Attractors for Closed Relations on Compact Hausdorff Spaces

RICHARD MCGEHEE

CONTENTS

Mathematicians have been iterating diffeomorphisms for about a century. Early activity centered around diffeomorphisms as section maps arising from ordinary differential equations, but the viewpoint gradually evolved until the iteration of diffeomorphisms became interesting in its own right. Recently, research into the dynamics of noninvertible maps has blossomed as an integral part of the study of chaos and fractals. Even more recently, the iteration of finite sets of contraction maps has been studied.

The next step of the evolution is to consider the iteration of relations. A relation on a space is a subset of the Cartesian product of the space with itself. The graph of a map is an example of a relation. Since the graph of a continuous map is a closed subset of the Cartesian product, it seems natural to study the iteration of closed relations. Although it is certainly important to study dynamics on noncompact spaces, only compact Hausdorff spaces are considered here.

At first glance, this setting seems hopelessly general, and one is tempted to conclude immediately that no interesting structures exist at this level of abstraction. However, it is demonstrated below that the concept of an

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"attractor" generalizes to this setting and that some of the important properties of attractors generalize as well. In particular, it is shown that every attractor for a relation on a compact Hausdorff space has an associated attractor block. Indeed, the proof in this case seems conceptually simpler than the proof for maps.

A bijective map f always has an inverse f^{-1}. This fact provides an elegant symmetry that pervades the terminology that evolved to study diffeomorphisms. For example, an "orbit" is a sequence p_i satisfying $p_{i+1} = f(p_i)$. Through every point there is a unique orbit which extends infinitely both backward and forward.

However, a more general map does not necessarily have an inverse, a fact which leads to asymmetry. For example, a unique infinite forward orbit exists through every point, but a backward orbit through a point may not exist, or, if it does exist, it may not be unique.

The symmetry is restored by considering relations. It then becomes possible to define the analog of the inverse of a bijective map, although it is less confusing to call it the "transpose." If f is a relation, then the transpose of f is the relation obtained by reversing the ordinate and the abscissa. More precisely, the transpose of f is defined as

$$f^* \equiv \{(x,y) : (y,x) \in f\}.$$

An "orbit" in this setting is a sequence of points p_i satisfying $(p_i, p_{i+1}) \in f$. Although the symmetry is restored, the cost is high. Now a forward orbit through a point may not exist, or, if it does exist, it may not be unique.

Another useful aspect of the symmetry is that a "repeller" can be defined to be an attractor for f^*. Every attractor has a dual repeller, defined as the set of all points not attracted to the attractor. For example, the so-called "filled Julia set" for the map $f(z) = z^2 + c$ on the Riemann sphere is the repeller dual to the attracting point at infinity. Indeed, the Julia set is an attractor for the transpose of f, which is a relation but not a map. This viewpoint is discussed further in Section 11.

Besides the restoration of symmetry, there is another reason to iterate relations. If one considers a map f on a metric space, then an ε-pseudo-orbit for f is a sequence p_i satisfying $d(p_{i+1}, f(p_i)) \leq \varepsilon$. One sees easily that an ε-pseudo-orbit is an orbit for the relation

$$f_\varepsilon \equiv \{(x,y) : d(y, f(x)) \leq \varepsilon\}.$$

Thus, in addition to providing an abstract framework for generalizing properties of iterates of maps, the study of iteration of relations includes pseudo-orbits as a special case.

Pseudo-orbits show up repeatedly in this paper in a slightly disguised form. Although there is no metric in a Hausdorff space, one can consider a pseudo-orbit for a relation f to be an orbit for a neighborhood g of f. It is important to note that a neighborhood of a relation is just another relation and therefore can be iterated.

The dynamics of relations have been studied by other authors. Bullett considered the iteration of certain quadratic correspondences [4]. In a series of papers, Langevin, Walczak, Przytycki, and Nitecki developed a theory of entropy for relations [15,14,19]. Akin has developed a general theory for iteration of relations on compact metric spaces [1]. There is considerable overlap between Akin's work and this paper. In particular, the ideas about attractors, attractor blocks, repellers, and connecting orbits all can be found in Akin's manuscript. However, the terminology, the development of the ideas, and the proofs of the theorems are substantially different. For example, the theory developed here applies to Hausdorff spaces, so the independence from the metric is automatic. Also, the development of the results on restrictions of relations appears to be unique to this paper, as does the use of the omega limit set of a set as a major tool in the proofs.

1. Relations on Sets

It is interesting to begin by developing some basic properties of relations on arbitrary sets with no additional structure. The results are stated without proof and are all elementary; some can be found in Chapter 0 of Kelley's *General Topology* [13]. Throughout this section, X will denote a set.

Definition. A *relation* on a set X is a subset of $X \times X$.

Recall that a map is a relation f with the additional property that, for every $x \in X$, there exists a unique $y \in X$ satisfying $(x,y) \in f$. The image of a set under a map has a direct analog for relations.

Definition. Let f be a relation on X, and let $S \subset X$. The *image* of S under f is the set

$$f(S) \equiv \{y \in X : \exists x \in S \text{ satisfying } (x,y) \in f\}.$$

Implication (1) of the following lemma is a direct generalization of the same property for maps. Implication (2) has no analog for maps. Both implications follow immediately from the definition.

Lemma 1.1. *If f and g are relations on X, and if S and T are subsets of X, then*

(1) $S \subset T \implies f(S) \subset f(T)$ *and*
(2) $f \subset g \implies f(S) \subset g(S)$.

The standard projection maps will be useful in the discussion. For $i = 1$ or 2, denote

$$\pi_i : X \times X \to X : (x_1, x_2) \mapsto x_i.$$

Lemma 1.2. *If f is a relation on X and if $S \subset X$, then*

$$f(S) = \pi_2(f \cap \pi_1^{-1}(S)).$$

The complement of a set S will be denoted S^c.

Lemma 1.3. *If f is a relation on X and if S and T are subsets of X, then*

$$f(S) \subset T \iff f \cap (S \times T^c) = \emptyset.$$

The image of a relation distributes through intersection and union in the same way as for maps.

Lemma 1.4. *If f is a relation on X, and if $\mathfrak{S}$ is a set of subsets of X, then*

(1) $f(\bigcup\{S : S \in \mathfrak{S}\}) = \bigcup\{f(S) : S \in \mathfrak{S}\}$, *and*
(2) $f(\bigcap\{S : S \in \mathfrak{S}\}) \subset \bigcap\{f(S) : S \in \mathfrak{S}\}$.

The following convenient notation will be used for a relation f and a point $x \in X$.

$$f(x) \equiv f(\{x\}).$$

A relation f therefore can be thought of as a set-valued function on X, the image of a point x being the set of all points related to x under f.

$$y \in f(x) \iff (x,y) \in f. \tag{1-1}$$

The inverse image of a set S under a map is the set of all points which map to S. One way to generalize this concept is the following.

Definition. If f is a relation on X and if $S \subset X$, then the *inverse image* of S is the set

$$f^{-1}(S) \equiv \{x \in X : f(x) \subset S\}.$$

Note that, in the case that f is a map, this definition reduces to the standard one. For maps, the inclusions given in the following lemma hold. The lemma shows that they hold for relations as well.

Lemma 1.5. *If f is a relation on X and if $S \subset X$, then*

$$f(f^{-1}(S)) \subset S \subset f^{-1}(f(S)).$$

Neither inclusion of the previous lemma can be replaced by equality, even in the case of maps. Recall, however, that $f(f^{-1}(S)) = S$ for surjective maps, while $f^{-1}(f(S)) = S$ for injective maps.

The following lemma is an immediate consequence of the definition.

Lemma 1.6. *If f is a relation on X and if $S \subset T \subset X$, then $f^{-1}(S) \subset f^{-1}(T)$.*

The following example shows that f^{-1} is not, in general, generated by a relation on X.

Example 1.7. Let $X = \{0,1\}$, let $f = \{(0,0),(0,1),(1,0)\}$, let $S_0 = \{0\}$, and let $S_1 = \{1\}$. Were f^{-1} generated by a relation, property (1) of Lemma 1.1 would imply tha t $f^{-1}(S_0 \cup S_1) = f^{-1}(S_0) \cup f^{-1}(S_1)$. However, one checks that $f^{-1}(S_0 \cup S_1) = f^{-1}(X) = X$, while $f^{-1}(S_0) \cup f^{-1}(S_1) = \{1\} \cup \emptyset = \{1\} \neq X$.

Although the definition of inverse image will be useful in what follows, it is not the only way to generalize the inverse image of a map. The following definition of the transpose of a relation is another generalization with the additional property that it is itself a relation.

Definition. If f is a relation on X, then the *transpose* of f is the relation

$$f^* \equiv \{(x,y) \in X \times X : (y,x) \in f\}.$$

Note that $(f^*)^* = f$. Note also that, if f is a function, then both $f^*(S)$ and $f^{-1}(S)$ give the usual definition of the inverse image of the set S, as stated next.

Lemma 1.8. *If f is a map on X and if $S \subset X$, then $f^*(S) = f^{-1}(S)$.*

Although the inverse image and the transpose are not identical for general relations, they are related by the equality given in the following lemma. This equality will be useful later.

Lemma 1.9. *If f is a relation on X and if $S \subset X$, then*

$$f^{-1}(S)^c = f^*(S^c).$$

To be able to iterate something, one must be able to compose it with itself. The following definition generalizes the definition of composition of maps.

Definition If f and g are relations on X, then the *composition* of f with g is the relation

$$f \circ g \equiv \{(x,z) \in X \times X : \exists y \text{ such that } (x,y) \in g \text{ and } (y,z) \in f\}$$

There are two important properties of composition which allow for the iteration of relations. The first is that composition is associative. The second is that the image under composition is the successive images under the relations. These properties are given in the following lemma.

Lemma 1.10. *If f, g, and h are relations on X, and if $S \subset X$, then both of the following properties hold.*

(1) $f \circ (g \circ h) = (f \circ g) \circ h$.
(2) $(f \circ g)(S) = f(g(S))$.

The composition of two relations can be characterized in terms of the following projection maps. For each of the three pairs (i,j), where $1 \leq i < j \leq 3$, let

$$\pi_{ij} : X \times X \times X \to X \times X : (x_1, x_2, x_3) \mapsto (x_i, x_j).$$

The characterization is given in the next lemma.

Lemma 1.11. *If f and g are relations, then*

$$f \circ g = \pi_{13}(\pi_{23}^{-1}(f) \cap \pi_{12}^{-1}(g)).$$

It is also useful to note that composition preserves inclusion in the sense given by the following lemma.

Lemma 1.12. *If f, g, f', and g' are relations and if $f \subset f'$ and $g \subset g'$, then $f \circ g \subset f' \circ g'$.*

If f is a bijective map on a set X and if $Y \subset X$, then f restricted to Y is again a bijective map if and only if $f(Y) = Y$. If f is a map on X, then f restricted to Y is again a map so long as $f(Y) = Y$. However, in the case of relations on X, any relation can be restricted to any subset Y to form a relation on Y.

Definition. If f is a relation on a set X and if $Y \subset X$, then $f|_Y$ is the relation on Y satisfying

$$f|_Y = f \cap (Y \times Y).$$

There is a subtle distinction between $f|_Y$ and $f \cap (Y \times Y)$. The set $f \cap (Y \times Y)$ is a subset of $X \times X$ and hence is considered to be a relation on X. On the other hand, $f|_Y$ is a relation on Y and hence is a subset of $Y \times Y$. This distinction will make a difference later when X is a topological space and Y carries the relative topology. Then care must be taken since a closed (or an open) subset of Y might not be a closed (or an open) subset of X.

Even with this subtle distinction in mind, the next lemma is a complete triviality. It is stated explicitly here for use in Section 5, where the distinction becomes consequential.

Lemma 1.13. *If f is a relation on a set X and if $S \subset Y \subset X$, then*

$$f|_Y(S) = (f \cap (Y \times Y))(S) = f(S) \cap Y.$$

The next two lemmas of this section are also trivial but will be useful in later sections.

Lemma 1.14. *If f is a relation on a set X and if $Y \subset X$, then*

$$(f \cap (Y \times Y))|_Y = f|_Y.$$

Lemma 1.15. *If f is a relation on a set X and if Y and Z are subsets of X, then*

$$(f|_Y)|_{Y \cap Z} = (f|_Z)|_{Y \cap Z} = f|_{Y \cap Z}.$$

The final lemma of this section states an elementary but valuable formula.

Lemma 1.16. *If f is a relation on a set X and if S and Y are subsets of X, then*

$$f|_Y(S \cap Y) = f(S \cap Y) \cap Y.$$

2. Relations on Compact Hausdorff Spaces

The properties discussed so far apply to relations on an arbitrary set X. In this section, properties for relations on a compact Hausdorff space will be developed.

If S is a subset of a topological space X, then the closure of S is denoted by $\overline{S}$, and the interior of S is denoted by S°. A *neighborhood* of S is a set U containing S in its interior. That is, there exists an open set V satisfying $S \subset V \subset U$. If U itself is open, it is called an *open neighborhood* of S; if U is closed, it is called a *closed neighborhood* of S. Since a subset K of a compact Hausdorff space is closed if and only if it is compact, a closed neighborhood in a compact Hausdorff space is automatically compact.

The following notation will be useful in this and other sections.

$$\mathfrak{N}(S;X) \equiv \{U : U \text{ is a neighborhood of } S \text{ in } X\},$$
$$\overline{\mathfrak{N}}(S;X) \equiv \{G : G \text{ is a closed neighborhood of } S \text{ in } X\}.$$

For a given symbol, such as S, denoting a subset of some ambient space, the identity of the ambient space is usually implicit in the text, in which case the set of neighborhoods of S in the ambient space will be denoted simply as $\mathfrak{N}(S)$. Similarly, the set of closed neighborhoods of S will be denoted as $\overline{\mathfrak{N}}(S)$. The explicit reference to the ambient space will be written only when there is a possibility of confusion or when special emphasis is intended.

A relation f on a compact Hausdorff space X is called *closed* if f is a closed subset of $X \times X$.

Here are the main results of this section.

