A New Proof of the Stable Manifold Theorem

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Abstract

We give a new proof of the stable manifold theorem for hyperbolic fixed points of smooth maps. This proof shows that the local stable and unstable manifolds are projections of a relation obtained as a limit of the graphs of the iterates of the map. The same proof generalizes to the setting of stable and unstable manifolds for smooth relations.

1 Introduction

The stable manifold theorem states that for a smooth map, near a hyperbolic fixed point, the stable manifold, points whose forward orbit converges to the fixed point, and the unstable manifold, points with backward orbit converging to the fixed point, are both smooth manifolds. This paper presents a new proof of the stable manifold theorem. The theorem is proved in the context of hyperbolic fixed points of "smooth relations" [1], [4], a generalization which includes as special cases hyperbolic fixed points of both invertible [3], [5] and noninvertible [6] maps. However, this is not merely a generalization of the standard theorem. The new approach restores to the noninvertible case the symmetry between the stable and unstable manifolds as is seen in the diffeomorphism case. In addition, it provides a new geometric way of looking at the local stable and unstable manifolds of a map; namely, they are both projections of an object one can think of as the "infinite iterate" of the graph of the map.

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The key to this new proof is that rather than looking at stable and unstable manifolds as subsets of the state space, we view them as projections of a smooth manifold in higher dimensions arising from the graph of the original map. More precisely, near a hyperbolic fixed point, the graph of a map and the graphs of its iterates can be expressed in an appropriate coordinate system as graphs of smooth contractions. The limit of these contractions exists and is smooth. The graph of this limit projects to the stable and unstable manifolds.

The derivative of a smooth map on R^n at a hyperbolic fixed point has no eigenvalues on the unit circle. Thus locally, in coordinates given by the stable and unstable directions X and Y, a map can be expressed as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Ax + g_1(x,y) \\ By + g_2(x,y) \end{pmatrix} \tag{1}$$

where x and y are vectors in X and Y , A and B matrices with $|A| < 1, |B^{-1}| < 1$, and g_1 and g_2 are Lipschitz with small Lipschitz constant.

By the Implicit Function Theorem, we can locally change to a skewed coordinate system such that in these new coordinates, we have a local contraction. Namely, we can write:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} Ax + \hat{g}_1(x,y') \\ B^{-1}y' + \hat{g}_2(x,y') \end{pmatrix} \tag{2}$$

$\hat{g}_1$ and $\hat{g}_2$ are again Lipschitz with small Lipschitz constant.

The proof presented here capitalizes on the fact that the map and all its iterates are local contractions when written in this skewed coordinate system. Before presenting the proof, we illustrate the ideas with some simple examples.

2 Some Simple Examples

1. Consider the graph of the following linear diffeomorphism on R^2 with hyperbolic fixed point $(0,0)$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{for } 0 < a < 1 < b \tag{3}$$

Since the x-axis and the y-axis are respectively the one-dimensional stable and unstable directions, we choose them to be the directions X and Y respectively in the skewed coordinates. Call the new function resulting from writing f in skewed coordinates ϕ_1. It is written as follows:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \begin{pmatrix} x \\ y' \end{pmatrix} \tag{4}$$

The k^{th} iterate of the original map is

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

Writing the k^{th} iterate in skewed coordinates, gives the following function ϕ_k. Note that ϕ_k is found by looking at f^k and *not* by iterating ϕ_1.

$$\begin{pmatrix} x_k \\ y \end{pmatrix} = \begin{pmatrix} a^k & 0 \\ 0 & \frac{1}{b^k} \end{pmatrix} \begin{pmatrix} x \\ y_k \end{pmatrix} \tag{6}$$

Consider the limit of the ϕ_k; it exists and is equal to the map which is identically zero; explicitly, $\lim_{k\to\infty} \phi_k$ is the following map in skewed coordinates on R^2:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y' \end{pmatrix} \tag{7}$$

Notice that this limit map in the skewed coordinate system does not correspond to a function in the original coordinates. However, we can gain information about the stable and unstable manifolds from its graph. Namely, the projection of the graph to the xy- plane is the x-axis, the stable manifold. The projection of the graph to the $x'y'$- plane is y'-axis, the unstable manifold.

2. The trick in Example 1 still works if the linear map is noninvertible; i.e. if $a = 0$. The map becomes:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ by \end{pmatrix}, \qquad \text{for } 1 < b \tag{8}$$

which can still be expressed in the same skewed coordinates as before:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{b}y' \end{pmatrix} \tag{9}$$

The limit of the ϕ_k, the k^{th} iterate written in skewed coordinates, is the same as before. Indeed, the stable and unstable manifolds are once again the x-axis and y-axis respectively.

3. If we allow the stretching term b in Example 1 to increase without bound, the graph of the map converges to $\{(u, 0, au, v) : (u, v) \in R^2\}$. This is no longer the graph of a function but is only a relation;

Definition 1 (Relation) *A relation on a space Z is a subset of $Z \times Z$. Viewing this in terms of iteration, an iterate of z under relation F is a point z' such that $(z, z') \in F$. Notice that iterates of a point are not necessarily unique; nor do iterates necessarily exist.*

The relation in this example is a two-dimensional plane which is a subset of R^4 with second coordinate always equal to 0. A point $(x, y) \in R^2$ has no iterates unless $y = 0$. A point $(x, 0)$ has as iterates every point of the form $(ax, y'),\ y' \in R$. Thus the origin is still a "fixed" point under iteration. Since points on the x-axis have k^{th} iterates of the form $(a^k x, 0)$, which converge to the origin, the x-axis is in (and in fact equal to) the stable manifold. Likewise, every point on the y-axis is an iterate of the origin. Thus the y-axis is contained in (and in fact equal to) the unstable manifold.

We can also use the technique in Examples 1 and 2 to see this; although there is no longer a map, limit of b increasing without bound corresponds to $b = \infty$; i.e. $\frac{1}{b} = 0$. Thus although our example is no longer a map, it is the graph of a function in skewed coordinates:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} ax \\ 0 \end{pmatrix} \tag{10}$$

In this case, as in Examples 1 and 2, the limit of the iterates as expressed in skewed coordinates exists and is equal to the zero function. Again the projections of the graph of this zero function are the stable and unstable manifolds.

