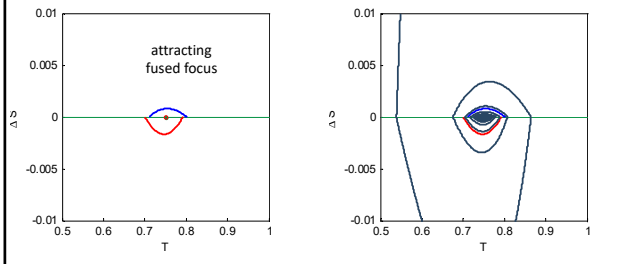
$\varepsilon = 0$



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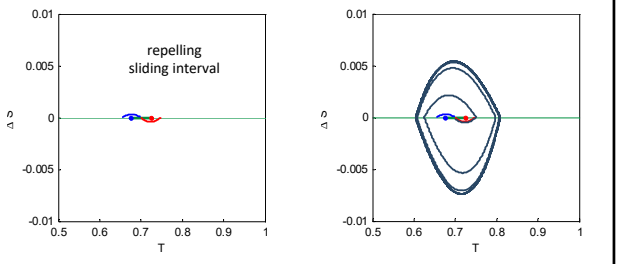
$\varepsilon = -0.02$



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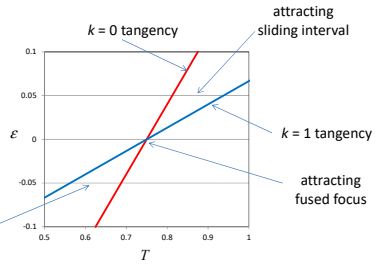
$$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$

As ε goes from positive to negative, two invisible tangencies pass through each other. The sliding region goes from attracting to repelling through an attracting fused focus at $\varepsilon = 0$.

Looks like a supercritical Hopf bifurcation.



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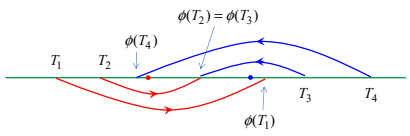
Welander's Model

Section Map

Follow the solution from when it crosses the discontinuity line until the next time it crosses the line.

$$T \mapsto \phi(T)$$

Not 1-1 for an attracting sliding region



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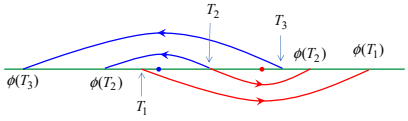
Welander's Model

Section Map

$$T \mapsto \phi(T)$$

Multi-valued for an repelling sliding region



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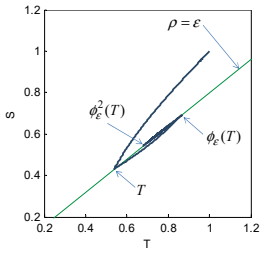
$$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$

Follow the solution from when it crosses the discontinuity line ($\rho = \varepsilon$) until the next time it crosses the line. Use T as a coordinate on the line.

$$T \mapsto \phi_\varepsilon(T)$$



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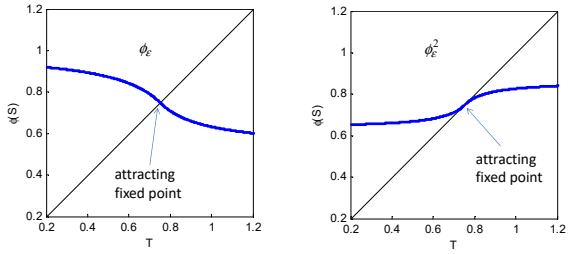
Welander's Model

Section Map

$$T \mapsto \phi_\varepsilon(T)$$

$$\varepsilon = 0$$



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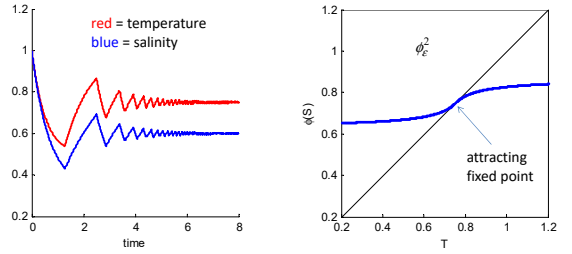
Welander's Model

Section Map

$$T \mapsto \phi_\varepsilon(T)$$

$$\varepsilon = 0$$

red = temperature
blue = salinity



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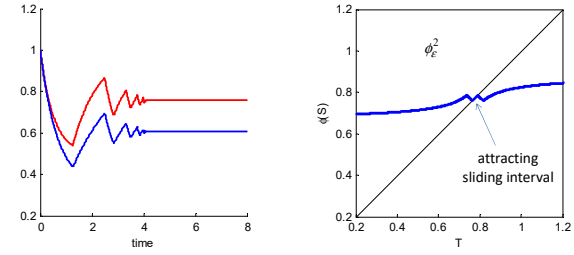
Welander's Model