Theorem 2.1. *If f is a closed relation on a compact Hausdorff space X and if K is a compact subset of X, then $f(K)$ is compact.*

Theorem 2.2. *If f and g are closed relations on a compact Hausdorff space, then $f \circ g$ is closed.*

Theorem 2.3. *If f is a closed relation on a compact Hausdorff space X, if K is a compact subset of X, and if $U \in \mathfrak{N}(f(K))$, then there exists a $V \in \mathfrak{N}(K)$ such that $f(V) \subset U$.*

Theorem 2.4. *If f is a closed relation on a compact Hausdorff space X, if K is a compact subset of X, and if $U \in \mathfrak{N}(f(K);X)$, then there exists a $g \in \overline{\mathfrak{N}}(f;X \times X)$ such that $g(K) \subset U$.*

Theorem 2.5. *If f and g are closed relations on a compact Hausdorff space X and if $h \in \mathfrak{N}(f \circ g; X \times X)$, then there exist $f' \in \overline{\mathfrak{N}}(f; X \times X)$ and $g' \in \overline{\mathfrak{N}}(g; X \times X)$ such that $f' \circ g' \subset h$.*

Recall that the image under a continuous map of a compact set is compact. Theorem 2.1 states that the same result holds for closed relations on compact Hausdorff spaces. Similarly, Theorems 2.2 and 2.3 are analogs of results for continuous maps. Theorems 2.4 and 2.5 are also analogs of results for maps in the sense that g can be considered as a perturbation of f. Note, however, that the topology of the appropriate function space has been replaced by the topology of $X \times X$.

The proofs of these theorems will be given at the end of this section after some standard results are recalled in the form of the following lemmas. These lemmas are all standard, and the proofs are omitted. The first is an elementary result about intersections.

Lemma 2.6. *If $\mathfrak{S}$ and $\mathfrak{T}$ are sets of subsets of a set X, then the following statements hold.*

(1) *If $\mathfrak{S} \subset \mathfrak{T}$, then $\bigcap \mathfrak{T} \subset \bigcap \mathfrak{S}$.*
(2) *If, for every $T \in \mathfrak{T}$, there is an $S \in \mathfrak{S}$ such that $S \subset T$, then $\bigcap \mathfrak{S} \subset \bigcap \mathfrak{T}$.*

The next lemma is a standard result about the product topology.

Lemma 2.7. *If K and L are compact subsets of a topological space X and if $U \in \mathfrak{N}(K \times L; X \times X)$, then there exist a $V \in \mathfrak{N}(K; X)$ and a $W \in \mathfrak{N}(L; X)$ such that $V \times W \subset U$.*

The next lemma is equivalent to the finite intersection property for compact topological spaces.

Lemma 2.8. *If $\mathfrak{K}$ is a set of closed subsets of a compact topological space and if U is a neighborhood of $\bigcap \mathfrak{K}$, then there exists a finite subset $\mathfrak{F}$ of $\mathfrak{K}$ such that $\bigcap \mathfrak{F} \subset U$.*

The next lemma is a minor restatement of the basic result that a compact Hausdorff space is normal. It is followed by a corollary, which is also a simple restatement of normality.

Lemma 2.9. *If K and L are disjoint closed subsets of a compact Hausdorff space, then there exists a $G \in \overline{\mathfrak{N}}(K)$ such that G and L are disjoint.*

Corollary 2.10. *If K is a closed subset of a compact Hausdorff space and if U is a neighborhood of K, then there exists a closed neighborhood G of K satisfying $G \subset U$.*

The next lemma is a simple exercise using normality and the definition of relative topology.

Lemma 2.11. *If X is a compact Hausdorff space, if Y is a closed subset of X, if K is a closed subset of Y, if $U \in \mathfrak{N}(Y;X)$, and if $V \in \mathfrak{N}(K;Y)$, then there exists a $G \in \overline{\mathfrak{N}}(K;X)$ such that $G \subset U$, and $G \cap Y \subset V$.*

The final lemma of this section is a standard result stating that a closed set is determined by its neighborhoods.

Lemma 2.12. *If K is a closed subset of a compact Hausdorff space, then $K = \bigcap \mathfrak{N}(K) = \bigcap \overline{\mathfrak{N}}(K)$.*

The remainder of this section consists of the proofs of the five theorems stated above.

Proof of Theorem 2.1. Lemma 1.2 implies that $f(K) = \pi_2(f \cap \pi_1^{-1}(K))$. Since π_1 is continuous and since K is closed, $\pi_1^{-1}(K)$ is closed, hence compact. Therefore, $f \cap \pi_1^{-1}(K)$ is compact, which, since π_2 is continuous, implies that $f(K)$ is compact. □

Proof of Theorem 2.2. Lemma 1.11 implies that $f \circ g = \pi_{13}(\pi_{23}^{-1}(f) \cap \pi_{12}^{-1}(g))$. Since π_{23} and π_{12} are continuous and since f and g are closed, $\pi_{23}^{-1}(f)$ and $\pi_{23}^{-1}(g)$ are both closed. Therefore, $\pi_{23}^{-1}(f) \cap \pi_{23}^{-1}(g)$ is closed, hence compact. Since π_{13} is continuous, $f \circ g$ is compact, hence closed. □

Proof of Theorem 2.3. It suffices to establish the conclusion under the assumption that U is open. Since $f(K) \subset U$, Lemma 1.3 implies that $f \cap (K \times U^c) = \emptyset$, which implies that $K \times U^c \subset f^c$. Since f is closed, f^c is open. Since U is open, U^c is closed, hence compact. Lemma 2.7 implies the existence of neighborhoods V of K and W of U^c such that $V \times W \subset f^c$. It then follows that $f \cap (V \times U^c) \subset f \cap (V \times W) = \emptyset$, from which Lemma 1.3 implies that $f(V) \subset U$. □

Proof of Theorem 2.4. As before, it is sufficient to assume that U is open. Since $f(K) \subset U$, Lemma 1.3 implies that $f \cap (K \times U^c) = \emptyset$. Since f and $K \times U^c$ are closed, Lemma 2.9 implies the existence of a closed neighborhood g of f such that $g \cap (K \times U^c) = \emptyset$. Lemma 1.3 implies that $g(K) \subset U$. □

Proof of Theorem 2.5. As usual, it suffices to assume that h is open. Lemma 2.12 implies that $f = \bigcap \overline{\mathfrak{N}}(f)$ and $g = \bigcap \overline{\mathfrak{N}}(g)$. It follows that

$$\pi_{23}^{-1}(f) = \bigcap \{ \pi_{23}^{-1}(F) : F \in \overline{\mathfrak{N}}(f) \}, \qquad \text{that}$$

$$\pi_{12}^{-1}(g) = \bigcap \{ \pi_{12}^{-1}(G) : G \in \overline{\mathfrak{N}}(g) \}, \qquad \text{and that}$$

$$\varphi = \bigcap \mathfrak{K}, \qquad \text{where}$$

$$\varphi \equiv \pi_{23}^{-1}(f) \cap \pi_{12}^{-1}(g), \qquad \text{and where}$$

$$\mathfrak{K} \equiv \{ \pi_{23}^{-1}(F) \cap \pi_{12}^{-1}(G) : F \in \overline{\mathfrak{N}}(f) \text{ and } G \in \overline{\mathfrak{N}}(g) \}.$$

Now let

$$\psi \equiv \pi_{13}^{-1}(h).$$

Since h is open and since π_{13} is continuous, it follows that ψ is open. Lemma 1.11 implies that $f \circ g = \pi_{13}(\varphi)$, which, since $f \circ g \subset h$, implies that $\varphi \subset \pi_{13}^{-1}(f \circ g) \subset \pi_{13}^{-1}(h) = \psi$. Therefore, ψ is a neighborhood of φ. Since F and G are closed and since π_{23} and π_{12} are continuous, each $\pi_{23}^{-1}(F) \cap \pi_{12}^{-1}(G)$ is closed; hence $\mathfrak{K}$ is a set of closed subsets of $X \times X \times X$. Since $\varphi = \bigcap \mathfrak{K}$, Lemma 2.8 implies the existence of a finite subset of $\mathfrak{K}$ whose intersection is a subset of ψ. Therefore,

$$\bigcap_{k=1}^{n} \pi_{23}^{-1}(F_k) \cap \pi_{12}^{-1}(G_k) \subset \psi,$$

where $F_k \subset \overline{\mathfrak{N}}(f)$ and $G_k \subset \overline{\mathfrak{N}}(g)$ for $k = 1, \ldots, n$.

Now let $f' \equiv \bigcap_{k=1}^{n} F_k$ and $g' \equiv \bigcap_{k=1}^{n} G_k$. Note that f' is a closed neighborhood of f and that g' is a closed neighborhood of g. Furthermore,

$$\begin{aligned} \pi_{23}^{-1}(f') \cap \pi_{12}^{-1}(g') &= \pi_{23}^{-1}\Big(\bigcap_{k=1}^{n} F_k\Big) \cap \pi_{12}^{-1}\Big(\bigcap_{j=1}^{n} G_j\Big) = \bigcap_{k=1}^{n} \pi_{23}^{-1}(F_k) \cap \bigcap_{j=1}^{n} \pi_{12}^{-1}(G_j) \\ &= \bigcap_{k=1}^{n} \bigcap_{j=1}^{n} \pi_{23}^{-1}(F_k) \cap \pi_{12}^{-1}(G_j) \subset \bigcap_{k=1}^{n} \pi_{23}^{-1}(F_k) \cap \pi_{12}^{-1}(G_k) \subset \psi. \end{aligned}$$

Therefore, Lemma 1.11 implies that

$$f' \circ g' = \pi_{13}(\pi_{23}^{-1}(f') \cap \pi_{12}^{-1}(g')) \subset \pi_{13}(\psi) = h,$$

and the proof is complete. □

3. ITERATION AND ORBITS

One of the fundamental concepts in dynamical systems is that of an orbit. For a map, an orbit is simply the succession of images of a point. For a relation, a point may have no image point, or it may have many image points. An orbit for a relation is one of the many possible successions of images.

For a relation f on a set, the following notation is suggestive of orbits, and will be used frequently.

$$x \overset{f}{\longmapsto} y \iff (x,y) \in f.$$

If the relation f is understood from the context, as it is throughout most of this paper, then its explicit reference is dropped. Thus

$$x \mapsto y \iff x \overset{f}{\longmapsto} y.$$

Definition. If f is a relation on a set X, then an *orbit* for f is a pair (p, I), where I is an interval of $\mathbb{Z}$, either finite of infinite, and where $p : I \to X$ satisfies

$$p_i \mapsto p_{i+1}, \quad \text{whenever } i \in I \text{ and } i+1 \in I$$

If (p,I) is an orbit and if J is an interval of $\mathbb{Z}$, then

$$p_J \equiv \{p_i : i \in I \cap J\}.$$

Note that, if $I \cap J = \emptyset$, then $p_J = \emptyset$.

The composition of relations has a sufficient number of pleasant properties to allow for the iteration of relations. To define precisely the nth iterate of a relation it is convenient to introduce the diagonal, or identity, relation defined as

$$\mathrm{id} \equiv \{ (x,y) \in X \times X : x = y \}.$$

This relation has the property that, for any relation f,

$$f \circ \mathrm{id} = \mathrm{id} \circ f = f.$$

Definition. If f is a relation on a set and if n is a nonnegative integer, then the relation f^n is defined inductively by

$$f^0 = \mathrm{id}, \quad \text{and} \quad f^n = f \circ f^{n-1}, \quad \text{for } n = 1,2,3, \dots .$$

There is, of course, a connection between iteration of a relation and orbits of a relation, as shown in the following theorem.

Theorem 3.1. *If f is a relation on a set X, if I is an interval of $\mathbb{Z}$, and if $p : I \to X$, then the following statements are equivalent.*

(1) *(p,I) is an orbit for f.*
(2) *$p_{i+j} \in f^j(p_i)$ whenever $j \geq 0$, $i \in I$, and $i+j \in I$.*
(3) *$p_i \overset{f^j}{\longmapsto} p_{i+j}$ whenever $j \geq 0$, $i \in I$, and $i+j \in I$.*

Proof. The equivalence of (2) and (3) is an immediate consequence of statement (1-1). Statement (3) reduces to the definition of orbit when $j = 1$, so the implication (3) $\Longrightarrow$ (1) is trivial. There remains only to show that (1) $\Longrightarrow$ (2)

Assume that (p,I) is an orbit for f, and fix $i \in I$. Statement (2) will be proved by induction on j. First note that the statement is trivial if $j = 0$, and assume that the statement has been established for $j = k$. If $i + k + 1 \notin I$, then, since I is an interval, it follows that $i + j \notin I$ for all $j \geq k + 1$ and hence that the conclusion holds. Assume therefore that $i + k + 1 \in I$. By the inductive hypothesis, $\{p_{i+k}\} \subset f^k(\{p_i\})$. Lemmas 1.1 and 1.10 and the definition of f^n imply that

$$f(\{p_{i+k}\}) \subset f(f^k(\{p_i\})) = (f \circ f^k)(\{p_i\}) = f^{k+1}(\{p_i\}).$$

Since (p,I) is an orbit, it follows that $(p_{i+k}, p_{i+k+1}) \in f$. Therefore, $p_{i+k+1} \in f(p_{i+k}) \subset f^{k+1}(p_i)$, and the proof is complete. □

The remainder of this section is devoted to the development of some elementary properties of iteration. Analogues of these lemmas for compact metric spaces can be found in Akin's manuscript [1]. A standard induction argument and the associativity of composition imply the first lemma.

Lemma 3.2. *If f is a relation on a set and if n and m are nonnegative integers, then $f^{n+m} = f^n \circ f^m$.*

The proofs of the next two lemmas are simple induction arguments, the first using Lemma 1.12 and the second using Lemma 1.13.

Lemma 3.3. *If f and g are relations on a set, if $f \subset g$, and if $n \geq 0$, then $f^n \subset g^n$.*

Lemma 3.4. *If f is a relation on a set X, if $S \subset Y \subset X$, and if $n \geq 0$, then*

$$(f|_Y)^n(S) = (f \cap (Y \times Y))^n(S).$$

Since the composition of closed relations on a compact Hausdorff space is closed, induction also implies that the iterates of a closed relation are closed.

Lemma 3.5. *If f is a closed relation on a compact Hausdorff space, then f^n is closed for all $n \geq 0$.*

Closed relations have a semicontinuity property with respect to iteration. If a relation changes by a small amount, then its n^{th} iterate changes by only a small amount, as stated precisely in the next lemma.