4. Here is a contrived quadratic example to illustrate the same idea in a nonlinear case. Note that the map f on R^2 has a hyperbolic fixed point (0,0):

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax \\ b(y + cx^2) \end{pmatrix}, \qquad \text{for } 0 < a < 1 < b \tag{11}$$

Since the axes are again the stable and unstable directions, we choose the axes for the skewed coordinate directions as before. The map represented in the skewed coordinate system gives the following function ϕ_1:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} ax \\ \frac{1}{b}y' - cx^2 \end{pmatrix} \tag{12}$$

Figure 1 shows the graph of ϕ_1 with domain $[-.3, .3] \times [-.3, .3]$. By the fact that ϕ_1 is a contraction, this figure is the same as the graph of f with both domain and range restricted to $[-.3, .3] \times [-.3, .3]$. Since the graph of

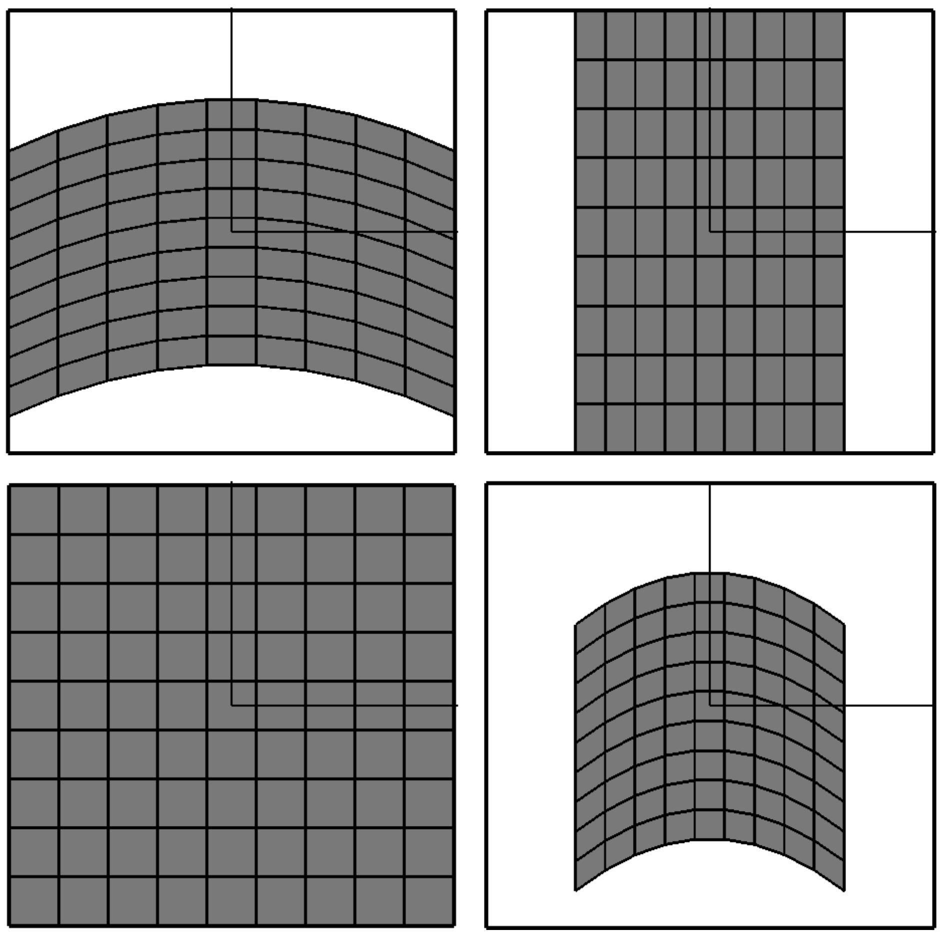

Figure 1: Projections of the graph of ϕ_1 resulting from the map in Example 4: Domain and range are $[-.3, .3] \times [-.3, .3]$, $a = .7$, $b = 1.43$ $c = 1$. Top left is the xy- plane, top right the $x'y'$- plane, bottom left the xy'- plane, bottom right the $x'y$- plane.

a map from R^2 to R^2 is in R^4, the figure consists of projections of the graph to coordinate planes. The projections have the following relationship to the maps f and ϕ_1: f maps the region in the xy- plane to the region in the $x'y'$-plane. ϕ_1 maps the region in the xy'- plane to the region in the $x'y$- plane.

The k^{th} iterate f^k is:

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} a^k x \\ b^k(y + c(\frac{1-\mu^k}{1-\mu})x^2) \end{pmatrix}, \qquad \text{where } \mu = \frac{a^2}{b} \tag{13}$$

Represented in skewed coordinates, it gives the following function ϕ_k:

$$\begin{pmatrix} x_k \\ y \end{pmatrix} = \begin{pmatrix} a^k x \\ \frac{1}{b^k} y_k - c(\frac{1-\mu^k}{1-\mu})x^2 \end{pmatrix}, \qquad \text{where } \mu = \frac{a^2}{b} \tag{14}$$

Figure 2 shows the graph of ϕ_{20} for the same domain and constants as in Figure 1. Again, f^{20} maps the region in the xy- plane to the region in the $x'y'$- plane; ϕ_{20} maps the region in the xy'- plane to the region in the $x'y$- plane.

The limit $\lim_{k\to\infty} \phi_k$ exists. It is given by:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{c}{1-\frac{a^2}{b}} x^2 \end{pmatrix}, \tag{15}$$

As in the previous examples, the projections of the limit map to the xy- and $x'y'$- planes are respectively the local stable and unstable manifolds for f.

Since the convergence to the limit function is exponentially fast, the graph of ϕ_{20} in Figure 2 is visually indistinguishable from the graph of $\lim_{k\to\infty} \phi_k$. This is why three of the projections appear to be curves. However, the graphs of both ϕ_{20} and the limit function are two-dimensional surfaces in R^4. To emphasize this point, Figure 3 shows projections of the same surface after it has been rotated in R^4.[2]

We now show that the result from the above examples generalizes to a certain class of relations. In Section 3 we give basic definitions for the dynamics of relations and state the stable manifold theorem in this general setting. In Section 4, we outline the proof of the stable manifold theorem. Finally, in Section 5 we give the full details of the proof outlined in the previous section.

