A PROOF OF THE MANDELBROT N^2 CONJECTURE

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1. The Quadratic Family

We consider the well-studied one-parameter family of quadratic maps of the complex plane

$$f_c(z) = f(z,c) = z^2 + c ,$$

where z is the state variable, c is the parameter, and both are complex numbers. Holding c constant, we iterate the map with respect to z and denote the k^{th} iterate by $f_c^k(z)$. We are concerned with one aspect of the structure of the "Mandelbrot set"

$$M \equiv \{c \in \mathbb{C} : f_c^k(0) \text{ is bounded}\} .$$

A <u>periodic point</u> for the map f_c is a point p in $\mathbb{C}$ such that $f_c^n(p) = p$ for some $n \geq 1$. The <u>period</u> of p is the least such positive integer n. The <u>orbit</u> of p is the finite set of all iterates of p. The <u>eigenvalue</u> of p is the complex number $(f_c^n)'(p)$. It is easy to see that $(f_c^n)'(p_1) = (f_c^n)'(p_2)$ for all p_1 and p_2 on the same periodic orbit. Therefore we can speak of the eigenvalue of the orbit. We say that p is an <u>attracting</u> periodic point if the eigenvalue of p is strictly inside the unit circle. Let

$$M' \equiv \{c \in \mathbb{C} : f_c \text{ has an attracting periodic orbit}\} ,$$

and note that M' is open.

A theorem of Julia [3] implies that, for each value of c , f_c has at most one attracting periodic orbit and that $z = 0$ is attracted to that orbit if it exists. Thus M' is a subset of M .

Let Ω be a component of M' , and let D° denote the open unit disc. For each point c in Ω , define $\rho_\Omega(c)$ to be the eigenvalue of the unique attracting periodic orbit of f_c . A theorem of Douady and Hubbard [1] implies that $\rho_\Omega : \Omega \to D^\circ$ is an analytic homeomorphism which extends analytically to the boundary except possibly at the point c where $\rho_\Omega(c) = 1$.

In this note we prove the following theorem.

<u>THEOREM A</u>: <u>Let Ω be any component of M' , let λ_0 be a positive n^{th} root of unity, and let $c_0 \equiv \rho_\Omega^{-1}(\lambda_0)$. Then there exists a component Λ of M' such that</u>

(1) $c_0 \in \overline{\Omega} \cap \overline{\Lambda}$,

(2) ρ_Λ <u>extends analytically to $\overline{\Lambda}$, and</u>

(3) $\rho_\Lambda'(c_0) = -n^2 \overline{\lambda}_0 \rho_\Omega'(c_0)$.

The component Λ corresponds to a periodic orbit with period n times the period of the orbit corresponding to Ω . This n-fold bifurcation is well-known, and its proof is included only as part of the proof of Theorem A. Conclusion (3) of Theorem A is the "Mandelbrot N^2 conjecture" [4]. More precisely, Mandelbrot conjectured that the directional derivative of $|\rho_\Lambda|$ in the direction of the outward normal of $\overline{\Lambda}$ at the boundary point c_0 is n^2 times the corresponding directional derivative of $|\rho_\Omega|$ at c_0 . Conclusion (3) implies that the boundaries of $\overline{\Omega}$ and $\overline{\Lambda}$ are tangent at c_0 and that this conjecture of Mandelbrot's is true.

2. One-parameter Families

Theorem A will be proved by showing that the family f_c satisfies the hypotheses of a more general bifurcation theorem, which will be stated and proved in Section 4. This generalization will have two hypotheses of nondegeneracy. The first will be discussed in this section and involves the dependence on the parameter. The second will be discussed in the next section and involves only the map at the fixed parameter value at which the bifurcation occurs.

Let F denote a family of maps parameterized by a single complex parameter c, and write

$$F_c(z) \equiv F(z,c) ,$$

where F is holomorphic on some open subset of $\mathbb{C} \times \mathbb{C}$. The following definition will be used later to state a generalization of Theorem A.