Lemma 3.6. *If f is a closed relation on a compact Hausdorff space X, if $n \geq 0$, and if $h \in \mathfrak{N}(f^n; X \times X)$, then there exists a $g \in \overline{\mathfrak{N}}(f; X \times X)$ such that $g^n \subset h$.*

Proof. The lemma is trivial for $n = 0$. As an inductive hypothesis, assume that the lemma is true for $n = k$. Let h be a neighborhood of f^{k+1}, and write $f^{k+1} = f \circ f^k$. Theorem 2.5 implies the existence of closed neighborhoods φ of f and ψ of f^k such that $\varphi \circ \psi \subset h$. By inductive hypothesis, there exists a closed neighborhood γ of f such that $\gamma^k \subset \psi$. Let $g = \varphi \cap \gamma$, and note that g is a closed neighborhood of f. Lemma 3.3 implies that $g^k \subset \gamma^k \subset \psi$. Lemma 1.12 implies that $g^{k+1} = g \circ g^k \subset \varphi \circ \psi \subset h$. Therefore, the lemma is true for $n = k + 1$, and the proof is complete. □

Another semicontinuity property is that small changes in the relation produce only small changes in the image of a compact set under the nth iterate of the relation.

Lemma 3.7. *If f is a closed relation on a compact Hausdorff space X, if K is a compact subset of X, if $n \geq 0$, and if $U \in \mathfrak{N}(f^n(K); X)$, then there exists a $g \in \overline{\mathfrak{N}}(f; X \times X)$ such that $g^n(K) \subset U$.*

Proof. Since f^n is a closed relation, Theorem 2.4 implies the existence of a closed neighborhood h of f^n such that $h(K) \subset U$. Lemma 3.6 then implies the existence of a closed neighborhood g of f such that $g^n \subset h$. Finally, Lemma 1.1 implies that $g^n(K) \subset h(K) \subset U$. □

4. Invariance and Related Properties

For a bijective map f, a set S is called "invariant" if $f(S) = S$. It is usually called "forward invariant," or "positively invariant," if $f(S) \subset S$, while it is called "backward invariant," or "negatively invariant," if $f^{-1}(S) \subset S$. Confusion develops when one tries to apply these terms to relations. For example, since the properties $f(S) = S$ and $f^*(S) = S$, which are identical for bijective maps, are logically independent for relations, it seems reasonable to call the first "forward invariant" and the second "backward invariant." However, this terminology certainly would lead to unnecessary confusion. In this paper, the terminology in the following definition will be used. Justification for this choice of names is given below as various properties are developed. The section ends with an attempt to maintain a foothold in the more familiar territory of maps.

Definition. If f is a relation on X and if $S \subset X$, then the following terminology will be used.

(1) If $f(S) \subset S$, then S is called a *confining* set for f.
(2) If $f^*(S) \subset S$, then S is called a *rejecting* set for f.
(3) If $f(S) \supset S$, then S is called a *backward complete* set for f.
(4) If $f^*(S) \supset S$, then S is called a *forward complete* set for f.
(5) If $f(S) = S$, then S is called an *invariant* set for f.
(6) If $f^*(S) = S$, then S is called a $*$*-invariant* set for f.

There are some trivial observations to make. A set is invariant if and only if it is both confining and backward complete. A set is $*$-invariant if and only if it is both rejecting and forward complete. A set is rejecting for f if and only if it is confining for f^*. A set is forward complete for f if and only if it is backward complete for f^*. A set is $*$-invariant for f if and only if it is invariant for f^*.

The following lemma characterizes the above definitions in terms of the notation given at the beginning of this section. The proof is elementary and will be omitted.

Lemma 4.1. *If f is a relation on a set X and if $S \subset X$, then the following statements hold.*

(1) *S is confining if and only if $x \in S$ and $x \mapsto y \implies y \in S$.*
(2) *S is rejecting if and only if $y \in S$ and $x \mapsto y \implies x \in S$.*
(3) *S is backward complete if and only if $y \in S \implies \exists x \in S$ such that $x \mapsto y$.*
(4) *S is forward complete if and only if $x \in S \implies \exists y \in S$ such that $x \mapsto y$.*

Statement (1) gives the motivation for the term "confining," since it states that all forward images of points in S are confined to S. The term "invariant" is the commonly accepted term in the case that f is a map; it is adopted here in the more general setting. The term $*$-invariant is simply shorthand for "invariant under f^*." The other terminology will be justified as the theory is developed.

Lemma 4.2. *If f is a relation on a set X and if $S \subset X$, then S is confining if and only if $S \subset f^{-1}(S)$.*

Proof. If S is confining, then Lemmas 1.5 and 1.6 imply that $S \subset f^{-1}(f(S)) \subset f^{-1}(S)$. On the other hand, if $S \subset f^{-1}(S)$, then Lemmas 1.1 and 1.5 imply that $f(S) \subset f(f^{-1}(S)) \subset S$, which implies that S is confining and completes the proof. □

Lemma 4.3. *If f is a relation on a set X and if $S \subset X$, then S is rejecting if and only if S^c if confining.*

Proof. If S is rejecting, then, by definition, $f^*(S) \subset S$. Therefore, Lemma 1.9 implies that $S^c \subset f^*(S)^c = f^{-1}(S^c)$, which, by Lemma 4.2 implies that S^c is confining. On the other hand, if S^c is confining, then Lemma 4.2 implies that $S^c \subset f^{-1}(S^c)$. Therefore, Lemma 1.9 implies that $f^*(S) = f^{-1}(S^c)^c \subset S$, which is the definition that S is rejecting. □

The previous lemma justifies the use of the term "rejecting." When combined with Lemma 4.1, it states that S is a rejecting set if and only if $x \in S^c$ and $x \mapsto y$ implies that $y \in S^c$. In other words, all forward iterates of points not in S are rejected by S. The terms "forward complete" and "backward complete" are justified by the following lemma. It states, for example, that there is a complete forward orbit through every point in a forward complete set.

Lemma 4.4. *If f be a relation on a set X and if $S \subset X$, then the following two statements hold.*

(1) *S is forward complete if and only if, for every $x \in S$, there exists an orbit $(p, [0, \infty))$ with $p_0 = x$ and $p_{[0,\infty)} \subset S$.*
(2) *S is backward complete if and only if, for every $x \in S$, there exists an orbit $(p, (-\infty, 0])$ with $p_0 = x$ and $p_{(-\infty,0]} \subset S$.*

Proof. The proof is given only for statement (1). The proof of statement (2) is similar.

Assume that, for every $x \in S$, there exists an orbit $(p, [0, \infty))$ with $p_0 = x$ and $p_{[0,\infty)} \subset S$. Let $y = p_1$. Then $y \in S$ and $x \mapsto y$. Therefore, Lemma 4.1 implies that S is forward complete.

Now assume that S is forward complete, let $x \in S$, and let $p_0 = x$. As an inductive hypothesis, assume that p_i has been defined for $i \in [0, n]$ such that $(p, [0, n])$ is an orbit satisfying $p_{[0,n]} \subset S$. Lemma 4.1 implies the existence of a $y \in S$ such that $p_n \mapsto y$. Let $p_{n+1} = y$. Then $(p, [0, n+1])$ is an orbit satisfying $p_{[0,n+1]} \subset S$. Therefore, p_i can be defined for all $i \in [0, \infty)$, and the proof is complete. □

The next lemma states that all of the invariance properties in the above definition are inherited by iterates. It is followed by a lemma which states that the image of a confining set under an iterate of the relation is again a confining set. The proofs of the all the implications in both lemmas are simple induction arguments using the definition of f^n and Lemma 1.1. They are omitted.

Lemma 4.5. *If f is a relation on a set X, if n is a nonnegative integer, and if $S \subset X$, then each of the following implications holds.*

(1) *If S is confining for f, then S is confining for f^n.*
(2) *If S is rejecting for f, then S is rejecting for f^n.*
(3) *If S is backward complete for f, then S is backward complete for f^n.*
(4) *If S is forward complete for f, then S is forward complete for f^n.*
(5) *If S is invariant for f, then S is invariant for f^n.*
(6) *If S is $*$-invariant for f, then S is $*$-invariant for f^n.*

Lemma 4.6. *If f is a relation on a set, if S is confining for f, and if n is a nonnegative integer, then $f^n(S)$ is confining for f.*

The next two lemmas will be needed in Section 5. The first is a trivial observation about confining sets. Its proof is a simple induction argument using the definition of f^n, the definition of "confining," and Lemma 1.1. The second lemma states that the intersection of confining sets is confining. Its proof is an immediate consequence of Lemma 1.4. Both proofs are omitted.

Lemma 4.7. *If f is a relation on a set X, if K is confining for f, and if $S \subset X$ satisfies $f^n(S) \subset K$ for some $n \geq 0$, then $f^k(S) \subset K$ for all $k \geq n$.*

Lemma 4.8. *If f is a relation on a set X and if $\mathfrak{S}$ is a set of subsets of X such that each $S \in \mathfrak{S}$ is confining for f, then $\bigcap \mathfrak{S}$ is confining for f.*

The next series of lemmas concern confining sets and relations restricted to subsets. They also will be used in later sections. The first gives a sufficient condition for a set to be confining for a restriction. It is a simple consequence of Lemma 1.16.

Lemma 4.9. *If f is a relation on a set X, if $Y \subset X$, and if S is confining for f, then $S \cap Y$ is confining for $f|_Y$.*

The next two lemmas give sufficient conditions for a relation and its restriction to agree on a subset. Both lemmas follow immediately from Lemma 1.16.

Lemma 4.10. *If f is a relation on a set X, if $S \subset Y \subset X$, and if S is confining for f, then $f|_Y(S) = f(S)$.*

Lemma 4.11. *If f is a relation on a set X, if $S \subset Y \subset X$, and if Y is confining for f, then $f|_Y(S) = f(S)$.*

The next lemma gives a sufficient condition for equality between an iterate of a relation and the same iterate of the relation restricted to a subset.

Lemma 4.12. *If f is a relation on a set X, if $S \subset Y \subset X$, if S is confining for f, and if n is a nonnegative integer, then*

$$(f|_Y)^n(S) = f^n(S).$$

Proof. The statement is trivial for $n = 0$. As an inductive hypothesis, assume that $(f|_Y)^k(S) = f^k(S)$. Note that $f^k(S) \subset Y$. Lemma 4.6 implies that $f^k(S)$ is confining for f, so Lemma 4.10 implies that $f|_Y(f^k(S)) = f(f^k(S))$. The definition of iteration, Lemma 1.10, and the inductive hypothesis therefore imply that

$$(f|_Y)^{k+1}(S) = (f|_Y)\big((f|_Y)^k(S)\big) = f|_Y(f^k(S)) = f(f^k(S)) = f^{k+1}(S),$$

which completes the proof. □

The properties "confining," "rejecting," "backward complete," and "forward complete" are logically independent in the sense that no combination of these properties implies any other. This independence is shown in the following examples.

Example 4.13. Let $X = \{0,1\}$, let $f = \{(0,0)\}$, and let $S = X$. Then $f(S) = f^*(S) = \{0\} \subset S$, so S is confining and rejecting but neither backward complete nor forward complete.

Example 4.14. Let $X = \{0,1\}$, let $f = \{(0,0),(0,1),(1,0)\}$, and let $S = \{0\}$. Then $f(S) = f^*(S) = X \supset S$, so S is backward complete and forward complete but neither confining nor rejecting.

Example 4.15. Let $X = \{0,1,2\}$, let $f = \{(0,0),(0,1),(2,0)\}$, and let $S = \{0,1\}$. Then $f(S) = S$ and $f^*(S) = \{0,2\}$. Therefore, S is invariant (hence confining and backward complete) but neither rejecting nor forward complete. For the relation f^*, S is $*$-invariant (hence rejecting and forward complete) but neither confining nor backward complete.

Example 4.16. Let $X = [-2,2]$, let $f = \{(x,y) \in X \times X : y = x^2 - 2\}$, and let $S = \{2\}$. Then $f(S) = S$ and $f^*(S) = \{-2,2\} \supset S$. Therefore, S is invariant (hence confining and backward complete) and forward complete but not rejecting. For the relation f^*, S is $*$-invariant (hence rejecting and forward complete) and backward complete but not confining.

Example 4.17. Let X and f be as in the previous example, but take $S = [0,2]$. Then $f(S) = X \supset S$ and $f^*(S) = [-2,-\sqrt{2}] \cup [\sqrt{2},2]$. Therefore, S is backward complete but neither confining nor rejecting nor forward complete. For the relation f^*, S is forward complete but neither confining nor rejecting nor backward complete.

Example 4.18. Let $X = [0,1]$, let $f = \{(x,y) \in X \times X : y = ax\}$, where $0 < a < 1$, and let $S = X$. Then $f(S) = [0,a] \subset S$ and $f^*(S) = S$. Therefore,

S is confining and $*$-invariant (hence rejecting and forward complete) but not backward complete. For the relation f^*, S is rejecting and invariant (hence confining and backward complete) but not forward complete.

Example 4.19. Let X, a, and f be as in the previous example, but take $S = [b,1]$, where $a < b < 1$. Then $f(S) = [ab,a]$ and $f^*(S) = \emptyset \subset S$. Therefore, S is rejecting but neither confining nor backward complete nor forward complete. For the relation f^*, S is confining but neither rejecting nor backward complete nor forward complete.

Example 4.20. Let $X = [0,1]$, let $f = \{(x,y) \in X \times X : y = x^2\}$, and let $S = [0,a]$, where $0 < a < 1$. Then $f(S) = [0,a^2] \subset S$ and $f^*(S) = [0,\sqrt{a}] \supset S$. Therefore, S is confining and forward complete but neither rejecting nor backward complete. For the relation f^*, S is rejecting and backward complete but neither confining nor forward complete.

Example 4.21. Let X and f be as in the previous example, but take $S = \{0\}$. Then $f(S) = f^*(S) = S$, so S is confining, rejecting, backward complete, and forward complete.

Example 4.22. Let X and f be as in Example 4.20, but take $S = [a,b]$, where $0 < a < b < 1$. Then $f(S) = [a^2,b^2]$ and $f^*(S) = [\sqrt{a},\sqrt{b}]$, so S is neither confining, rejecting, backward complete, nor forward complete.

The remainder of this section is a digression to explore some distinctions between maps and relations.