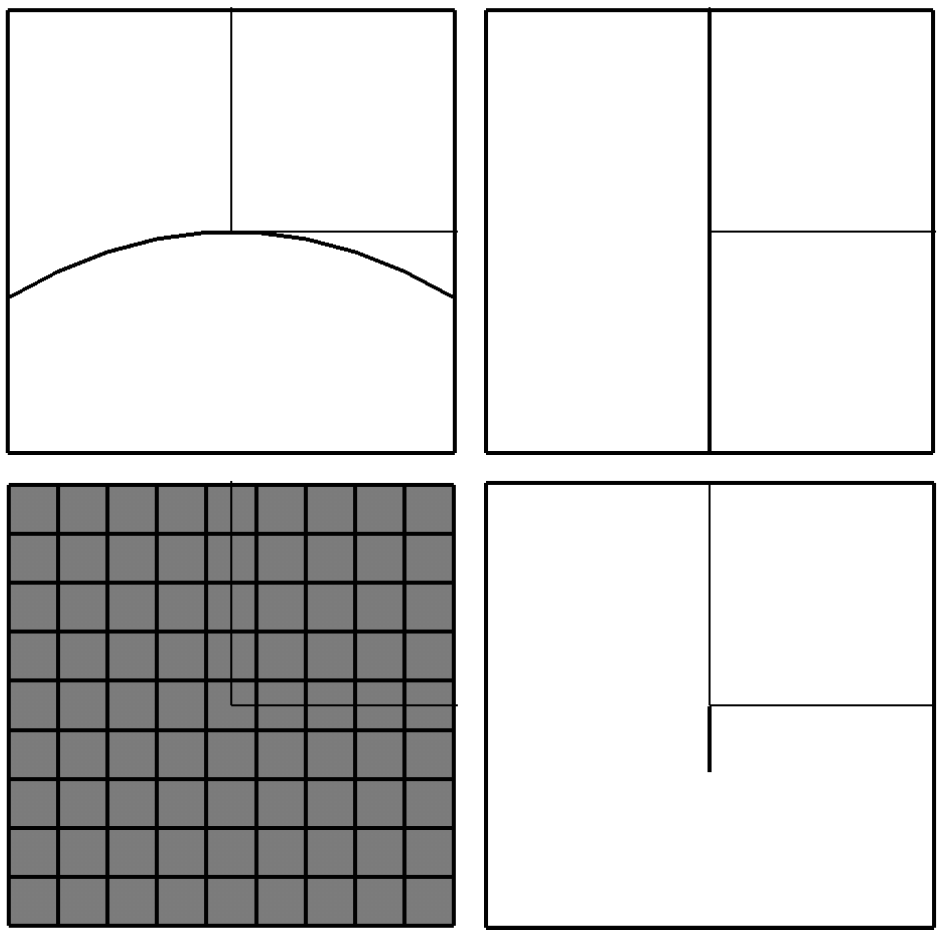

Figure 2: Same projections as figure 1, this time of ϕ_{20}, the skewed function of the twentieth iterate of f. This is very close to the limit case. Although three of the projections look like curves, they are actually projections of a surface. See Figure 3.

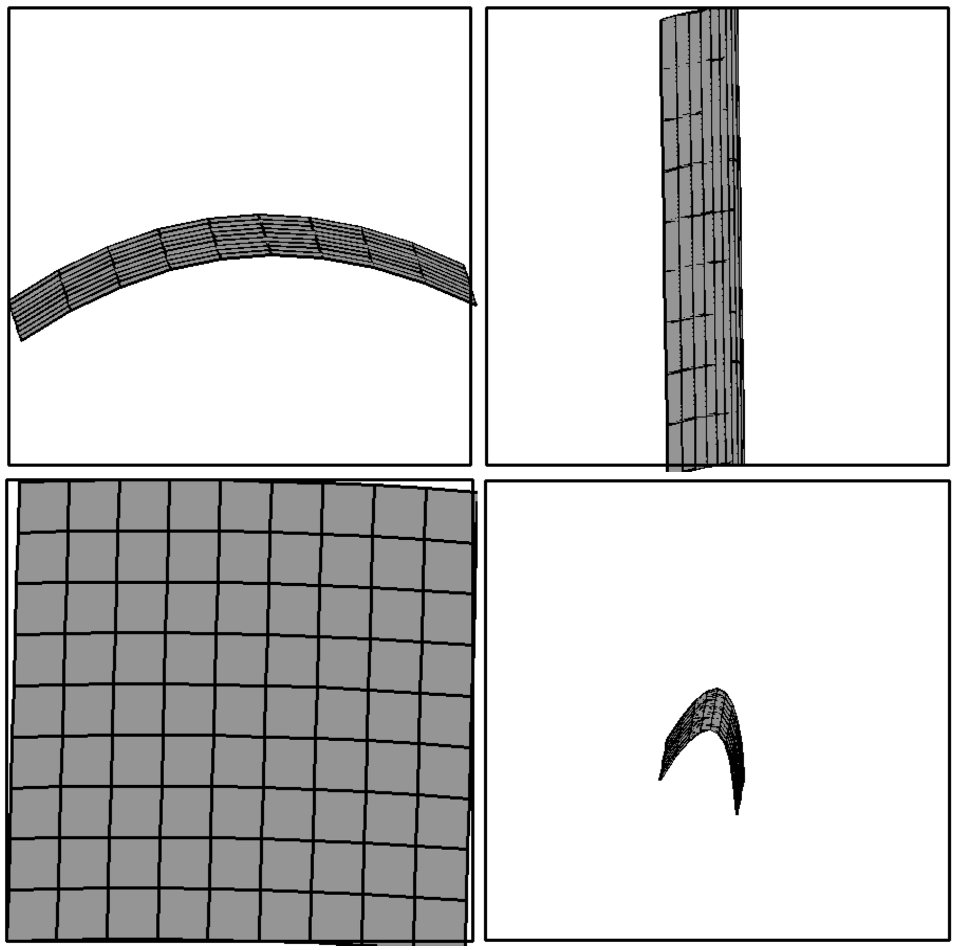

Figure 3: The graph of ϕ_{20} shown in Figure 2 after it has been rotated slightly in R^4. Same projections as before. This figure illustrates that although three of the four projections in Figure 2 appear to be curves, the graph is actually a surface in R^4.