<u>Definition</u>: Assume that z_0 is a fixed point of F_{c_0}. We say that F has a <u>simple</u> n-<u>fold</u> <u>bifurcation</u> at (z_0,c_0) if there exist neighborhoods U of c_0 and V of z_0 such that the following two conditions hold.

(1) For $c \in U$, F_c has a unique fixed point in V, whose eigenvalue we denote by $\rho(c)$.

(2) For $c \in U - \{c_0\}$, F_c has a unique periodic orbit of period n in V, whose eigenvalue we denote by $\sigma(c)$.

If, in addition, the following condition holds, then we say that the bifurcation <u>satisfies</u> <u>the</u> <u>N^2-rule</u>.

(3) σ extends to a holomorphic function on U satisfying

$$\sigma'(c_0) = -\, n^2\, \overline{\rho(c_0)}\, \rho'(c_0) .$$

Our plan is to give sufficient conditions for a simple n-fold bifurcation to occur and to show that, under these conditions, a simple n-fold bifurcation satisfies the N^2-rule. We begin by discussing the dependence on the parameter.

If F_{c_0} has a fixed point z_0, and if the eigenvalue λ_0 at this fixed point is not equal to one, then the implicit function theorem gives us a unique fixed point $z = q(c)$ which is a continuation of z_0 and which is a holomorphic function of the parameter in a neighborhood of c_0. In other words, q is the unique function satisfying

$$F(q(c),c) = q(c) \ , \quad q(c_0) = z_0 \ .$$

Denote by $\rho(c)$ the eigenvalue of $q(c)$, i.e.

$$\rho(c) \equiv F_c'(q(c)) \ .$$

If we assume that $\rho'(c_0) \neq 0$, then ρ maps a neighborhood of c_0 conformally onto a neighborhood of λ_0. Just to emphasize the nature of this last assumption, we write it

$$\rho'(c_0) = \frac{1}{(1-\lambda_0)} \frac{\partial F}{\partial c}(z_0,c_0) \frac{\partial^2 F}{\partial z^2}(z_0,c_0) + \frac{\partial^2 F}{\partial z \partial c}(z_0,c_0) \neq 0 \ . \tag{2.1}$$

These considerations serve to motivate the following definition.

Definition: Let $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be holomorphic in a neighborhood of the point (z_0,c_0) in $\mathbb{C} \times \mathbb{C}$. Assume that z_0 is a fixed point for F_{c_0} with eigenvalue $\lambda_0 \neq 1$. We say that F is a regular parameterization at (z_0,c_0) if (2.1) holds.

The theorem of Douady and Hubbard discussed in Section 1 immediately yields the following result.

LEMMA 1: *Let* $f_c(z) = z^2 + c$, *and assume that* z_0 *is a periodic point of* f_{c_0} *with period* k *and with a primitive* n^{th} *root of unity as an eigenvalue. Let* $F(z,c) \equiv f_c^k(z)$. *Then* F *is a regular parameterization at* (z_0,c_0) .

3. Nondegeneracy of the Normal Form

The second nondegeneracy hypothesis involves only a property of the map F_{c_0} at the fixed parameter value c_0 at which the bifurcation occurs.

Definition: Let g satisfy $g(z_0) = z_0$ and $g'(z_0) = \lambda_0$, where λ_0 is a primitive n^{th} root of unity. We say that g has *nondegenerate normal form* at z_0 if $(g^n)'(z_0) \neq 0$.

In the following discussion we assume that $z_0 = 0$. If $g(0) = 0$ and $g'(0) = \lambda_0$, then we can write g as

$$g(z) = \lambda_0 z + \sum_{j=2}^{\infty} c_j z^j . \tag{3.1}$$

If λ_0 is a primitive n^{th} root of unity, then a straightforward computation due to Fatou [2] shows that either g^n is the identity or there exists a positive integer m such that

$$g^n(z) = z + \sum_{j=mn+1}^{\infty} b_j z^j , \tag{3.2}$$

where $b_{mn+1} \neq 0$. The hypothesis that g has nondegenerate normal form is simply the hypothesis that $m = 1$, i.e. the hypothesis that

$$g^n(z) = z + \sum_{j=n+1}^{\infty} b_j z^j , \tag{3.3}$$

where $b_{n+1} \neq 0$.