Lemma 4.23. *If f is a map on a set X and if $S \subset X$, then S is confining if and only if S is forward complete.*

Proof. Lemma 1.8 implies that $f^*(S) = f^{-1}(S)$. Therefore, Lemma 4.2 implies that S is confining if and only if $S \subset f^{-1}(S) = f^*(S)$, which is the definition that S is forward complete. □

The equivalence between a confining set and a forward complete set is the only dependence among the four properties. Examples 4.16 through 4.22 show that the three properties "confining," "rejecting," and "backward complete" are logically independent for maps. Note that the relation f in each of these examples is a map. Furthermore, the relation f^* in Example 4.20 is also a map.

For surjective maps, there is an additional implication.

Lemma 4.24. *If f is a surjective map on a set X and if $S \subset X$ is rejecting, then S is backward complete.*

Proof. If S is rejecting, then $f^{-1}(S) = f^*(S) \subset S$. Since f is surjective, $S = f(f^{-1}(S)) \subset f(S)$, so S is backward complete. □

Examples 4.16, 4.17, and 4.20 through 4.22 show that Lemma 4.24 is the only additional implication for surjective maps. Note that the relation f in each of these examples is a surjective map. Furthermore, the relation f^* in Example 4.20 is also a surjective map.

If S is $*$-invariant, then it is both rejecting and forward complete. Lemma 4.23 implies that, if f is a map, then S is confining, while Lemma 4.24 implies that, if f is a surjective map, then S is also backward complete. The following corollary summarizes these statements.

Corollary 4.25. *If f is a surjective map on a set X and if $S \subset X$ is $*$-invariant, then S is invariant.*

A set which is both invariant and $*$-invariant is sometimes called "completely invariant" [10]. The corollary states that, for surjective maps, a $*$-invariant set is always completely invariant. Example 4.16 shows that the converse of the corollary is not true.

For injective maps, the converse of the implication of Lemma 4.24 holds.

Lemma 4.26. *If f is an injective map on a set X and if $S \subset X$ is backward complete, then S is rejecting.*

Proof. If S is backward complete, then $f(S) \supset S$. Since f is injective, $S = f^{-1}(f(S)) = f^*(f(S)) \supset f^*(S)$. Therefore, S is rejecting. □

Examples 4.18 through 4.22 show that Lemma 4.26 is the only additional implication for injective maps. Note that the relation f in each of these examples is an injective map. Note also that the relation f^* in Example 4.20 is also an injective map.

For bijective maps, Lemmas 4.24 and 4.26 imply that a set S is rejecting if and only if it is backward complete. Lemma 4.23 therefore implies that there are only two independent properties for bijective maps, confining and rejecting. Examples 4.20 through 4.22 show that these two properties are indeed logically independent.

5. Omega Limit Sets for Sets

Throughout this section, indeed, throughout the remainder of this paper, X will denote a compact Hausdorff space, and f will denote a closed relation on X.

The omega limit set of an orbit for a flow or a map is an important tool in the study of dynamical systems. A closely related concept is that of the omega limit set of a set, introduced by Conley for flows on metric spaces [6]. Both concepts are useful in the study of relations. Omega limits sets for orbits will be discussed in the next section; this section is concerned with the generalization of Conley's definition to relations and with the development of properties for the omega limit set of a set.

Conley's definition has at least two generalizations to relations. The first is the following direct analogue.

Definition. The *strict omega limit set* of a set S under the relation f is the set

$$\hat{\omega}(S;f) \equiv \bigcap_{n\geq 0} \overline{\bigcup_{k\geq n} f^k(S)}.$$

A property used extensively by Conley, and one which will be useful in the remainder of this paper, is that the omega limit set of a set is invariant. Unfortunately, this property does not hold for the strict omega limit set, even if S is a single point, and even if the relation f is the inverse of a continuous surjective map on X, as seen below in Example 5.2. It is convenient, therefore, to introduce the following definition of omega limit set. Let

$$\mathfrak{K}(S;f) \equiv \{K : K \text{ is a closed confining set satisfying } f^n(S) \subset K \text{ for some } n \geq 0\}.$$

Definition. The *omega limit set* of a set S under the relation f is the set

$$\omega(S;f) \equiv \bigcap \mathfrak{K}(S;f).$$

It will be seen below in Theorem 5.9 that the omega limit set of a set is indeed invariant. Furthermore, as shown in Theorem 5.3, the omega limit set and the strict omega limit set agree if f is a map. Thus both are generalizations of Conley's definition.

Since the particular relation often will be clear from the context, it is sometimes convenient to drop the explicit reference to it. Thus, $\omega(S)$ will denote the omega limit set of S, $\hat{\omega}(S)$ will denote the strict omega limit set of S, and $\mathfrak{K}(S)$ will denote the set $\mathfrak{K}(S;f)$. Also, it is often inconsequential to distinguish between a point and the set consisting of that point. In particular, the following notation will be used, as will similar notations.

$$\omega(x) \equiv \omega(\{x\}).$$

The strict omega limit set is always a subset of the omega limit set, as shown in the following theorem. The opposite inclusion is not always true, as shown below in Example 5.2.

Theorem 5.1. *If f is a closed relation on a compact Hausdorff space X and if $S \subset X$, then $\hat{\omega}(S) \subset \omega(S)$.*

Proof. It will be shown that, for every $K \in \mathfrak{K}(S)$, there exist s a nonnegative integer n such that

$$\overline{\bigcup_{k\geq n} f^k(S)} \subset K. \tag{5-1}$$

Lemma 2.6 will then imply the result.

Let $K \in \mathfrak{K}(S)$. By definition, there is a nonnegative integer n such that $f^n(S) \subset K$. Since K is confining, Lemma 4.7 implies that $f^k(S) \subset K$, for every $k \geq n$. Therefore, $\bigcup_{k\geq n} f^k(S) \subset K$, which, since K is closed, implies inclusion (5-1) and completes the proof. □

Example 5.2. Let $X \equiv [-2, +\infty]$, i.e., X is the one-point compactification of the interval $[-2, +\infty)$. Consider the following continuous surjective map on X.

$$\varphi(x) = \begin{cases} x^2 - 2, & \text{if } x \in [-2, +\infty) \\ +\infty, & \text{if } x = +\infty \end{cases}$$

Consider φ as a relation, let $f = \varphi^*$, and let $2 < \xi < \infty$. It is well-known that the set of pre-images of the point 0 under the map φ is dense in $[-2,2]$ [5]. Since $0 \in f^2(2)$, it follows that the minimal closed confining set for f containing the point 2 is $[-2,2]$. Therefore, $\omega(\xi; f) = [-2,2] \neq \{2\} = \hat{\omega}(\xi; f)$, which shows that the strict omega limit set is not always identical to the omega limit set. Furthermore, $\{2\}$ is not invariant, which shows that the strict omega limit set is not necessarily invariant.

The relation f also provides an example which shows that the closure of a confining set is not necessarily confining. If $S = (2, \xi]$, then $f(S) = (2, \sqrt{\xi + 2}] \subset S$, but $f(\overline{S}) = [2, \sqrt{\xi+2}] \cup \{-2\} \not\subset \overline{S}$.

The next theorem shows that the omega limit set and the strict omega limit set agree if the relation is a map.

Theorem 5.3. *If g is a continuous map on a compact Hausdorff space X and if $S \subset X$, then*

$$\omega(S; g) = \hat{\omega}(S; g)$$

Proof. For each $n \geq 0$, let

$$G_n \equiv \bigcup_{k \geq n} g^k(S) \quad \text{and} \quad \mathfrak{G} \equiv \{\overline{G_n} : n \geq 0\}.$$

Note that $g(G_n) = G_{n+1} \subset G_n$. Therefore,

$$g(\overline{G_n}) \subset \overline{g(G_n)} \subset \overline{G_n}, \tag{5-2}$$

which states that $\overline{G_n}$ is confining. Furthermore, since $\overline{G_n}$ is closed and since $g^n(S) \subset G_n \subset \overline{G_n}$, it follows that $\overline{G_n} \in \mathfrak{K}(S; g)$. Therefore, $\mathfrak{G} \subset \mathfrak{K}(S; g)$, which, with Lemma 2.6, implies that

$$\omega(S; g) = \bigcap \mathfrak{K}(S; g) \subset \bigcap \mathfrak{G} = \hat{\omega}(S; g).$$

The opposite inclusion was proved in Theorem 5.1, so the proof is complete. □

The previous proof fails at step (5-2) for closed relations because the inclusion $g(\overline{S}) \subset \overline{g(S)}$ is true for maps but not for relations. Indeed, the lemma itself fails for closed relations, as was shown in Example 5.2.

The next theorem shows that, for a closed confining set, the omega limit set and the strict omega limit set agree and are given by a simple formula.

Theorem 5.4. *If G is a closed confining set for a closed relation on a compact Hausdorff space, then*

$$\hat{\omega}(G) = \omega(G) = \bigcap_{n\geq 0} f^n(G).$$

Proof. Theorem 5.1 implies that

$$\hat{\omega}(G) \subset \omega(G).$$

For each $n \geq 0$, Lemma 4.6 implies that $f^n(G)$ is confining, while Theorem 2.1 implies that $f^n(G)$ is closed. Since $f^n(G)$ is a subset of itself, $f^n(G) \in \mathfrak{K}(G)$. Lemma 2.6 therefore implies that

$$\omega(G) = \bigcap \mathfrak{K}(G) \subset \bigcap_{n\geq 0} f^n(G).$$

The inclusion $f^n(G) \subset \bigcup_{k\geq n} f^k(G)$ is trivial. Therefore,

$$\bigcap_{n\geq 0} f^n(G) \subset \bigcap_{n\geq 0} \overline{\bigcup_{k\geq n} f^k(G)} = \hat{\omega}(G),$$

and the proof is complete. □

The following lemma establishes some properties of the set $\mathfrak{K}(S)$.

Lemma 5.5. *If S is a subset of X, then the following properties hold.*

(1) $S' \subset S \implies \mathfrak{K}(S) \subset \mathfrak{K}(S')$.
(2) $g \subset f \implies \mathfrak{K}(S;f) \subset \mathfrak{K}(S;g)$.
(3) $\mathfrak{K}(f(S)) = \mathfrak{K}(S)$.
(4) $K \in \mathfrak{K}(S) \implies f(K) \in \mathfrak{K}(S)$.
(5) *If $\mathfrak{F}$ is a finite subset of $\mathfrak{K}(S)$, then $\bigcap \mathfrak{F} \in \mathfrak{K}(S)$.*

Proof. Let $S' \subset S$, and let $K \in \mathfrak{K}(S)$. Then K is a closed confining set satisfying $f^n(S) \subset K$ for some $n \geq 0$. Lemma 1.1 implies that $f^n(S') \subset f^n(S) \subset K$, which implies that $K \in \mathfrak{K}(S')$ and establishes property (1).

Now assume that $g \subset f$, and let $K \in \mathfrak{K}(S;f)$. Again, K is a closed confining set for f satisfying $f^n(S) \subset K$ for some $n \geq 0$. Lemma 1.1 implies that $g(K) \subset f(K) \subset K$, while the same lemma, together with Lemma 3.3, implies that $g^n(S) \subset f^n(S) \subset K$. Therefore, $K \in \mathfrak{K}(S;g)$, which establishes property (2).

Now let $K \in \mathfrak{K}(f(S))$. Then K is a closed confining set satisfying $f^{n+1}(S) = f^n(f(S)) \subset K$ for some $n \geq 0$. Therefore, $K \in \mathfrak{K}(S)$, which establishes that

$$\mathfrak{K}(f(S)) \subset \mathfrak{K}(S).$$

Now let $K \in \mathfrak{K}(S)$. Then K is a closed confining set satisfying $f^n(S) \subset K$. Since K is confining, it follows that $f^n(f(S)) = f(f^n(S)) \subset f(K) \subset K$, which establishes that

$$\mathfrak{K}(S) \subset \mathfrak{K}(f(S))$$

and completes the proof of property (3).

Again let $K \in \mathfrak{K}(S)$. Since K is closed, hence compact, Theorem 2.1 implies that $f(K)$ is compact, hence closed. Since K is confining, Lemma 4.6 implies that $f(K)$ is confining. Since $f^n(S) \subset K$ for some $n \geq 0$, it follows that $f^{n+1}(S) = f(f^n(S)) \subset f(K)$, which implies that $f(K) \in \mathfrak{K}(S)$ and establishes property (4).

Finally, let $\mathfrak{F}$ be a finite subset of $\mathfrak{K}(S)$, and write $G \equiv \bigcap \mathfrak{F}$. Since G is the intersection of closed sets, it is closed. Since G is the intersection of confining sets, Lemma 4.8 implies that G is confining. For each $K \in \mathfrak{F}$, there is a nonnegative integer n_K such that $f^{n_K}(S) \subset K$. Let $n = \max\{n_K : K \in \mathfrak{F}\}$. Then Lemmas 3.2 and 4.5 imply that

$$f^n(S) = f^{n-n_K}(f^{n_K}(S)) \subset f^{n-n_K}(K) \subset K.$$

Therefore, $f^n(S) \subset \bigcap \mathfrak{F} = G$, which implies that $G \in \mathfrak{K}(S)$ and establishes property (5). □

The last property has a corollary which states that an element of $\mathfrak{K}(S)$ can be found in an arbitrary neighborhood of $\omega(S)$.

Corollary 5.6. *If $S \subset X$ and if U is a neighborhood of $\omega(S)$, then there exists a $K \in \mathfrak{K}(S)$ such that $K \subset U$.*

Proof. Lemma 2.8 implies the existence of a finite subset $\mathfrak{F}$ of $\mathfrak{K}(S)$ such that $\bigcap \mathfrak{F} \subset U$. Property (5) of Lemma 5.5 implies that $K \equiv \bigcap \mathfrak{F} \in \mathfrak{K}(S)$. □

The next lemma establishes some properties of the set $\mathfrak{K}(S)$ for restrictions of relations. These properties will be exploited below in the proofs of Lemma 5.13 and Theorems 5.14 and 5.15. Note that the ambient space for the relation has become important.

Lemma 5.7. *If Y is a closed subset of X and if $S \subset Y$, then the following properties hold.*

(1) $\mathfrak{K}(S; f|_Y) \subset \mathfrak{K}(S; f \cap (Y \times Y))$.
(2) $K \in \mathfrak{K}(S; f \cap (Y \times Y)) \implies K \cap Y \in \mathfrak{K}(S; f|_Y)$.