3 Basic Definitions

In the previous section, relations on Z were defined as subsets of $Z \times Z$ and were viewed in terms of iteration. Here are some definitions in this context. We denote z having an iterate z' under relation f by $z \overset{f}{\mapsto} z'$.

Definition 2 (Fixed Point) *Given a relation f on set Z, $z \in Z$ is a fixed point of f if $(z, z) \in f$.*

Definition 3 (Composition for Relations) *Given relations g and h on set Z, $h \circ g$ is the relation given by*

$$\{(z, z'') : \exists\ z' \in Z, (z, z') \in g \text{ and } (z', z'') \in h\} \tag{16}$$

Notation: If I is an interval of integers and $z_k \in Z$ for all $k \in I$ is a sequence of points in Z, then we denote

$$\{z_k\}_{k \in I} = \left\{ \begin{array}{c} (z_i, z_{i+1}, ..., z_j), \text{ if } I = [i, j] \\ (...z_{j-1}, z_j), \text{ if } I = (-\infty, j] \\ (z_i, z_{i+1}, ...), \text{ if } I = [i, \infty) \end{array} \right\} \tag{17}$$

Definition 4 (Orbits for Relations) *Given relation f on space Z, an orbit through z is a sequence $\{z_k\}_{k \in I}$ such that $z = z_i$ for some $i \in I$, and $(z_k, z_{k+1}) \in f$ whenever $k, k+1 \in I$. If $I = [i, \infty)$ then $\{z_k\}$ is called an infinite forward orbit. If $I = (-\infty, i]$ then $\{z_k\}$ is called an infinite backward orbit.*

Definition 5 (Stable and Unstable Manifolds) *For a relation f on metric space Z with fixed point z_o, the stable and unstable manifolds $W^s(z_o)$ and $W^u(z_o)$ are defined by:*

$W^s(z_o) = \{z \in Z :$ there exists an infinite forward orbit $\{z_k\}$ through z such that $z_k \to z_o$ as $k \to \infty\}$.

$W^s(z_o) = \{z \in Z :$ there exists an infinite backward orbit $\{z_k\}$ through z such that $z_k \to z_o$ as $k \to -\infty\}$.

Definition 6 (C^r Relations) *If f is a relation on a smooth manifold Z, then f is C^r when it is a C^r embedded submanifold of $Z \times Z$.*

Definition 7 (Linear Relations) *If f is a relation in a vector space Z, then f is a linear relation if it is a linear subspace of $Z \times Z$.*

Definition 8 (Hyperbolic Linear Relations) *If f is an n-dimensional linear relation on an n-dimensional vector space Z, then f is hyperbolic when there is a splitting $Z = E^s \times E^u$ such that under this splitting, f is of the form*

$$\left\{ \begin{pmatrix} x \\ by' \\ ax \\ y' \end{pmatrix} : x \in E^s, y' \in E^u \right\} \tag{18}$$

where a and b are matrices, and $|a|, |b| < 1$.

Note that the graph of any hyperbolic linear map is a hyperbolic linear relation. See Example 1 for the case of a saddle in R^2.

Definition 9 (C^r Hyperbolic Relations) *A C^r relation f on a smooth manifold Z has a hyperbolic fixed point z_o when $T_{(z_o,z_o)}f$, its tangent plane at (z_o, z_o), is a hyperbolic linear relation.*

Note that the graph of a map with hyperbolic fixed point z_o is a relation which has hyperbolic fixed point z_o.

We can now state the main theorem of the paper.

Theorem 1 (Stable Manifold Theorem for Relations) *If f is a C^r relation on R^n, and f has hyperbolic fixed point z_o, then near z_o, $W^s(z_o)$ and $W^u(z_o)$ are graphs of C^r functions.*

4 Outline of the Proof of the Main Theorem

The following definitions and lemmas outline the proof of the main theorem. The proofs of the lemmas are in the next section.

First note that for a relation f on R^n with a C^r hyperbolic fixed point z_o, f is locally the graph of a function $\underline{f}$. More precisely, for any μ and $k \leq r$, there is a neighborhood of z_o such that for some splitting $R^n = E^s \times E^u$ on this neighborhood, f is the graph of function $\underline{f}$, which is of the following form:

$$\underline{f} \begin{pmatrix} x \\ y' \end{pmatrix} = \begin{pmatrix} ax + \underline{g}_1(x, y') \\ by' + \underline{g}_2(x, y') \end{pmatrix} \tag{19}$$

where $x \in E^s$, $y' \in E^u$, a and b matrices, $|a|$, $|b| < 1$, and $\underline{g}_1$ and $\underline{g}_2$ functions which have all derivatives of order $\leq k$ Lipschitz with Lipschitz constant μ.

Motivated by this local expression of a hyperbolic relation as the graph of a function, we consider some definitions for relations on Euclidean space Z which are graphs of functions with certain properties for some coordinate system. We call these functions "associated" functions and call the coordinates "skewed" coordinates, represented by X and Y, where $Z = X \times Y$, and X and Y are Euclidean. Note that not every relation is the graph of such an associated function; these definitions are specifically intended for working with relations with hyperbolic fixed points. Also notice that the skewed coordinate system is not unique in any of the definitions below. However, once we choose a coordinate system, if there is an associated function in the coordinate system, then it is unique.

Notation: In all that follows, a relation is represented by a letter, and an associated function for this relation by the same letter underlined.

Definition 10 (Lipschitz Relations) *A relation f is Lipschitz of order λ, or $f \in Lip_\lambda$, when there is an associated function $\underline{f} \in Lip_\lambda$ such that $(x, y) \overset{f}{\mapsto} (x', y') \Leftrightarrow \underline{f}(x, y') = (x', y)$.*

Lemma 2 *Suppose a relation f in Lip_λ, $\lambda < 1$ has an associated Lipschitz function $\underline{f}$ described in the above definition. Then the relation f is C^r exactly when the associated function $\underline{f}$ is C^r.*

The proof of the above lemma follows from the implicit function theorem. It is tacitly assumed in the following lemma, which states that the composition of two Lipschitz and C^r relations gives another Lipschitz and C^r relation.