The term "normal form" in the above definition was chosen because expansion (3.2) is intimately related to the normal form of g . We digress briefly to discuss this relationship.

Consider a polynomial of the form

$$\phi(\zeta) = \zeta + \sum_{j=2}^{\infty} \delta_j \zeta^j . \tag{3.4}$$

We wish to choose ϕ so that

$$h(\zeta) \equiv \phi^{-1} \circ g \circ \phi(\zeta) = \lambda_0 \zeta + a_{n+1} \zeta^{n+1} + O(\zeta^{n+2}) . \tag{3.5}$$

If we expand the expression $\phi \circ h = g \circ \phi$ in power series and equate coefficients, we find that there is a unique polynomial of the form (3.4) which satisfies (3.5). We also find that the number a_{n+1} is uniquely determined by the coefficients λ_0 , c_2 , ..., c_{n+1} in expansion (3.1). If we now take the n^{th} iterate of both sides of (3.5), we find that

$$h^n(\zeta) = \phi^{-1} \circ g^n \circ \phi(\zeta) = \zeta + n \, \overline{\lambda}_0 a_{n+1} \zeta^{n+1} + O(\zeta^{n+2}) .$$

However, (3.3) implies that $\phi^{-1} \circ g^n \circ \phi(\zeta) = g^n(\zeta) + O(\zeta^{n+2})$. Therefore $b_{n+1} = n \, \overline{\lambda}_0 a_{n+1}$, so b_{n+1} and a_{n+1} must be either both zero or both nonzero. Thus we see that g has nondegenerate normal form if and only if the coefficient of order $n+1$ in the normal form of g does not vanish.

We now show that the quadratic family statisfies this second nondegeneracy condition.

LEMMA 2: Let $f_c(z) = z^2 + c$, and assume that z_0 is a periodic point of f_{c_0} with period k and with a primitive n^{th} root of unity as an eigenvalue. Then $f_{c_0}^k$ has nondegenerate normal form at z_0.

Proof: We rely on a theorem of Fatou [2] concerning the domain of attraction for a periodic orbit whose eigenvalue is a root of unity. The basic fact is the following: if $g(z)$ is a rational function whose Taylor series at the origin is $g(z) = z + bz^{\ell+1} + O(z^{\ell+2})$, with $b \neq 0$, then the domain of attraction of the origin contains ℓ distinct components $P_1, \ldots, P_\ell$ (called "petals" by Thurston) for which $0 \in \overline{P}_i$. Moreover, each P_i contains a critical point of g.

Since $f_{c_0}^{nk}$ is a polynomial which is not the identity, its expansion about each of the periodic points has the form (3.2). Therefore, each point x_i in the periodic orbit has mn invariant domains $D_1^i, \ldots, D_{mn}^i$, each containing a critical point of $f_{c_0}^{nk}$, and such that, for $y \in D_\ell^i$, $f^{kj}(y) \to x_i$ as $j \to \infty$. Thus $D_{\ell_1}^{i_1} \neq D_{\ell_2}^{i_2}$ if $i_1 \neq i_2$ or $\ell_1 \neq \ell_2$. If D is the invariant domain D_ℓ^i containing 0 (the critical point of f_{c_0}), then each D_ℓ^i is $f^j(D)$ for some $0 \leq j < kn$. By counting, we conclude that $mkn \leq kn$ and hence that $m = 1$. Thus the normal form for $f_{c_0}^k$ is nondegenerate at z_0, and the proof is complete.