Proof. Let $K \in \mathfrak{K}(S; f|_Y)$. Then K is a closed confining set for $f|_Y$ satisfying $(f|_Y)^n(S) \subset K$ for some $n \geq 0$. Since K is closed in Y, which is itself closed in X, it follows that K is closed in X. Lemma 1.13 implies that K is confining for $f \cap (Y \times Y)$, while Lemma 3.4 implies that $(f \cap (Y \times Y))^n(S) \subset K$. Therefore, $K \in \mathfrak{K}(S; f \cap (Y \times Y))$, and property (1) is established.

Now let $K \in \mathfrak{K}(S; f \cap (Y \times Y))$. Then K is a closed confining set for $f \cap (Y \times Y)$ satisfying $(f \cap (Y \times Y))^n(S) \subset K$ for some $n \geq 0$. Since K is closed in X, $K \cap Y$ is closed in Y. Since K is confining for $f \cap (Y \times Y)$, Lemmas 4.9 and 1.14 imply that $K \cap Y$ is confining for $(f \cap (Y \times Y))|_Y = f|_Y$. Lemma 3.4 implies that $(f|_Y)^n(S) \subset K$, which, since $(f|_Y)^n(S) \subset Y$, implies that $(f|_Y)^n(S) \subset K \cap Y$. Therefore, $K \cap Y \in \mathfrak{K}(S; f|_Y)$, which establishes property (2) and completes the proof. □

The next lemma states that the image of S under sufficiently high iterates of f can be found in any neighborhood of $\omega(S)$.

Lemma 5.8. *If $S \subset X$ and if U is a neighborhood of $\omega(S)$, then there exists a nonnegative integer n such that $f^k(S) \subset U$ for all $k \geq n$.*

Proof. Corollary 5.6 implies the existence of a $K \in \mathfrak{K}(S)$ such that $K \subset U$. The definition of $\mathfrak{K}(S)$ implies the existence of an $n \geq 0$ such that $f^n(S) \subset K$. Lemma 4.7 implies that, for $k \geq n$, $f^k(S) \subset K \subset U$. □

The next theorem, which establishes that the omega limit set of a set is a closed invariant set, is one of the fundamental results of this section.

Theorem 5.9. *If f is a closed relation on a compact Hausdorff space X and if $S \subset X$, then $\omega(S)$ is a closed invariant set.*

Proof. Since $\omega(S)$ is the intersection of closed subsets, it is closed. Since it is the intersection of confining sets, Lemma 4.8 implies that it is confining. There remains only to show that $\omega(S)$ is backward complete.

It will be shown that

$$\mathfrak{N}(f(\omega(S))) \subset \mathfrak{N}(\omega(S)). \tag{5-3}$$

Lemmas 2.12 and 2.6 will then imply that

$$\omega(S) = \bigcap \mathfrak{N}(\omega(S)) \subset \bigcap \mathfrak{N}(f(\omega(S))) = f(\omega(S))$$

and hence that $\omega(S)$ is backward complete.

Let $U \in \mathfrak{N}(f(\omega(S)))$. Theorem 2.3 implies the existence of a neighborhood V of $\omega(S)$ such that $f(V) \subset U^\circ$. Corollary 5.6 implies the existence of a $K \in \mathfrak{K}(S)$ such that $K \subset V$. Property (4) of Lemma 5.5 implies that $f(K) \in \mathfrak{K}(S)$. Therefore, $\omega(S) \subset f(K) \subset f(V) \subset U^\circ$, which implies that $U \in \mathfrak{N}(\omega(S))$, establishes inclusion (5-3), and completes the proof. □

The next two results are simple, but extremely useful. They are combined into one theorem, which states that the omega limit set is monotone in its arguments.

Theorem 5.10. *If f is a closed relation on a compact Hausdorff space X and if $S \subset X$, then the following properties hold.*

(1) $S' \subset S \implies \omega(S'; f) \subset \omega(S; f)$.
(2) $g \subset f \implies \omega(S; g) \subset \omega(S; f)$.

Proof. Assume that $S' \subset S$ and that $g \subset f$. Properties (1) and (2) of Lemma 5.5 state that $\mathfrak{K}(S;f) \subset \mathfrak{K}(S';f)$ and that $\mathfrak{K}(S;g) \subset \mathfrak{K}(S;f)$. Lemma 2.6 therefore implies that

$$\omega(S';f) = \bigcap \mathfrak{K}(S';f) \subset \bigcap \mathfrak{K}(S;f) = \omega(S;f) \qquad \text{and}$$
$$\omega(S;g) = \bigcap \mathfrak{K}(S;g) \subset \bigcap \mathfrak{K}(S;f) = \omega(S;f),$$

which completes the proof. □

The omega limit set of a set S depends only on the ultimate behavior of the image of S under high iterates of the relation. The next theorem establishes that the omega limit set of S is the same as the omega limit set of the image of S under any iterate of the relation.

Theorem 5.11. *If f is a relation on a compact Hausdorff space X, if $S \subset X$, and if n is a nonnegative integer, then*

$$\omega(f^n(S)) = \omega(S).$$

Proof. Property (3) of Lemma 5.5 implies that $\mathfrak{K}(f(S)) = \mathfrak{K}(S)$, which implies that

$$\omega(f(S)) = \bigcap \mathfrak{K}(f(S)) = \bigcap \mathfrak{K}(S) = \omega(S).$$

A simple induction argument completes the proof. □

The next theorem gathers together some elementary results.

Theorem 5.12. *If f is a closed relation on a compact Hausdorff space X and if S is a subset of X, then the following statements hold.*

(1) *If S is a closed confining set, then $\omega(S) \subset S$.*
(2) *If S is a backward complete set, then $\omega(S) \supset S$.*
(3) *If S is a closed invariant set, then $\omega(S) = S$.*
(4) $\omega(\omega(S)) = \omega(S)$.

Proof. If S is a closed confining set, then $f^n(S) \subset S$ for $n = 0$, so $S \subset \mathfrak{K}(S)$. Therefore, $\omega(S) \subset S$, which establishes statement (1).

Now assume that S is backward complete. Lemma 4.5 implies that $S \subset f^n(S)$, for all $n \geq 0$. Therefore, $S \subset K$, for every $K \in \mathfrak{K}(S)$, which implies that $S \subset \bigcap \mathfrak{K}(S) = \omega(S)$ and establishes statement (2).

Statement (3) follows immediately from statements (1) and (2) and the definition of invariant, while statement (4) is a consequence of statement (3) and Theorem 5.9. □

The final results of this section concern omega limit sets for relations restricted to subsets. The next lemma is simple, but it is not as trivial as it appears, since each half of the equality lives in a different space.

Lemma 5.13. *If Y is a closed subset of X and if $S \subset Y$, then*

$$\omega(S; f|_Y) = \omega(S; f \cap (Y \times Y)).$$

Proof. Statement (1) of Lemma 2.6, combined with inclusion (1) of Lemma 5.7, implies that

$$\omega(S; f \cap (Y \times Y)) \subset \omega(S; f|_Y),$$

while statement (2) of Lemma 2.6, combined with inclusion (2) of Lemma 5.7, implies that

$$\omega(S; f|_Y) \subset \omega(S; f \cap (Y \times Y)),$$

and the proof is complete. □

The next theorem states that the omega limit set for a restricted relation is always a subset of the omega limit set for the relation.

Theorem 5.14. *If f is a closed relation on a compact Hausdorff space X, if Y is a closed subset of X, and if $S \subset Y$, then*

$$\omega(S; f|_Y) \subset \omega(S; f).$$

Proof. Lemma 5.13 and Property (2) of Theorem 5.10 imply that

$$\omega(S; f|_Y) = \omega(S; f \cap (Y \times Y)) \subset \omega(S; f),$$

which completes the proof. □

The final theorem states that equality is achieved in the previous theorem if S is a closed confining set.

Theorem 5.15. *If f is a closed relation on a compact Hausdorff space, if K is a closed confining set for f, and if Y is a closed set satisfying $K \subset Y$, then*

$$\omega(K; f|_Y) = \omega(K; f).$$

Proof. Lemma 4.9 implies that K is confining for $f|_Y$. Since K is a closed subset of the ambient space, it is also a closed subset of Y. Theorem 5.4 and Lemma 4.12 therefore imply that

$$\omega(K; f|_Y) = \bigcap_{n \geq 0} (f|_Y)^n(K) = \bigcap_{n \geq 0} f^n(K) = \omega(K; f),$$

and the proof is complete. □

This section ends with the definition of the alpha limit set of a set, which is simply the omega limit set for the transpose relation.

Definition. The *alpha limit set* of a set S for the relation f is the set

$$\omega^*(S;f) \equiv \omega(S;f^*).$$

As above, it is often convenient to drop the explicit reference to the relation and to denote the alpha limit set of a set S simply by $\omega^*(S)$. Of course, all the previous theorems have analogs for the alpha limit set. For example, for a set S, $\omega^*(S)$ is a closed $*$-invariant set.

6. Omega Limit Sets for Orbits

The omega limit set of an orbit of a relation is defined in exactly the same way as for maps; the omega limit set of an orbit is the set of limit points of the forward part of the orbit. The alpha limit set of an orbit is similar.

Definition. If (p,I) is an orbit, then the *omega limit set* of (p,I) is the set

$$\omega((p,I)) \equiv \bigcap_{k\in\mathbb{Z}} \overline{p_{[k,\infty)}},$$

while the *alpha limit set* of (p,I) is the set

$$\omega^*((p,I)) \equiv \bigcap_{k\in\mathbb{Z}} \overline{p_{(-\infty,k]}}.$$

Note that the alpha limit set of an orbit is simply the omega limit set for the orbit running backward in time, which is an orbit for the transpose relation.

In the preceeding section, the invariance of the omega limit set of a set was stressed. In this light, it should be noted that the omega limit set of an orbit is not invariant, as seen below in Example 6.3.

It is also important to recall that a forward orbit through a point does not necessarily exist. Even if it does exist, it is not necessarily unique. Therefore, there is an important distinction between the omega limit set of a set consisting of a single point and the omega limit set of an orbit through that point. However, as shown below in Theorem 6.2, the omega limit set of the orbit is a subset of the omega limit set of the point.

This section is devoted to establishing some elementary properties of the omega limit set of an orbit. First it is shown that, if an orbit is forward finite, then its omega limit set is empty, while, if it is forward infinite, then its omega limit set is nonempty.

Lemma 6.1. *If $P \equiv (p,I)$ is an orbit and if K is a closed set, then the following statements hold.*

(1) *If I is bounded above, then $\omega(P) = \emptyset$.*
(2) *If I is unbounded above, then $\omega(P)$ is a nonempty compact set.*
(3) *If $p_I \subset K$, then $\omega(P) \subset K$.*

Proof. If I is bounded above, then $p_{[k,\infty)} = \emptyset$ for sufficiently large k, which implies that $\omega(P) = \emptyset$ and establishes statement (1). On the other hand, if I is unbounded above, then $p_{[k,\infty)} \neq \emptyset$ for all k. Therefore, $\overline{p_{[k,\infty)}}$ is a nested sequence of nonempty compact sets, which implies that $\omega(P)$ is a nonempty compact set and establishes statement (2). Finally, if $p_I \subset K$, then $p_{[k,\infty)} \subset K$ for all k, which, since K is closed, implies that $\overline{p_{[k,\infty)}} \subset K$ for all k. Therefore, $\omega(P) \subset K$, which establishes statement (3) and completes the proof. □

The next theorem shows that the omega limit set of an orbit is a subset of the strict omega limit set of any point along the orbit. The opposite inclusion is not generally true, as is shown in Example 6.3 below.

Theorem 6.2. *If (p,I) is an orbit for a closed relation on a compact Hausdorff space and if $i \in I$, then*

$$\omega((p,I)) \subset \hat{\omega}(p_i) \subset \omega(p_i).$$

Proof. Let f be a closed relation on a compact Hausdorff space X, let (p,I) be an orbit for f, and let n be a nonnegative integer. Statement (2) of Theorem 3.1 implies that $p_{[n+i,\infty)} \subset \bigcup_{j\geq n} f^j(p_i)$. Indeed, the same inclusion holds with closure on both sides. Therefore, $\omega((p,I)) = \bigcap_{k\in\mathbb{Z}} \overline{p_{[k,\infty)}} \subset \overline{\bigcup_{j\geq n} f^j(p_i)}$. Since n is arbitrary, $\omega((p,I)) \subset \bigcap_{n\geq 0} \overline{\bigcup_{k\geq n} f^k(S)} = \hat{\omega}(p_i)$, which establishes the first inclusion. The second inclusion is an immediate consequence of Theorem 5.1, and the proof is complete. □

Example 6.3. Consider the surjective map φ defined in Example 5.2, and, as before, let $f = \varphi^*$. Let $(p,\mathbb{Z})$ be the orbit for f defined by $p_i = 2$, for all i. Then $\omega((p,\mathbb{Z})) = \{2\} \neq [-2,2] = \hat{\omega}(2;f)$, which shows that the omega limit set of an orbit is not always identical to the strict omega limit set of a point on that orbit. Furthermore, $\{2\}$ is not invariant, which shows that the omega limit set of an orbit is not necessarily invariant.

Although the omega limit set of an orbit is not necessarily invariant, it is always both backward and forward complete.