Lemma 3 *Let $\alpha < 1$ and $r \geq 0$. If g, Γ relations in Lip_α and C^r on $Z = X \times Y$, with associated functions in the same skewed coordinates, then $\Gamma \circ g \in Lip_\alpha$ and C^r as well.*

Given relation f, for a relation ϕ, define G by $G(\phi) = f \circ \phi \circ f$. The following lemma says that for f with a hyperbolic fixed point and certain ϕ, G is a contraction.

Lemma 4 *Let f satisfy the hypotheses of theorem 1 and $\alpha < 1$. For suitably small neighborhood of the fixed point, assume ϕ is Lip_α with associated function in the same skewed coordinates as f. Note that $\{\phi\}$ lies in the Banach space of Lip_α relations in a fixed skewed coordinate system with the norm being the sup norm on the associated functions. Then G is a contraction in the sup norm on the associated functions.*

Since G is a contraction in the space of Lip_α relations, G has a unique fixed point which is also in the space of Lip_α relations, and any such relation converges to this fixed point. In fact we can choose a neighborhood Ω such that f is an appropriate Lipschitz relation in the domain of G. Thus on this neighborhood, the fixed point is equal to $\lim_{k\to\infty} f^k$. Call this fixed point relation h and its associated function $\underline{h}$. The above lemma guarantees that h is Lipschitz on Ω. In fact, h is also C^r on Ω, as is restated below.

Lemma 5 *Assume $f \in C^r$ satisfying the hypotheses in theorem 1, and for Ω a neighborhood of z_o such that on Ω the fixed point relation is Lipschitz and equal to $\lim_{k\to\infty} f^k$. Then h is C^r on Ω.*

Definition 11 (ω-limit relation) *Given the relation f on compact metric space Z,*

$$f^\omega = \cap_{n\geq 1}\overline{\cup_{k\geq n} f^k},$$

where f^k is the composition of k copies of f.

The following lemma states that the relation h defined above is equal to the ω-limit relation:

Lemma 6 $f^\omega = h$.

The next two lemmas state that ω-limit relation is locally the cross product of the stable and unstable manifolds.

Lemma 7 *For a relation f satisfying the hypotheses of theorem 1, there is a neighborhood of the fixed point such that if $u \in W^s(z_o)$ and $v \in W^u(z_o)$, then (u, z_o) and (z_o, v) are contained in f^ω.*

In fact, a stronger statement holds; the following lemma states that f^ω relates every point in $W^s(z_o)$ to every point of $W^u(z_o)$.

Lemma 8 *For a relation f satisfying the hypotheses of theorem 1, there is a neighborhood of the fixed point such that $u \in W^s(z_o)$ and $v \in W^u(z_o) \Leftrightarrow (u, v) \in f^\omega$.*

Proof of theorem 1: By lemma 8, the stable an unstable manifolds are projections of f^ω. Precisely, $f^\omega = W^s(z_o) \times W^u(z_o)$. By lemma 7, this set equals $\{u : (u, z_o) \in f^\omega\} \times \{v : (z_o, v) \in f^\omega\}$. By lemma 6, $h = f^\omega$; by lemmas 5 and 2, h has an associated C^r function $\underline{h}$. In terms of the splitting, denote $z_o = (x_o, y_o)$. $W^s \times W^u = \{(x, y) : \underline{h}(x, y_o) = (x_o, y)\} \times \{(z, w) : \underline{h}(x_o, w) = (z, y_o)\}$. Thus both W^s and W^u are locally the graphs of C^r functions. $\square$

5 Proofs of lemmas

Proof of lemma 3: The proof is an application of the C^r and Lipschitz implicit function theorems. Since it is less common than the C^r implicit function theorem, we state the Lipschitz version here.

Theorem 9 (Lipschitz implicit function theorem) *If X and Y are metric spaces, and $\underline{F} : X \times Y \to X$ is a continuous mapping $\underline{F} \in Lip_\lambda$, $\lambda < 1$, then there exists function $\underline{g} : Y \to X$, $\underline{g} \in Lip_\lambda$ such that*

$$\underline{F}(x, y) = x \Leftrightarrow x = \underline{g}(y)$$

Proceeding with the proof of lemma 3, we need to show that if $g, \Gamma \in$Lip$_\alpha$ and C^r, then there exists a Lip$_\alpha$ and C^r function $\underline{\Gamma \circ g}$ such that $(x, y, x'', y'') \in \Gamma \circ g$ exactly when $\underline{\Gamma \circ g}(x, y'') = (x'', y)$. Define a function $\underline{F} : Z \times Z \times Z \to Z \times Z$ by

$$\underline{F}((x'', y'), (x', y), (x, y'')) = (\underline{\Gamma}(x', y''), \underline{g}(x, y')) \tag{20}$$

Since $\underline{F}$ is Lip$_\alpha$ and has no unit norm eigenvalues, by the implicit function theorem, there exists a Lip$_\alpha$ and C^r function $\underline{m} : Z \to Z \times Z$ such that $F(x'', y', x', y, x, y'') = (x'', y', x', y)$ exactly when $\underline{m}(x, y'') = (x'', y', x', y)$. Thus $(\underline{m}_1, \underline{m}_4) = \underline{\Gamma \circ g}$. □

Proof of lemma 4: This proof is a series of estimates. The key to the estimates is that f and the domain of G are Lipschitz.

Assume that f is as in the theorem, and we have picked a neighborhood and splitting so that equation 19 holds and $\underline{g}_1, \underline{g}_2$ are Lip$_\mu$ functions. Let $\lambda = \max(|a|, |b|) + \mu$. Assume we have chosen a small enough neighborhood that $\lambda + \alpha\mu < 1$.

(Note that $f \in$ Lip$_\lambda$.)

For a relation ψ with associated function $\underline{\psi}$, let $\|.\|$ denote the sup norm, and let $\underline{\psi} = (\psi_1, \psi_2)$ be the components of the associated function.

We want to show that for any relations $\phi, \psi \in$ Lip$_\alpha$, there is some uniform constant $\theta < 1$ such that $\|\underline{G(\psi)} - \underline{G(\phi)}\| < \theta\|\underline{\psi} - \underline{\phi}\|$. This is equivalent to showing that $\sup_{x=\xi, y'''=\eta'''} \overline{|(x, y, x''', y''') - (\xi, \eta, \xi''', \eta''')|} < \theta\|\underline{\psi} - \underline{\phi}\|$, where $(x, y, x''', y''') \in G(\phi)$, and $(\xi, \eta, \xi''', \eta''') \in G(\psi)$.