Section Map

$$T \mapsto \phi_\varepsilon(T)$$

$$\varepsilon = 0.01$$



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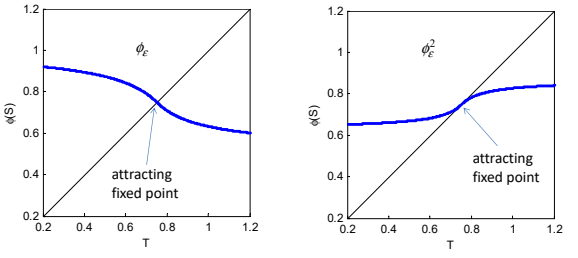
Welander's Model

Section Map

$$T \mapsto \phi_\varepsilon(T)$$

$$\varepsilon = 0$$



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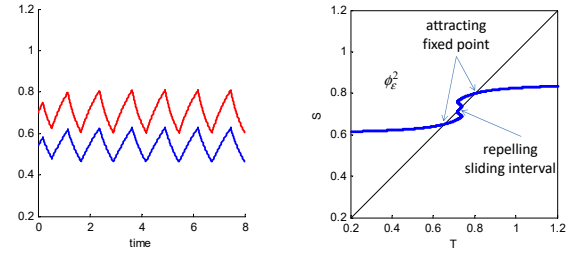
Welander's Model

Section Map

$$T \mapsto \phi_\varepsilon(T)$$

$$\varepsilon = -0.01$$



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Welander's Model

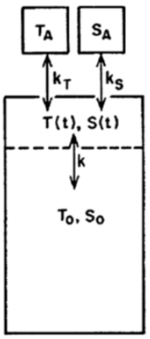
$$\dot{T} = k_T(T_A - T) - k(\rho)T$$
$$\dot{S} = k_S(S_A - S) - k(\rho)S$$
$$\rho = -\alpha T + \gamma S$$

Smooth dependence on density:

$$k(\rho) = \left(\frac{1}{\pi}\tan^{-1}\left(\frac{\rho-\varepsilon}{a}\right)+\frac{1}{2}\right)(k_0+k_1)$$

Singular limit:

$$k(\rho) = \begin{cases} k_0, & \rho < \varepsilon \\ k_1, & \rho > \varepsilon \end{cases}$$



Welander assumed that the self-oscillations he found in his discontinuous model would hold held for a nearby smooth system.

Juliann Leifeld, PhD 2016:

Welander's assumption was correct, for small ε.

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Welander's Model

$$\dot{T} = k_T(T_A - T) - k(\rho)T$$
$$\dot{S} = k_S(S_A - S) - k(\rho)S$$
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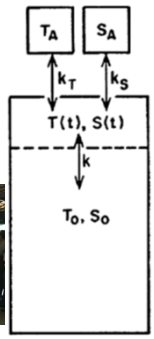
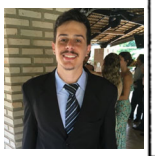


Carvalho and Gouveia showed uniqueness of the Filippov periodic orbit for large ε.

Yagor Carvalho

Luis Gouveia

Postdocs, 2023



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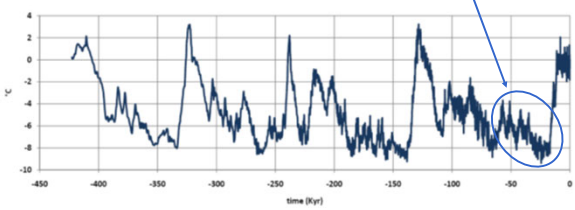
Welander's Model

Recent (last 400 Kyr) Temperature Cycles

Vostok Ice Core Data

Explained by Welander?



J.R. Petit, *et al* (1999) Climate and atmospheric history of the past 420,000 years from the Vostok ice core, Antarctica, *Nature* 399, 429-436.

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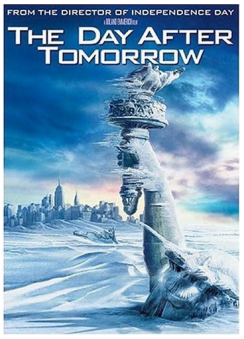
Welander's Model

Moral

Surprisingly, the moral of the film was NOT that everyone should study the Welander model.



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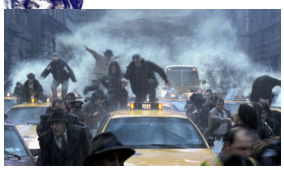
Welander's Model



After Los Angeles and Tokoyo were decimated, and as a tidal wave was about to inundate Manhattan, a conversation ensued between the Vice President of the United States and the leading paleoclimatologist.

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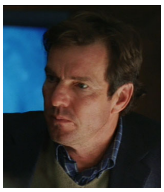
Welander's Model

Moral

Vice President: *"Maybe you should stick to science and leave policy to us."*



Scientist: *"Well, we tried that approach. You didn't want to hear about the science when it could have made a difference."*

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