Theorem 6.4. *If P is an orbit for a closed relation on a compact Hausdorff space, then $\omega(P)$ is both backward complete and forward complete.*

Proof. Let f be a closed relation on a compact Hausdorff space X, and let $P \equiv (p,I)$ be an orbit for f. If I is bounded above, then the conclusion is trivial, so assume that I is unbounded above. For $i \in I$, let

$$g_i \equiv \bigcup_{k\geq i} \{(p_k,p_{k+1})\}, \quad \text{and let} \quad g \equiv \bigcap_{i\in I} \overline{g_i}.$$

Note that, since $\{(p_k,p_{k+1})\} \subset f$ for all $k \in I$, $g_i \subset f$ for all $i \in I$. Since f is closed, it follows that $g \subset f$. Note also that

$$\pi_1 g_i = p_{[i,\infty)} \quad \text{and} \quad \pi_2 g_i = p_{[i+1,\infty)}.$$

Since X is compact, it follows that

$$\pi_1\overline{g_i} = \overline{p_{[i,\infty)}} \quad \text{and} \quad \pi_2\overline{g_i} = \overline{p_{[i+1,\infty)}}.$$

Since $\overline{g_i}$ is a nested sequence of compact sets and since π_1 and π_2 are continuous, it follows that

$$\pi_1 g = \pi_1 \bigcap_{i\in I} \overline{g_i} = \bigcap_{i\in I} \pi_1\overline{g_i} = \bigcap_{i\in I} \overline{p_{[i,\infty)}} = \omega(P) \qquad \text{and that}$$
$$\pi_2 g = \pi_2 \bigcap_{i\in I} \overline{g_i} = \bigcap_{i\in I} \pi_2\overline{g_i} = \bigcap_{i\in I} \overline{p_{[i+1,\infty)}} = \omega(P).$$

Therefore, $g(\omega(P)) = g^*(\omega(P)) = \omega(P)$. Since $g \subset f$, Lemma 1.1 implies that

$$\omega(P) = g(\omega(P)) \subset f(\omega(P)) \qquad \text{and that}$$
$$\omega(P) = g^*(\omega(P)) \subset f^*(\omega(P)),$$

which shows that $\omega(P)$ is both backward complete and forward complete. □

This section ends with two results about omega limit sets for forward complete sets. The first is that the omega limit set of any point in a closed forward complete set K has a nonempty intersection with K.

Lemma 6.5. *If K is closed and forward complete and if $x \in K$, then $\omega(x) \cap K \neq \emptyset$.*

Proof. Since K is forward complete, Lemma 4.4 implies the existence of an orbit $P \equiv (p, [0, \infty))$ such that $p_0 = x$ and $p_{[0,\infty)} \subset K$. Lemma 6.1 implies that $\omega(P)$ is nonempty and that $\omega(P) \subset K$. Since Theorem 6.2 implies that $\omega(P) \subset \omega(x)$, it follows that $\omega(x) \cap K \neq \emptyset$, and the proof is complete. □

The previous lemma has a corollary, which states that the omega limit set of a nonempty closed forward complete set is itself nonempty.

Corollary 6.6. *If K is a nonempty closed forward complete set, then $\omega(K) \neq \emptyset$.*

Proof. Let $x \in K$. Then Lemma 6.5 implies that $\emptyset \neq \omega(x) \cap K \subset \omega(K) \cap K$, which implies that $\omega(K) \neq \emptyset$. □

7. ATTRACTORS

Although the concept of attraction is widely used, there is little agreement on its exact definition. For a survey of the concept and for a good list of references, the reader should consult Milnor's paper [16]. The treatment of attractors presented here is analogous to that developed by Conley [6] for flows, that is, an attractor is an invariant set which is the omega limit set of a neighborhood of itself.

Definition. A set A is called an *attractor* for f if there exists a neighborhood U of A such that

$$\omega(U) = A$$

Theorem 5.9 immediately implies that an attractor is a closed invariant set.

Lemma 7.1. *If A is an attractor, then A is a closed invariant set.*

An important property of an attractor is that it can be surrounded by a neighborhood whose image lies strictly inside itself. Such a neighborhood will be called an "attractor block" in this presentation, following terminology used by Conley and Easton [9].

Definition. A set B is called an *attractor block* for f if

$$f(\overline{B}) \subset B^{\circ}.$$

The following three theorems state the main results of this section. The proofs are given below, as the appropriate machinery is developed. The proof of Theorem 7.2 follows Lemma 7.6, that of Theorem 7.3 follows Lemma 7.12, while that of Theorem 7.4 follows Lemma 7.13.

Theorem 7.2. *If f is a closed relation on a compact Hausdorff space and if B is an attractor block for f, then B is a neighborhood of $\omega(B)$ and, hence, $\omega(B)$ is an attractor for f.*

Theorem 7.3. *If f is a closed relation on a compact Hausdorff space, if A is an attractor for f, and if V is a neighborhood of A, then there exists a closed attractor block B for f such that $B \subset V$ and $\omega(B) = A$.*

Theorem 7.4. *If f is a closed relation on a compact Hausdorff space, if A is an attractor for f, and if A' is an attractor for $f|_A$, then A' is an attractor for f.*

Suppose that A is an attractor, that B is an attractor block, and that $A = \omega(B)$. Then A will be called the attractor associated with B, while B will be called an attractor block associated with A. Theorem 7.2 states that an attractor block always has an associated attractor; indeed, the associated attractor is unique. Theorem 7.3 states that an attractor always has an associated attractor block in any neighborhood. Note that the attractor block associated with a given attractor usually will not be unique.

Some authors use the attractor block to define the attractor [11]. In view of Theorems 7.2 and 7.3, this formulation is equivalent to the definition given above.

The following corollary is a simple restatement of Theorem 7.2. It will be used in later sections.

Corollary 7.5. *If A is an attractor and if B is an attractor block associated with A, then B is a neighborhood of A.*

The next lemma develops some elementary properties of attractor blocks.

Lemma 7.6. *If B is an attractor block, then the following statements are true.*

(1) *B is a confining set.*
(2) *B° and $\overline{B}$ are attractor blocks.*
(3) $\omega(B^\circ) = \omega(B) = \omega(\overline{B}) \subset \overline{B}$.

Proof. Statement (1) follows from Lemma 1.1, since

$$f(B) \subset f(\overline{B}) \subset B^\circ \subset B.$$

Statement (2) follows from the same lemma, since

$$f\left(\overline{B^\circ}\right) \subset f(\overline{B}) \subset B^\circ = (B^\circ)^\circ \qquad \text{and}$$
$$f\left(\overline{\overline{B}}\right) = f(\overline{B}) \subset B^\circ \subset \overline{B}^\circ.$$

Statements (1) and (2) imply that $\overline{B}$ is confining. Since $\overline{B}$ is also closed, Theorem 5.12 implies that $\omega(\overline{B}) \subset \overline{B}$. Theorem 5.10 implies that $\omega(B^\circ) \subset \omega(B) \subset \omega(\overline{B})$. Theorems 5.11 and 5.10 imply that $\omega(\overline{B}) = \omega(f(\overline{B})) \subset \omega(B^\circ)$, which establishes statement (3) and completes the proof. □

Proof of Theorem 7.2. It suffices to show that $\omega(B) \subset B^\circ$. Theorem 5.9 implies that $\omega(B)$ is invariant. This invariance, together with statement (3) of Lemma 7.6, implies that $\omega(B) = f(\omega(B)) \subset f(\overline{B}) \subset B^\circ$, which completes the proof. □

An attractor block can be characterized by the property that the closure of the Cartesian product of itself with its complement does not intersect the relation f.

Lemma 7.7. *B is an attractor block for f if and only if $f \cap (\overline{B} \times \overline{B^c}) = \emptyset$.*

Proof. Lemma 1.3 implies that B is an attractor block if and only if $f \cap (\overline{B} \times (B^\circ)^c) = \emptyset$. Since $(B^\circ)^c = \overline{B^c}$, the proof is complete. □

The symmetry between a relation and its transpose implies that a set is an attractor block for a relation if and only if the complement of the set is an attractor block for the transpose of the relation. This fact will be used in a later section.

Corollary 7.8. *B is an attractor block for f if and only if B^c is an attractor block for f^*.*

Proof. According to the preceeding lemma, f is an attractor block for f if and only if $f \cap (\overline{B} \times \overline{B^c}) = \emptyset$. This statement is equivalent to $f^* \cap (\overline{B^c} \times \overline{B}) = \emptyset$. Since $\overline{B} = \overline{(B^c)^c}$, the preceeding lemma implies that the last statement is equivalent to the statement that B^c is an attractor block for f^*. □

The next lemma is elementary, but it provides a useful tool for the construction of attractor blocks. It states that a confining set for a neighborhood of a relation is an attractor block for the relation. It is the first time in this paper that essential use is made of the fact that a neighborhood of a relation is itself a relation.

Lemma 7.9. *If g is a closed neighborhood of f and if B is confining for g, then B is an attractor block for f.*

Proof. Since B is confining for g, Lemma 1.3 implies that $g \cap (B \times B^c) = \emptyset$. Since g is a neighborhood of f, it follows that $f \cap (\overline{B} \times \overline{B^c}) = f \cap (\overline{B \times B^c}) = \emptyset$. Therefore, Lemma 7.7 implies that B is an attractor block for f. □

Example 5.2 shows that the closure of a confining set for f may not be confining for f. However, the following corollary shows that, if a set is confining for a closed neighborhood of f, then its closure is confining for f.

Corollary 7.10. *If g is a closed neighborhood of f and if B is confining for g, then $\overline{B}$ is confining for f.*

Proof. Lemma 7.9 implies that B is an attractor block for f. Lemma 7.6 implies that $\overline{B}$ is also an attractor block for f and hence that $\overline{B}$ is confining for f. □

The following lemma shows that the definition of attractor is a local condition in the sense that a neighborhood W satisfying $\omega(W) = A$ exists arbitrarily close to A. It also shows that the neighborhood W can be chosen to be closed.

Lemma 7.11. *If A is an attractor and if V is a neighborhood of A, then there exists a closed neighborhood W of A such that $W \subset V$ and $\omega(W) = A$.*

Proof. Since A is an attractor, there exists a neighborhood U of A such that $\omega(U) = A$. Corollary 2.10 implies the existence of a closed neighborhood W of A such that $W \subset V \cap U$. Theorem 5.10 and statement (3) of Theorem 5.12 imply that $A = \omega(A) \subset \omega(W) \subset \omega(U) = A$, which completes the proof. □

The next lemma is the key to the proof of Theorem 7.3. It states that, if A is an attractor for a relation f and if g is a small neighborhood of f, then the omega limit set of A under the relation g is close to the omega limit set of A under f. Since $\omega(A; g)$ is an attractor block for f, the proof of the existence of an attractor block becomes an almost trivial consequence of the lemma.

One should contrast the proof of Lemma 7.12 with Hurley's proof of the existence of an attractor block in an arbitrary neighborhood of the attractor [11]. Although his definitions and motivation are different, his proof can be reformulated to provide an alternate proof of this lemma.

Lemma 7.12. *If A is an attractor for f and if V is a neighborhood of A, then there exists a closed neighborhood g of f such that $\omega(A;g) \subset V$.*

Proof. In view of Lemma 7.11, it suffices to assume that V is closed and that $\omega(V;f) = A$. Lemma 5.8 implies the existence of a positive integer m such that $f^m(V) \subset V^\circ$.

Since A is invariant, $f^k(A) = A \subset V^\circ$, for all k. Therefore, Lemma 3.7, applied $m+1$ times, implies the existence of a closed neighborhood h of f such that

$$h^m(V) \subset V^\circ \text{ and } h^k(A) \subset V^\circ, \quad \text{for } 0 \leq k \leq m-1.$$

As an inductive hypothesis, assume that $h^k(A) \subset V^\circ$, for $0 \leq k \leq n-1$, where $n \geq m$. Then Lemmas 3.2 and 1.1 imply that

$$h^n(A) = h^m(h^{n-m}(A)) \subset h^m(V^\circ) \subset h^m(V) \subset V^\circ,$$

which, by induction, establishes that

$$h^n(A) \subset V^\circ \quad \text{for } n \geq 0. \tag{7-1}$$

Now let

$$K \equiv \bigcup_{n \geq 0} h^n(A).$$

Lemma 1.4 implies that K is confining for h, since $h(K) = \bigcup_{n\geq 0} h^{n+1}(A) \subset K$. Since h° is a neighborhood of f, Corollary 2.10 implies the existence of a closed neighborhood g of f satisfying $g \subset h^\circ$. Therefore, h is a neighborhood of g, which, in turn, is a neighborhood of f. Corollary 7.10 implies that $\overline{K}$ is confining for g. Inclusion (7-1) implies that $K \subset V^\circ$. Since V is closed, $\overline{K} \subset V$. By definition of K, $A \subset K \subset \overline{K}$. Since $\overline{K}$ is a closed confining set for g, Theorems 5.10 and 5.12 imply that $\omega(A;g) \subset \omega(\overline{K};g) \subset \overline{K} \subset V$, and the proof is complete. □

Proof of Theorem 7.3. In view of Lemma 7.11, it suffices to assume that V is closed and that $\omega(V;f) = A$. Lemma 7.12 implies the existence of a closed neighborhood g of f such that $\omega(A;g) \subset V$. Let

$$B \equiv \omega(A;g) \subset V. \tag{7-2}$$

Since $\omega(A;g)$ is invariant under g, B is confining for g. Lemma 7.9 implies that B is an attractor block for f, while Theorem 5.9 implies that B is closed. Since A is invariant for f, Lemma 1.1 implies that $g(A) \supset f(A) = A$ and hence that A is backward complete for g. Statement (2) of Theorem 5.12 therefore implies that $\omega(A;g) \supset A$, which, combined with inclusion (7-2), yields

$$A \subset B \subset V.$$

Therefore, $A = \omega(A;f) \subset \omega(B;f) \subset \omega(V;f) = A$, and the proof is complete. □

The final lemma of this section is another version of the locality of the property of attraction.

Lemma 7.13. *If f is a closed relation on a compact Hausdorff space, if A is a closed invariant set for f, and if Y is a closed neighborhood of A satisfying*

$$\omega(Y; f|_Y) \subset A,$$

then A is an attractor for f.

Proof. Since A is invariant for f, Y is a neighborhood of $f(A)$, so Theorem 2.3 implies the existence of a neighborhood V of A such that $f(V) \subset Y$.