4. The Simple n-fold Bifurcation

We conclude this paper with the following theorem, which specifies sufficient conditions for a one-parameter family to have a simple n-fold bifurcation. In view of Lemmas 1 and 2, this theorem immediately implies Theorem A.

THEOREM B: *Let F be holomorphic on a neighborhood of $(z_0, c_0) \in \mathbb{C} \times \mathbb{C}$. Assume that z_0 is a fixed point for F_{c_0} with a primitive n^{th} root of unity as its eigenvalue, and assume that F is a regular parameterization at (z_0, c_0). Assume also that the normal form of F_{c_0} is nondegenerate at z_0. Then F has a simple n-fold bifurcation at (z_0, c_0) which satisfies the N^2-rule.*

Proof: Let λ_0 denote the eigenvalue of the fixed point z_0 for F_{c_0}. Since $\lambda_0 \neq 1$, we know that z_0 is part of a holomorphic family of fixed points $z = q(c)$. Therefore we suffer no loss of generality by assuming that $q(c) = 0$ for all c, i.e. that $F_c(0) = 0$ for all c. The hypothesis that F is a regular parameterization implies that the eigenvalue $F_c'(0)$ and c are related by a conformal homeomorphism on a small open set. It will be convenient to introduce a new parameter ε related to c by

$$F_c'(0) = \lambda_0(1+\varepsilon) .$$

The previous remark insures that ε and c are related by a conformal homeomorphism. It therefore suffices to prove Theorem B for a family of the form

$$F_\varepsilon(z) = \lambda_0(1+\varepsilon)z + \sum_{j=2}^{\infty} c_j(\varepsilon)z^j . \tag{4.1}$$

Let U and V be neighborhoods of the origin such that $F : V \times U \to \mathbb{C}$ is holomorphic.

We now write the n^{th} iterate of F_ε as

$$F_\varepsilon^n(z) = (1+\varepsilon)^n z + \sum_{j=2}^{\infty} b_j(\varepsilon) z^j \; . \tag{4.2}$$

The considerations of the previous section imply that

$$b_j(0) = 0 \; , \; j = 2,\ldots,n \; .$$

The hypothesis that F_0 has nondegenerate normal form implies that

$$b_{n+1}(0) \neq 0 \; .$$

A periodic point of F_ε of period n will be a nontrivial fixed point of F_ε^n, i.e. a nonzero solution of $F_\varepsilon^n(z) = z$. Since $F_\varepsilon^n(0) = 0$ for all ε in U, $u(z,\varepsilon) \equiv \frac{1}{z} F_\varepsilon(z) - 1$ is holomorphic on $V \times U$. Note that

$$u(z,\varepsilon) = (1+\varepsilon)^n - 1 + \sum_{j=2}^{n} b_j(\varepsilon) z^{j-1} + b_{n+1}(\varepsilon) z^n + O(z^{n+1}) \; . \tag{4.3}$$

Let

$$\Gamma \equiv \{(z,\varepsilon) \in V \times U : u(z,\varepsilon) = 0\} \; .$$

One easily checks that

$$(z,\varepsilon) \in \Gamma - \{(0,0)\} \iff F_\varepsilon^n(z) = z \text{ and } z \neq 0 \; . \tag{4.4}$$

Denote the two projections of Γ onto V and U by

$$\pi_1 : \Gamma \to V : (z,\varepsilon) \to z \; ,$$

$$\pi_2 : \Gamma \to U : (z,\varepsilon) \to \varepsilon \; .$$

Also, let

$$\Lambda_\varepsilon \equiv \pi_1(\pi_2^{-1}(\varepsilon)) \subset V \; .$$

If z is a periodic point with period n for F_ε, then (4.4) implies that $(z,\varepsilon) \in \Gamma$. We will have shown that F has a simple n-fold bifurcation at $(0,0)$ once we have established that, for each $\varepsilon \neq 0$, Λ_ε consists of exactly one periodic orbit of period n for F_ε.