If $(x, y, x''', y''') \in G(\phi)$, and $(\xi, \eta, \xi''', \eta''') \in G(\psi)$, then there exist $x', y', x'', y'', \xi', \eta', \xi'', \eta''$ such that

$$\begin{aligned} (x, y) &\overset{f}{\mapsto} (x', y') \overset{\phi}{\mapsto} (x'', y'') \overset{f}{\mapsto} (x''', y''') \\ (\xi, \eta) &\overset{f}{\mapsto} (\xi', \eta') \overset{\psi}{\mapsto} (\xi'', \eta'') \overset{f}{\mapsto} (\xi''', \eta''') \end{aligned} \tag{21}$$

The following inequalities hold:

$$\begin{aligned} |x''' - \xi'''| &= |ax'' + \underline{g}_1(x'', y''') - a\xi'' - \underline{g}_1(\xi'', \eta''')| \qquad (22) \\ &\leq \lambda |x'' - \xi''|, \text{ since } y''' = \eta''' \\ \text{and } |x'' - \xi''| &= |\phi_1(x', y'') - \psi_1(\xi', \eta'')| \qquad (23) \\ &\leq \alpha \max(|x' - \xi'|,\ |y'' - \eta''|) + \|\underline{\phi} - \underline{\psi}\|, \text{ since } \psi, \phi \in \mathrm{Lip}_\alpha \end{aligned}$$

Similarly,

$$\begin{aligned} |x' - \xi'| &\leq \mu |y' - \eta'| \qquad (24) \\ |y'' - \eta''| &\leq \mu |x'' - \xi''| \qquad (25) \\ |y' - \eta'| &\leq \alpha \max(|x' - \xi'|,\ |y'' - \eta''|) + \|\underline{\phi} - \underline{\psi}\| \qquad (26) \\ |y - \eta| &\leq \lambda |y' - \eta'| \qquad (27) \end{aligned}$$

If we let $\Delta = \max(|x'' - \xi''|, |y' - \eta'|)$, then from the above equations, we have

$$\begin{aligned} \Delta &\leq \alpha\mu\Delta + \|\underline{\phi} - \underline{\psi}\|, \text{ so} \qquad (28) \\ \Delta &\leq \frac{1}{1 - \alpha\mu} \|\underline{\phi} - \underline{\psi}\| \end{aligned}$$

Thus for $\theta = \frac{\lambda}{1-\alpha\mu} < 1$, which is guaranteed by our original assumption, $\sup_{x=\xi, y'''=\eta'''} |(x, y, x''', y''') - (\xi, \eta, \xi''', \eta''')| < \theta \|\underline{\psi} - \underline{\phi}\|$. $\square$

Proof of lemma 5: To show that $h \in C^r$ when $f \in C^r$, we first show that there is a neighborhood of the fixed point of f such that the limit relation of f restricted to this neighborhood is C^r. To do this, we use the fiber contraction theorem [3] to show that the map G is a C^1 contraction when f is C^1. G is locally a C^r contraction when $f \in C^r$ by an induction argument. In order to show that h is a C^r relation on the original neighborhood, the relationship between h and the limit relation on a smaller neighborhood bears further comment. To this end, we prove that h is equal to the limit relation on the smaller neighborhood composed with finitely many C^r subsets of f. Therefore h is C^r on the entire original neighborhood. We use the following definitions and lemmas; the central proof follows their statements and proofs.

The following is a definition of a derivative relation of a smooth relation.

Definition 12 (Tangent Relation) *Given a smooth relation Γ on R^p, the tangent relation $T\Gamma$ on R^{2p} is the tangent bundle of Γ.*

If a relation has an associated function, then its tangent relation has an associated function, as described in the following lemma.

Lemma 10 *For a smooth relation Γ on $R^p = X \times Y$ with associated function $\underline{\Gamma}$, $T\Gamma$ is the graph of $(\underline{\Gamma}, D\underline{\Gamma})$. In other words, $(x, y, x', y', \xi, \eta, \xi', \eta') \in T\Gamma$ exactly when $(x, y, x', y') \in \Gamma$ and $D\underline{\Gamma}(x, y')(\xi, \eta') = (\xi', \eta)$.*

Proof This is due to the fact that a graph of a smooth function has tangent bundle equal to the graph of the derivative of the function. □

Lemma 11 (Derivatives and composition) *Assume that Γ and g are smooth relations with associated functions, and $(x, y, x'', y'') \in \Gamma \circ g$. Then locally there exist x' and y' such that the graph of $D(\underline{\Gamma \circ g})_{(x,y'')}$ equal to graph($D\underline{\Gamma}_{(x',y'')}$) $\circ$ graph($D\underline{g}_{(x,y')}$). In terms of tangent relations, locally $T\Gamma \circ Tg = T(\Gamma \circ g)$.*

Proof of lemma 11: In the proof of lemma 3, we showed that locally there are a unique x' and y' which are functions of (x, y'') such that

$$(x, y) \overset{g}{\mapsto} (x', y') \overset{\Gamma}{\mapsto} (x'', y'') \tag{29}$$

We know that $y' = \underline{\Gamma}_2(\underline{g}_1(x, y'), y'')$. For the coordinate system $R^n = E^s \times E^u$, write the derivative matrices in the form $D\underline{g} = \begin{pmatrix} D_1\underline{g}_1 & D_2\underline{g}_1 \\ D_1\underline{g}_2 & D_2\underline{g}_2 \end{pmatrix}$. Implicit differentiation gives

$$Dy' = (1 - D_1\underline{\Gamma}_2 D_2\underline{g}_1)^{-1}(D_1\underline{\Gamma}_2 D_1\underline{g}_1, D_2\underline{\Gamma}_2) \tag{30}$$

where all derivatives are evaluated at (x, y, x', y', x'', y''). It is now possible to write the derivative of $\Gamma \circ g$ explicitly. Comparing this derivative to the function associated with graph $D\underline{\Gamma}_{(x',y'')} \circ$ graph $D\underline{g}_{(x,y')}$ shows that they are equal. □

Lemma 12 *Let $\alpha < 1$, g and Γ both be Lip_α C^r relations on the compact set V; assume that in the coordinates $V = V_1 \times V_2$, there is a contraction $\underline{g}$ associated with g. Also assume that for $U_1 \subset V_1$ and $U_2 \subset V_2$, $\underline{g} : U_1 \times V_2 \to V$ and $\underline{g} : V_1 \times U_2 \to V$. Let Γ be a relation on V. Then $g \circ \Gamma \circ g$ is Lip_α and C^r on $U_1 \times U_2$.*

Proof of lemma 12: Define the function $\underline{F} : V \times V \times V \times U \to V \times V \times V$ as $\underline{F}((x'', y), (x', y''), (x''', y'), (x, y''')) = (\underline{g}(x'', y'''), \underline{\Gamma}(x', y''), \underline{g}(x, y'))$. Proceed using the implicit function theorem as in the proof of lemma 3. $\Box$

The final lemma is the fiber contraction theorem due to Hirsch and Pugh. Its proof can be found in [3].