Since A is confining for f, Lemma 4.10 implies that $f|_Y(A) = f(A) = A$. Since A is closed in the ambient space, it is a closed subset of Y, so Theorems 5.12 and 5.10 imply that

$$A = \omega(A; f|_Y) \subset \omega(Y; f|_Y) \subset A.$$

Therefore, $\omega(Y; f|_Y) = A$, which, since Y is trivially a neighborhood of A in Y, implies that A is an attractor for $f|_Y$. Theorem 7.3 therefore implies the existence of a closed attractor block B for $f|_Y$ such that $B \subset V \cap Y$ and $\omega(B; f|_Y) = A$. Since $B \subset Y$ and since $f(B) \subset f(V) \subset Y$, Lemma 1.13 implies that

$$f(B) = f(B) \cap Y = f|_Y(B) \subset B,$$

and hence that B is confining for f. Since B is a closed subset of Y, which is itself a closed subset of the ambient space, B is closed. Theorem 5.15 therefore implies that

$$\omega(B; f) = \omega(B; f|_Y) = A.$$

Since B is a neighborhood of A in Y and since Y is a neighborhood of A in the ambient space, it follows that B is a neighborhood of A and hence that A is an attractor for f. □

Proof of Theorem 7.4. Lemma 7.1 implies that A and A' both are compact, that A' is invariant for $f|_A$, and that A is invariant for f. In particular, A is confining for f, so Lemma 4.11 implies that $f(A') = f|_A(A') = A'$, and hence that A' is a closed invariant set for f. In view of Lemma 7.13, the proof will be complete once it is shown that there exists a $Y \in \overline{\mathfrak{N}}(A'; X)$ such that

$$\omega(Y; f|_Y) \subset A'. \tag{7-3}$$

The definition of attractor implies the existence of a $U \in \mathfrak{N}(A; X)$ and a $V \in \mathfrak{N}(A'; A)$ such that

$$\omega(U; f) = A \quad \text{and} \quad \omega(V; f|_A) = A'.$$

Lemma 2.11 implies the existence of a $Y \in \overline{\mathfrak{N}}(A'; X)$ such that

$$Y \subset U, \tag{7-4}$$

$$Y' \equiv Y \cap A \subset V. \tag{7-5}$$

Theorem 5.14 and inclusion (7-4) imply that $\omega(Y; f|_Y) \subset \omega(Y; f) \subset A$, which, since $\omega(Y; f|_Y) \subset Y$, implies that $\omega(Y; f|_Y) \subset Y'$. Theorems 5.12 and 5.10 then imply that

$$\omega(Y; f|_Y) = \omega(\omega(Y; f|_Y); f|_Y) \subset \omega(Y'; f|_Y). \tag{7-6}$$

Lemma 1.15 implies that

$$(f|_Y)|_{Y'} = (f|_A)|_{Y'} = f|_{Y'}.$$

Since A is confining for f, Lemma 4.9 implies that Y' is confining for $f|_Y$. Theorems 5.15 and 5.14 therefore imply that

$$\omega(Y'; f|_Y) = \omega(Y'; f|_{Y'}) \subset \omega(Y'; f|_A),$$

which, together with inclusions (7-5) and (7-6), implies inclusion (7-3) and completes the proof. □

8. BASINS

The basin of an attractor is the set of all points attracted to it in the sense that the omega limit set of the point is a subset of the attractor.

Definition. If A is an attractor for a relation f, then its *basin* is the set

$$\mathrm{B}(A; f) \equiv \{x : \omega(x; f) \subset A\}.$$

It should be noted that, if $x \in X$ satisfies $f(x) = \emptyset$, then $\omega(x; f) = \emptyset$ and hence x is in the basin of every attractor.

The set $\mathrm{B}(A; f)$ is sometimes called the "basin of attraction" of A. The relation f is often understood from the context, in which case explicit reference to it in the notation will be deleted.

$$\mathrm{B}(A) \equiv \mathrm{B}(A; f).$$

The main results of this section are given in the following theorem. It states that the basin of an attractor A is an open confining set containing A. Furthermore, the basin has a peculiar invariance property stated precisely as property (4). As will be shown in the next section, this property is equivalent to the statement that the complement of the basin is $*$-invariant. The proof of the theorem will be given below after the development of some preliminary results.

Theorem 8.1. *If A is an attractor for a relation f on a compact Hausdorff space, then the basin of A satisfies the following properties.*

(1) $A \subset \mathrm{B}(A)$.
(2) $\mathrm{B}(A)$ *is open.*
(3) $\mathrm{B}(A)$ *is confining.*
(4) $f^{-1}(\mathrm{B}(A)) = \mathrm{B}(A)$.

The first lemma reveals a local uniformity for the number of iterations it takes to confine the image of a point to a neighborhood of the attractor.

Lemma 8.2. *If A is an attractor for f, if $x \in \mathrm{B}(A)$, and if U is a neighborhood of A, then there exist a nonnegative integer m and a neighborhood V of x such that $f^m(V) \subset U$.*

Proof. By definition, $\omega(x) \subset A$. Since U is a neighborhood of $\omega(x)$, Lemma 5.8 implies the existence of a nonnegative integer m such that $f^m(x) \subset U$. Since $\{x\}$ is compact, Theorem 2.3 implies the existence of a neighborhood V of x such that $f^m(V) \subset U$. □

There is not much difference between a point and a compact set. In particular, in the present setting, the omega limit set of any compact subset of the basin is a subset of the attractor. Recall that, even for flows, the statement is not true for noncompact sets.

Lemma 8.3. *If A is an attractor and if K is a compact subset of $\mathrm{B}(A)$, then $\omega(K) \subset A$.*

Proof. Let B be an attractor block associated with A. For each $x \in K$, Lemma 8.2 implies the existence of a nonnegative integer m_x and a neighborhood V_x of x such that $f^{m_x}(V_x) \subset B$. Since K is compact, there exists a finite set F such that $\bigcup\{V_x : x \in F\} \supset K$. Let $m = \max\{m_x : x \in F\}$.

Now let $y \in K$. Choose $x \in F$ such that $y \in V_x$. Since $f^{m_x}(y) \subset f^{m_x}(V_x) \subset B$ and since B is confining, $f^m(y) = f^{m-m_x}(f^{m_x}(y)) \subset f^{m-m_x}(B) \subset B$. Therefore, $f^m(K) \subset B$. Theorem 5.11 implies that $\omega(K) = \omega(f^m(K)) \subset \omega(B) = A$, and the proof is complete. □

The next lemma is an immediate consequence of Theorem 5.10 and the definition of basin. It states that a set whose omega limit set is a subset of A must lie entirely in the basin of A. The corollary states that, in particular, any attractor block associated with A must lie in the basin.

Lemma 8.4. *If A is an attractor and if S is a set satisfying $\omega(S) \subset A$, then $S \subset \mathrm{B}(A)$.*

Corollary 8.5. *If A is an attractor and if B is an attractor block associated with A, then $B \subset \mathrm{B}(A)$.*

Proof of Theorem 8.1. If $x \in A$, then Theorems 5.10 and 5.12 imply that $\omega(x) \subset \omega(A) = A$, which implies that $x \in \mathrm{B}(A)$ and establishes property (1).

Now let $x \in \mathrm{B}(A)$, and let U be a neighborhood of A satisfying $\omega(U) = A$. Lemma 8.2 implies the existence of a nonnegative integer m and a neighborhood V of x such that $f^m(V) \subset U$. Theorem 5.11 implies that $\omega(V) = \omega(f^m(V)) \subset \omega(U) = A$. Lemma 8.4 implies that $V \subset \mathrm{B}(A)$, which implies that $\mathrm{B}(A)$ is open and establishes property (2).

Again let $x \in \mathrm{B}(A)$ and assume that $x \mapsto y$. Statement (1-1) implies that $y \in f(x)$. Therefore, Theorems 5.10 and 5.11 imply that $\omega(y) \subset \omega(f(x)) = \omega(x) \subset A$, which implies that $y \in \mathrm{B}(A)$. Statement (1) of Lemma 4.1 implies that $\mathrm{B}(A)$ is confining, which establishes property (3).

Now let $x \in f^{-1}(\mathrm{B}(A))$. By definition, $f(x) \subset \mathrm{B}(A)$. Since $f(x)$ is compact, Lemma 8.3 implies that $\omega(f(x)) \subset A$. Therefore, $\omega(x) \subset A$, which implies that $x \in \mathrm{B}(A)$ and hence that

$$f^{-1}(\mathrm{B}(A)) \subset \mathrm{B}(A).$$

Property (3) and Lemma 4.2 imply that

$$\mathrm{B}(A) \subset f^{-1}(\mathrm{B}(A)),$$

which establishes property (4) and completes the proof. □

9. Repellers

For flows on compact metric spaces, Conley introduced the notion of a repeller dual to a given attractor [6]. A repeller is simply an attractor for the flow followed backward in time, and Conley showed that the complement of the basin of an attractor is a repeller. These results have direct analogues for relations.

Definition. A *repeller* for f is a set R such that R is an attractor for f^*.

Definition. If R is a repeller for f, then the *basin* of R is the set

$$\mathrm{B}^*(R; f) \equiv \{x : \omega^*(x; f) \subset R\}.$$

The following lemma states that the basin $\mathrm{B}^*(R; f)$ of a repeller R for a relation f is the same as the basin $\mathrm{B}(R; f^*)$ of R when R is viewed as an attractor for f^*. One is tempted to call $\mathrm{B}^*(R; f)$ the "basin of repulsion," but this name has connotations of a value judgement inappropriate in the present context.

Lemma 9.1. *If A is an attractor for f and if R is a repeller for f, then the following properties hold.*

(1) $\mathrm{B}^*(R; f) = \mathrm{B}(R; f^*)$.
(2) $\mathrm{B}(A; f) = \mathrm{B}^*(A; f^*)$.

Proof. The definition of the alpha limit set implies that

$$\mathrm{B}^*(R; f) = \{x : \omega^*(x; f) \subset R\} = \{x : \omega(x; f^*) \subset R\} = \mathrm{B}(R; f^*),$$

which establishes property (1). Property (2) follows from property (1) and from the fact that $(f^*)^* = f$. □

As usual, if the relation f is understood from the context, its explicit reference will be deleted. Furthermore, it is often clear that the set R is a repeller, in which case it is convenient to use the same notation for the basin of a repeller and the basin of an attractor.

$$\mathrm{B}(R) = \mathrm{B}^*(R) = \mathrm{B}^*(R; f).$$

Definition. If A is an attractor for f, then the *dual repeller* of A for the relation f is the set

$$\mathrm{D}(A; f) \equiv \mathrm{B}(A; f)^c.$$

Definition. If R is a repeller for f, then the *dual attractor* of R for the relation f is the set

$$\mathrm{D}^*(R; f) \equiv \mathrm{B}^*(R; f)^c.$$

At this point in the discussion, it is not clear that the dual repeller is a repeller or that the dual attractor is an attractor. Theorems 9.3 and 9.6 below will show that this terminology is indeed appropriate, but first some abbreviated notation will be introduced.

The following lemma is an immediate consequence of Lemma 9.1. It states that the dual repeller of an attractor A is identical to the dual attractor of A when A is view as a repeller for the transpose relation.

Lemma 9.2. *If A is an attractor for f and if R is a repeller for f, then the following properties hold.*

(1) $\mathrm{D}^*(R; f) = \mathrm{D}(R; f^*)$.
(2) $\mathrm{D}(A; f) = \mathrm{D}^*(A; f^*)$.

Again, the relation f and the identity of the set in question as an attractor or a repeller are often understood from the context. In such cases, the notation will become

$$A^* \equiv \mathrm{D}(A; f),$$
$$R^* \equiv \mathrm{D}^*(R; f).$$

The following theorem is the main result of this section. Not only does it justify the choice of the term "repeller" in the name "dual repeller," but it also establishes that the basin of the dual repeller is the complement of the attractor.

Theorem 9.3. *If A is an attractor for a closed relation on a compact Hausdorff space, then A^* is a repeller, and* $\mathrm{B}(A^*) = A^c$.

The proof is based on the next two lemmas and will be given after the proofs of the lemmas. The first lemma states that the dual repeller is a closed $*$-invariant set.

Lemma 9.4. *If A is an attractor, then A^* is closed and $*$-invariant.*

Proof. Theorem 8.1 implies that $\mathrm{B}(A)$ is open, which implies that A^* is closed. Lemma 1.9 and property (4) of Theorem 8.1 imply that $f^*(A^*) = f^*(\mathrm{B}(A)^c) = f^{-1}(\mathrm{B}(A))^c = \mathrm{B}(A)^c = A^*$. □

It was shown in Corollary 7.8 that the complement of an attractor block B for a relation is an attractor block for the transpose relation. If A is the attractor associated with B, then not only is the dual repeller of A an attractor for the transpose, but it is the particular attractor associated with the attractor block B^c.

Lemma 9.5. *If A is an attractor for f and if B is an attractor block associated with A, then B^c is an attractor block for f^*, and $\omega^*(B^c) = A^*$.*

Proof. Corollary 7.8 and Theorem 7.2 imply that B^c is an attractor block for f^* with associated attractor $C \equiv \omega(B^c; f^*) = \omega^*(B^c; f) = \omega^*(B^c)$.

Let $x \in C$. Since C is a compact invariant set for f^*, it is $*$-invariant for f, hence forward complete for f. Since C is closed and forward complete, Lemma 6.5 implies that $\omega(x) \cap C \neq \emptyset$. Corollary 7.5 implies that $A \subset B$ and $C \subset B^c$. Therefore, $C \subset A^c$, which implies that $\omega(x) \cap A^c \neq \emptyset$ and hence that $\omega(x) \not\subset A$. It follows that $x \notin \mathrm{B}(A)$ and hence that $x \in \mathrm{B}(A)^c = A^*$. Therefore,

$$\omega^*(B^c) \subset A^*.$$

Corollary 8.5 implies that $B \subset \mathrm{B}(A)$, which implies that $A^* = \mathrm{B}(A)^c \subset B^c$. Statement (3) of Theorem 5.12 and Lemma 9.4 imply that $A^* = \omega(A^*; f^*) \subset \omega(B^c; f^*) = \omega^*(B^c)$. Therefore,

$$A^* \subset \omega^*(B^c),$$

and the proof is complete. □

Proof of Theorem 9.3. Lemma 9.5 and Theorem 7.2 imply that A^* is an attractor for f^*, which implies that A^* is a repeller for f.

Let $x \in A^c$. Since $\{x\}^c$ is a neighborhood of A, Theorem 7.3 implies the existence of an attractor block B associated with A such that $x \notin B$. Lemma 9.5 implies that B^c is an attractor block for f^* with associated attractor A^*. Corollary 8.5 implies that $B^c \subset \mathrm{B}(A^*)$, which, since $x \in B^c$, implies that $x \in \mathrm{B}(A^*)$. Therefore,

$$A^c \subset \mathrm{B}(A^*).$$

Now let $x \in A$. Lemma 7.1 implies that A is a compact invariant set for f, hence it is a closed set which is forward complete for f^*. Lemma 6.5 therefore implies that $\omega(x; f^*) \cap A \neq \emptyset$, which implies that $\omega^*(x) \cap A \neq \emptyset$, which implies that $\omega^*(x) \not\subset A^c$. Since $A \subset \mathrm{B}(A)$, it follows that $A^* \subset A^c$, which implies that $\omega^*(x) \not\subset A^*$. Therefore, $x \notin \mathrm{B}(A^*)$, which implies that

$$A \subset \mathrm{B}(A^*)^c$$

and completes the proof. □

The next theorem is the analogue of Theorem 9.3 for repellers. It shows the symmetry between attractors and repellers.