Using (4.3) we see that $\frac{\partial u}{\partial \varepsilon}(0,0) = n \neq 0$. The implicit function theorem therefore gives us a function $\gamma : V \to U$ such that

$$\Gamma = \{(z,\varepsilon) \in V \times U : \varepsilon = \gamma(z)\} .$$

Of course, we may have to shrink U and V. Note that

$$\pi_1^{-1}(z) = (z,\gamma(z)) .$$

Again using (4.3) we compute that

$$\gamma(z) = -\frac{b_{n+1}(0)}{n} z^n + O(z^{n+1}) . \tag{4.5}$$

Therefore we can again shrink U and V, if necessary, so that $\gamma : V \to U$ is an n-fold covering with a branch at 0. Note that $\gamma = \pi_2 \circ \pi_1^{-1}$ and hence that $\Lambda_\varepsilon = \gamma^{-1}(\varepsilon)$. Therefore, for $\varepsilon \neq 0$, Λ_ε consists of exactly n points.

Since F_ε and F_ε^n commute, the family F induces a map F_* on Γ defined by

$$F_*(z,\varepsilon) \equiv (F_\varepsilon(z),\varepsilon) .$$

Using z as a coordinate on Γ we can write F_* in this coordinate as $G : V \to \mathbb{C}$ defined by

$$G(z) \equiv (\pi_1 \circ F_* \circ \pi_1^{-1})(z) = F(z,\gamma(z)) .$$

One easily computes that $G(0) = 0$ and that $G'(0) = F_0'(0) = \lambda_0$. Since π_2 is invariant under F_*, $\gamma = \pi_2 \circ \pi_1^{-1}$ is invariant under G.

Since $\gamma^{-1}(\varepsilon)$ consists of exactly n points for $\varepsilon \neq 0$ and since λ_0 is a primitive n^{th} root of unity, $\Lambda_\varepsilon = \gamma^{-1}(\varepsilon)$ can only be a periodic orbit for G of period n. Since $G|_{\Lambda_\varepsilon} = F_\varepsilon|_{\Lambda_\varepsilon}$, Λ_ε is a periodic orbit for F_ε of period n, and hence F has a simple n-fold bifurcation at $(0,0)$.

We have only left to show that this bifurcation satisfies the N^2-rule. Let $(\zeta,\varepsilon) \in \Gamma$, with $\varepsilon \neq 0$. We have just seen that ζ lies on the unique periodic orbit of period n for F_ε in V. The eigenvalue of this orbit is just $\tilde{\sigma}(\zeta,\varepsilon) \equiv (F_\varepsilon^n)'(\zeta)$. Since it is well-defined along the orbit, $\tilde{\sigma} : \Gamma \to C$ projects to a function $\sigma : U - \{0\} \to \mathbb{C}$ so that $\tilde{\sigma} = \sigma \circ \pi_2$. Since Γ is an n-fold covering of U with a branch at the origin, the implicit function theorem implies that σ is holomorphic on $U - \{0\}$.

Using equation (4.2) we compute that

$$(F_\varepsilon^n)'(z) = (1+\varepsilon)^n + \sum_{j=2}^{\infty} j\, b_j(\varepsilon)\, z^{j-1} .$$

Combining this equation with (4.5) and recalling that $b_j(0) = 0$ for $j = 2,\ldots,n$, we find that

$$\tilde{\sigma}(\zeta,\varepsilon) = 1 - n^2\varepsilon + O(\zeta^{n+1}) .$$

Therefore σ extends to a continuous, and hence holomorphic, function on U, with $\sigma(0) = 1$ and $\sigma'(0) = -n^2$. From equation (4.1) we see that $\rho(\varepsilon) = F_\varepsilon'(0) = \lambda_0(1+\varepsilon)$ and hence that $\rho(0) = \rho'(0) = \lambda_0$. Therefore $\sigma'(0) = -n^2\, \overline{\rho(0)}\, \rho'(0)$, and the proof is complete.

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