Lemma 13 (Fiber contractions) *Let Ψ be a map on a space X with attractive fixed point p. For each $x \in X$, let Υ_x be a map on metric space Y such that $\Theta(x, y) = (\Psi(x), \Upsilon_x(y))$ is continuous on $X \times Y$. For fixed $\lambda < 1$ and each x, let each $\Upsilon_x \in Lip_\lambda$. Then there is an attracting fixed point (p, q) for Θ.*

Finally, the following definition makes the notation more convenient:

Definition 13 ((C^r,ϵ) and (Lip^r,ϵ) Small Relations) *A relation is (C^r, ϵ) small if there is some associated function which is (C^r, ϵ) small; in other words, there is an associated function which is C^r, and all its derivatives of order $< r$ are Lip_ϵ. A relation is (Lip^r,ϵ) small if it is (C^r, ϵ) small and the r^{th} derivative of the associated function is Lip_ϵ.*

Using these lemmas, the proof proceeds as follows: Let f be a C^r relation on R^n with hyperbolic fixed point at z_o, as in theorem 1. Let Ω be a neighborhood of the fixed point such that $\lim_{j\to\infty} f^j = h$, as described in the discussion after the statement of lemma 4. Let μ and λ be as in the proof of lemma 4, and let $z_o = (x_o, y_o)$ in terms of the splitting.

First we show G is a C^1 contraction when f is C^1 ($r = 1$). Since Tf is not necessarily Lipschitz, we cannot just apply lemma 4 on the tangent bundle. However, as in the diffeomorphism case, we can still prove the result using the fiber contraction theorem.

Let ϕ be a Lip$_\lambda$ relation on R^{2n} such that $\phi = (\rho, L)$, ρ a relation on R^n, and $\underline{L}(x, y', .)$ linear. Consider the map $TG : \phi \to Tf \circ \phi \circ Tf$. We verify the conditions for the fiber contraction theorem for TG; $\pi_1 TG$ has attractive fixed point h. Near (x_o, y_o), $D\underline{f}(x, y')$ is close to $D\underline{f}(x_o, y_o)$ in linear norm. Thus we can use estimates similar to those in the proof of lemma 4 on $\pi_2 TG$ on a neighborhood of $(x_o, y_o, 0, 0)$. On such a neighborhood, for fixed ρ and varying L, $\pi_2 TG$ is Lip$_\lambda$ in the sup norm. By the fiber contraction theorem, TG is a contraction in the sup norm on relations ϕ above. Thus TG is a contraction when $\phi = T\rho$, where ρ is a (Lip^1,μ) small relation on R^n. Therefore G is a C^1 contraction on (Lip^1,μ) small relations.

For the case $r > 1$, proceed by induction. Assume that for all relations $g \in C^{r-1}$ on with hyperbolic fixed point on R^p, and for ρ (Lip^{r-1},μ) small, that $\rho \to g \circ \rho \circ g$ is a C^{r-1} contraction. Choose $\tilde{g}$ C^r, hyperbolic. By our assumption, $T\tilde{\rho} \to T\tilde{g} \circ T\tilde{\rho} \circ T\tilde{g}$ is a C^{r-1} contraction when $\tilde{\rho}$ (Lip^r,μ) small relations. Therefore $\tilde{\rho} \to \tilde{g} \circ \tilde{\rho} \circ \tilde{g}$ is a C^r contraction.

We have so far shown that for $f \in C^r$, there is an ϵ such that on a ball of radius ϵ of the fixed point, G is a contraction in the C^r sup norm on (Lip^r, μ) small relations. The limit relation on this small ball is thus C^r. We now use the smoothness of the limit relation on the small balls to show that the relation h is C^r on all of Ω.

Choose r. Denote the ball of radius ϵ of the fixed point by $B^1_\epsilon \times B^2_\epsilon$. Let Γ be the C^r fixed point of G restricted to this ϵ ball. Note that $\Gamma \subset h$, since it can be described as the limit of iteration of the relation $f|_{B^1_\epsilon \times B^2_\epsilon}$.

For λ described in the proof of lemma 4, if $|(x,y) - (x_o, y_o)| < \frac{\epsilon}{\lambda}$, then $|\underline{f}(x,y) - (x_o,y_o)| < \epsilon$. Thus $\underline{f} : B^1_{\frac{\epsilon}{\lambda}} \times B^2_\epsilon \to B^1_\epsilon \times B^2_\epsilon$ and $\underline{f} : B^1_\epsilon \times B^2_{\frac{\epsilon}{\lambda}} \to B^1_\epsilon \times B^2_\epsilon$. Thus by lemma 12, $f \circ \Gamma \circ f$ is Lip$_\lambda$ and C^r when restricted to the set $B^1_{\frac{\epsilon}{\lambda}} \times B^2_{\frac{\epsilon}{\lambda}}$.

This new relation is also a subset of h since if $f \subset f'$ and $\Gamma \subset \Gamma'$, then $f \circ \Gamma \subset f' \circ \Gamma'$. We know that $\Gamma \subset h$. Thus $f \circ \Gamma \circ f \subset f \circ h \circ f = h$.

Now iterate this process of composing with f and restricting to a neighborhood; eventually we have a Lip$_\lambda, C^r$ relation on Ω. Since this relation is contained in h and both are associated with functions on Ω, the relations must be equal. Thus h is C^r on Ω. □

Proof of lemma 6: Assume that we have a neighborhood and splitting as in f equation 19. We show that there is a sequence in f^k converging to a limit point (x,y,z,w) exactly when there is a sequence k_i such that $\lim_{i\to\infty} \underline{f^{k_i}}(x,w) = (z,y)$.