Theorem 9.6. *If R is a repeller for a closed relation on a compact Hausdorff space, then R^* is an attractor, and* $\mathrm{B}(R^*) = R^c$.

Proof. If R is a repeller for f, then R is an attractor for f^*, so Theorem 9.3 implies that $S \equiv \mathrm{D}(R; f^*)$ is a repeller for f^* and that $\mathrm{B}^*(S; f^*) = R^c$. Lemma 9.2 implies that $S = \mathrm{D}^*(R; f) = R^*$, while Lemma 9.1 implies that $\mathrm{B}(S; f) = \mathrm{B}^*(S; f^*) = R^c$. Therefore, $R^* = S$ is an attractor for f, and $\mathrm{B}(R^*) = R^c$. □

The next theorem justifies the term "dual" in the names "dual attractor" and "dual repeller."

Theorem 9.7. *If A is an attractor and R is a repeller for a closed relation on a compact Hausdorff space, then $(A^*)^* = A$ and $(R^*)^* = R$.*

Proof. Let A be an attractor for f. Theorem 9.3 implies that A^* is a repeller for f and that $\mathrm{B}^*(A^*; f) = A^c$. Therefore, $A = \mathrm{B}^*(A^*; f)^c = \mathrm{D}^*(A^*; f) = (A^*)^*$.

Now let R be a repeller for f. Theorem 9.6 implies that R^* is an attractor for f and that $\mathrm{B}(R^*; f) = R^c$. Therefore, $R = \mathrm{B}(R^*; f)^c = \mathrm{D}(R^*; f) = (R^*)^*$. □

Note that the entire space X is always an attractor block for a relation f. Therefore, $\omega(X)$ is always an attractor. Its basin is the entire space, and its dual repeller is the empty set. Similarly, $\omega^*(X)$ is always a repeller, and its dual attractor is the empty set.

This section ends with four examples. The first three examples are actually maps on a larger space. However, when restricted to the space in question, they fail to be defined everywhere. Nonetheless, they are relations, and they illustrate some of the ideas in this section. The fourth example is a simple illustration of the symmetry between attractors and repellers. It is, in some sense, the canonical example of an attractor and its dual repeller.

Example 9.8. Let X be the real interval $[0,1]$, let $c > 0$, and let the relation f on X be defined as

$$f = \{(x,y) \in X \times X : y = x + c\}.$$

Note that $f^n(X) = [nc, nc+1] \cap [0,1]$ and hence that $f^n(X) = \emptyset$ for $n > 1/c$. Therefore, $\omega(X) = \emptyset$, so $\emptyset$ is the attractor dual to the repeller $\emptyset$.

Example 9.9. Let X be the real interval $[-1,1]$, let $\alpha > 1$, and let the relation f on X be defined as

$$f = \{(x,y) \in X \times X : y = \alpha x\}.$$

Note that $f(X) = X$ and hence that $\omega(X) = X$. Therefore, X is the attractor dual to the repeller $\emptyset$.

Note also that $(f^*)^n(X) = [-\alpha^{-n}, \alpha^{-n}]$ and hence that $\omega^*(X) = \{0\}$. Therefore, $\{0\}$ is the repeller dual to the attractor $\emptyset$.

Example 9.10. Let $X = [-1,1] \times [-1,1]$, let $0 < \beta < 1 < \alpha$, and let

$$f = \{(x,y) \in X \times X : y = Ax\},$$

where

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$

Note that $f^n(X) = \{(x_1,x_2) \in X : |x_2| \leq \beta^n\}$ and hence that $\omega(X) = \{(x_1,x_2) \in X : x_2 = 0\}$. Therefore, the horizontal axis A is the attractor dual to the repeller $\emptyset$. Similarly, the vertical axis R is the repeller dual to the attractor $\emptyset$.

Note also that A is invariant and rejecting, but not forward complete, and that R is $*$-invariant and confining, but not backward complete.

The previous example illustrates a more general phenomena. If one considers the relation defined as the restriction of a diffeomorphism to a neighborhood of a saddle point, then the local unstable manifold of the saddle is the attractor dual to $\emptyset$, while the stable manifold is the repeller dual to $\emptyset$.

Example 9.11. Let $X = [0,1]$, and let

$$f = \{(x,y) \in X \times X : y = 0 \text{ or } x = 1\}.$$

Then $\{0\}$ is an attractor with dual repeller $\{1\}$. Note that $\{0\}$ is invariant and forward complete, but not rejecting, and that $\{1\}$ is $*$-invariant and backward complete, but not confining.

10. Connecting Orbits

The notion that the set of orbits connecting an attractor with its dual repeller is a useful object of study was introduced by Conley [7]. This notion gave rise the the theory of connection matrices, which has found some interesting applications. (See the survey by Moeckel [18].) In this section it will be shown that the following basic property of attractors holds in the setting of relations on compact Hausdorff spaces: if an orbit contains a point which is neither in an attractor A nor in its dual repeller A^*, then the orbit must move from A^* to A,

Once again, throughout this section, f will denote a closed relation on a compact Hausdorff space X.

Definition. If A is an attractor for f, then the *set of connecting orbits* associated with A is given by

$$\mathrm{C}(A) \equiv (A \cup A^*)^c.$$

By definition, $A^* = \mathrm{B}(A)^c$, and, by Theorem 9.3, $A^c = \mathrm{B}(A^*)$. Therefore,

$$\mathrm{C}(A) = A^c \cap (A^*)^c = \mathrm{B}(A^*) \cap \mathrm{B}(A). \tag{10-1}$$

Theorem 8.1 implies that $A \subset \mathrm{B}(A) = (A^*)^c$ and hence that $A^* \subset A^c$. Therefore, the first equality of (10-1) implies that

$$A \cup \mathrm{C}(A) = A \cup (A^*)^c = (A^*)^c \tag{10-2}$$

and that

$$A^* \cup \mathrm{C}(A) = A^* \cup A^c = A^c. \tag{10-3}$$

Among other things, it has just been established that the space X breaks into the disjoint union of A, A^*, and $\mathrm{C}(A)$.

The following theorem justifies the terminology "connecting orbits." It states that any orbit intersecting $\mathrm{C}(A)$ moves from A^* to A.

Theorem 10.1. *If A is an attractor and if (p, I) is an orbit for a closed relation on a compact Hausdorff space, and if $p_I \cap \mathrm{C}(A) \neq \emptyset$, then $\omega((p, I)) \subset A$ and $\omega^*((p, I)) \subset A^*$.*

Proof. Choose $i \in I$ such that $p_i \in \mathrm{C}(A)$. Equality (10-1) implies that $p_i \in \mathrm{B}(A) \cap \mathrm{B}(A^*)$, which implies that $\omega(p_i) \subset A$ and $\omega^*(p_i) \subset A^*$. Theorem 6.2 implies that $\omega((p, I)) \subset \omega(p_i)$ and that $\omega^*((p, I)) \subset \omega^*(p_i)$, which, together with the previous two inclusions, imply that $\omega((p, I)) \subset A$ and that $\omega^*((p, I)) \subset A^*$. □

In the general setting of relations, one must allow for a possibility which cannot occur for maps. Since there may exist points $x \in A^*$ and $y \in A$ such that $x \mapsto y$, it is possible for an orbit to move from A^* to A without ever intersecting $\mathrm{C}(A)$. However, since A is invariant and since A^* is $*$-invariant, a jump from A^* to A can occur at most once. Thus an orbit which is not contained entirely in A or entirely in A^* has no choice but to move from A^* to A. These remarks are made precise in the theorem following the next lemma.

Lemma 10.2. *If A is an attractor for f and if (p, I) is an orbit for f, then the following implications hold.*

(1) $p_I \cap A \neq \emptyset \implies \omega((p, I)) \subset A$.
(2) $p_I \cap A^* \neq \emptyset \implies \omega^*((p, I)) \subset A^*$.

Proof. Suppose there exists an $i \in I$ such that $p_i \in A$. Theorems 6.2 and 5.12 imply that $\omega((p, I)) \subset \omega(p_i) \subset \omega(A) = A$, which establishes implication (1). Implication (2) follows by application of implication (1) to A^*, considered as an attractor for f^*. □

Theorem 10.3. *If A is an attractor and if (p, I) is an orbit for a closed relation on a compact Hausdorff space, then one of the following statements is true.*

(1) $p_I \subset A$.
(2) $p_I \subset A^*$.
(3) $\omega((p, I)) \subset A$ *and* $\omega^*((p, I)) \subset A^*$.

Proof. Let (p, I) be an orbit for the relation. There are only two possibilities. Either $p_I \cap \mathrm{C}(A) \neq \emptyset$, or $p_I \cap \mathrm{C}(A) = \emptyset$. In the first case, Theorem 10.1 implies that statement (3) holds, and the proof is complete. Therefore, it suffices to establish the result under the assumption that $p_I \cap \mathrm{C}(A) = \emptyset$.

Assume that both statements (1) and (2) are false. It will be shown that statement (3) must be true. Since statement (1) is false, it follows that $p_I \cap A^c \neq \emptyset$. Equality (10-3) therefore implies that $(p_I \cap A^*) \cup (p_I \cap \mathrm{C}(A)) \neq \emptyset$. Since it is assumed that $p_I \cap \mathrm{C}(A) = \emptyset$, it follows that $p_I \cap A^* \neq \emptyset$, which, by Lemma 10.2, implies that $\omega^*((p, I)) \subset A^*$. A similar argument using equality (10-2) implies that $\omega((p, I)) \subset A$. Therefore, statement (3) holds, and the proof is complete. □

11. Iterated Function Systems and Julia Sets

Barnsley has studied fractals generated by sets of contraction mappings [2]. The first goal of this section is to show that Barnsley's definition of attractor and the one given in this paper are identical whenever they both are defined.

The following definition of an "Iterated Function System" is paraphrased from that given in Barnsley's book [2].

Definition. If X is a metric space, then an *Iterated Function System* on X is a finite set $\{w_1, w_2, \ldots, w_N\}$ of contraction mappings on X.

An Iterated Function System $W \equiv \{w_1, w_2, \ldots, w_N\}$ on X determines a relation f on X in the obvious way:

$$f_W = \{(x, y) \in X \times X : y = w_i(x) \text{ for some } i = 1, \ldots, N\}.$$

The "Random Iteration Algorithm" [2] produces a sequence x_n, $n = 0, 1, 2, \ldots$, satisfying

$$x_n \in \{w_1(x_{n-1}), w_2(x_{n-1}), \ldots, w_N(x_{n-1})\}, \quad n = 1, 2, \ldots.$$

Note that $(x, [0, \infty))$ satisfies the definition of orbit given above in Section 4. That is, a sequence produced by the Random Iteration Algorithm is an orbit for the associated relation.

Hutchinson proved the following theorem [12].

Theorem 11.1. *If X is a nonempty complete metric space and if W is a nonempty finite set of contraction mappings on X, then there is a unique nonempty closed bounded set A such that $f_W(A) = A$.*

Barnsley calls the set A "the attractor" for the Iterated Function System. The following theorem shows that Barnsley's notion of attractor coincides with the definition given above in Section 7.

Theorem 11.2. *If X is a nonempty compact metric space and if W is a nonempty finite set of contraction mappings on X, then A is a nonempty attractor for f_W if and only if A is a nonempty compact invariant set for f_W.*

Proof. If A is an attractor for f_W, then Lemma 7.1 implies that A is a compact invariant set.

Now assume that A is a nonempty compact invariant set. Since X is a closed attractor block for f_W, $\omega(X)$ is an attractor for f_W. Theorem 5.9 implies that $\omega(X)$ is a compact invariant set. Note that, for every $x \in X$, $x \mapsto w(x)$, for each $w \in W$. Therefore, Lemma 4.1 implies that X is forward complete, and Corollary 6.6 implies that $\omega(X) \neq \emptyset$. Since X is a compact metric space, both A and $\omega(X)$ are nonempty closed bounded invariant sets. Hutchinson's theorem implies that $\omega(X) = A$ and hence that A is an attractor. □

Note that an Iterated Function System has only one nonempty attractor, the attractor dual to the repeller $\emptyset$.

Example 11.3. Let X be the real interval $[0,1]$, and let f be the relation on X defined by

$$f = \{ (x,y) \in X \times X : 3y = x \text{ or } 3y = x + 2 \}.$$

This relation corresponds to the Iterated Function System $\{w_1, w_2\}$, where

$$w_1(x) = \tfrac{1}{3}x,$$
$$w_2(x) = \tfrac{1}{3}x + \tfrac{2}{3}.$$

The nonempty attractor is the standard middle third Cantor set.

This paper ends with a short discussion of how Julia sets fit into the framework presented here. A rigorous introduction to Julia sets and Mandelbrot sets was written by Blanchard [3], where the reader can find proofs of the results implied in the following discussion.

Let X be the Riemann sphere and consider the standard quadratic map $Q_c(z) = z^2 + c$, where c is a complex number. Let

$$f_c = \{ (z,w) \in X \times X : w = Q_c(z) \}.$$

The point at infinity is an attractor A_∞, with dual repeller the so-called "filled Julia set."

For values of c in the complement of the Mandelbrot set, A_∞ is the only nontrivial attractor. In this case, A^*_∞ is the Julia set.

For many values of c in the Mandelbrot set, there is another attractor A_c, which is either a fixed point or a periodic orbit. Its dual repeller is the union of the Julia set with the basin of attraction of ∞. For these values of c, one can form another attractor by taking $A \equiv A_\infty \cup A_c$. The Julia set is the repeller dual to A.

Note that the Julia set is an attractor for ${f_c}^*$. This fact is the basis for one of the early algorithms for using a computer to visualize the Julia set [3].

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School of Mathematics
University of Minnesota
Minneapolis, MN 55455

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