First we show that $h \subset f^\omega$; from the definition,

$$f^\omega = \left\{u : u_{k_i} \to u \text{ for some } u_{k_i} \in f^{k_i}\right\}. \tag{31}$$

By lemma 4, $f \mapsto f \circ f \circ f \mapsto f^5 \mapsto ...$ maps to h in the sup norm on the associated functions. Thus for all (x,w) and odd k, $\underline{f^k}(x,w)$ has a limit, and the limit is equal to $\underline{h}(x,w)$. Thus $u_k = (x, \underline{f^k}_2(x,w), \underline{f^k}_1(x,w), w)$ shows that $h \subset f^\omega$.

Conversely, to show that $f^\omega \subset h$, suppose $\eta = (x,y,z,w) \in f^\omega$, and $u_{k_i} = (x_{k_i}, y_{k_i}, z_{k_i}, w_{k_i})$ is the sequence in f^{k_i} guaranteed by equation 31 such that $|u_{k_i} - \eta| \to 0$. Define $v_{k_i} = (x, \underline{f}^{k_i}_2(x,w), \underline{f}^{k_i}_1(x,w), w)$. Then

$$| \eta - v_{k_i} | \leq | \eta - u_{k_i} | + | u_{k_i} - v_{k_i} | . \quad (32)$$

The first term on the right goes to zero by construction. In addition, since f is in Lip_λ, the second term is less than or equal to the first term. Therefore it goes to zero as well. Thus $\eta \in h$. □

Proof of lemma 7 follows from lemma 8. □

Proof of lemma 8: Assume that we have a neighborhood and splitting described in equation 19 and that in terms of the splitting, the fixed point is denoted by (x_o, y_o). Assume $(x, y) \in W^s(x_o, y_o)$ and $(z, w) \in W^u(x_o, y_o)$. In the proof that follows, we look at the forward k-iterates of a neighborhood of (x, y) and the backward k-iterates of a neighborhood of (z, w). For large k, near the fixed point, a portion of the forward iterates form a Lipschitz "vertical" curve, and a portion of the backward iterates form a Lipschitz "horizontal" curve. The two curves are near each other, and thus intersect, implying the existence of a point near (x, y) with a $2k$-iterate near (z, w). More precisely, we use this idea to show that for any ϵ, K there exists $k > K$ and a point $(s, t, u, v) \in f^k$ such that $\mathrm{dist}((x, y, z, w), (s, t, u, v)) < \epsilon$ and thus $(x, y, z, w) \in f^\omega$.

Let ϵ be given. We know that there exist sequences (x_k, y_k), and (z_k, w_k) both converging to the fixed point, $(x, y, x_k, y_k) \in f^k$ and $(z_k, w_k, z, w) \in f^k$. For a small δ, let k be large enough that distance from (x_k, y_k) to (z_k, w_k) is less than δ.

Now look at an ϵ ball of y_k in Y. For the point (x, η'), where η' is in the ϵ ball, we have the point $(x, \eta, \xi', \eta') \in f^k$. The set of points (ξ', η') form the graph of a Lipschitz function from Y to X near (x_k, y_k), each point of which is related to a point near (x, y) by f^k. Similarly, there is a graph of a Lipschitz function from X to Y near (z_k, w_k), and a point near (z, w) is related to each of the points in this graph. But if δ is small enough, these Lipschitz graphs must intersect. Thus there is a point $(s, t, u, v) \in f^{2k}$ within ϵ of (x, y, z, w). We conclude that $(x, y, z, w) \in f^\omega$.

Conversely, assume $(x, y, z, w) \in f^\omega$. Therefore for any $k > 0$, there are points near (x, y) with k forward iterates. Using compactness, we show that (x, y) has an infinite forward orbit. Using the fact that $f \in Lip_\lambda$, we show that the forward orbit must converge to the fixed point, and thus $(x, y) \in W^s(x_o, y_o)$. Likewise, $(z, w) \in W^u(x_o, y_o)$.

Let $B_\epsilon(x,y)$ be the closed ϵ ball of (x,y), and define the set

$$S^k_\epsilon(x,y) = \{(\xi,\eta) \in B_\epsilon(x,y) : (\xi,\eta) \text{ has a } k^{th} \text{ iterate}\}$$

$S^k_\epsilon(x,y)$ is nonempty, by the assumption on (x,y). It is compact, since f is closed, which implies f^k closed [4] and thus compact. Thus $\cap_\epsilon S^k_\epsilon(x,y)$ is nonempty, since it is the intersection of non-empty nested compact sets. It is equal to $\{(x,y)\}$, since this is the only point it could contain. Therefore (x,y) has a k^{th} iterate (x_k,y_k) for every k. Thus there exists an infinite forward orbit starting at (x,y). By compactness, there exists a limit point (z',w'). Thus $(x_j,y_j,z',w') \in f^\omega$ for the same z',w' for all j.

Since $(x_j,y_j,z',w') \in f^\omega$ for all j, if (x_k,y_k) and (x_j,y_j) are in this forward orbit, then $|y_j - y_k| < \lambda|x_j - x_k|$.

Since f^k is Lip_λ for every k,$|x_{k+1}-x_k| < \lambda\max(|x_{k-1}-x_k|,|y_{k+1}-y_k|) < \lambda\max(|x_{k-1}-x_k|,\lambda|x_{k+1}-x_k|)$. Thus $|x_{k+1}-x_k| < \lambda|x_{k-1}-x_k|$. This Cauchy sequence implies that x_k converges to a unique z_o. Likewise, y_k converges to a unique w_o. This means that $(x_k,y_k,x_{k+1},y_{k+1}) \to (z_o,w_o,z_o,w_o)$. Since f is closed and $(x_k,y_k,x_{k+1},y_{k+1}) \in f$, $(z_o,w_o,z_o,w_o) \in f$ as well. The unique fixed point of f is (x_o,y_o). Therefore (x_k,y_k) converges to (x_o,y_o). Therefore $(x,y) \in W^s(x_o,y_o)$. Similarly, $(z,w) \in W^u(x_o,y_o)$. $\square$

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