A simplification of Budyko's ice-albedo feedback model

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Abstract

A classical model of ice-albedo feedback was recently augmented to incorporate ice line dynamics, and the resulting infinite dimensional dynamical system was reduced to a one-dimensional system using invariant manifold theory. Here we introduce an approximation to the model, which immediately produces a five-dimensional dynamical system having an analogous one-dimensional invariant manifold. We derive a simple ordinary differential equation approximating the system and use it to estimate the value of a parameter left unspecified in previous work.

Introduction

The sun provides energy affecting Earth's temperature and powering weather and climate. From a global perspective, the surface temperature is at equilibrium when the outgoing radiation into space balances the incoming radiation from the sun. This balance is affected by many factors; the one of interest here is the albedo, the proportion of sunlight reflected back into space. Ice has a higher albedo than either land or ocean, meaning that it reflects more of the Sun's energy, diminishing its effect of warming the surface.

Currently ice is confined mostly to the polar regions of the Earth, but it has covered much more of the surface periodically over the past several million years. "Ice-albedo feedback" is one of the factors that can explain the extent of the ice cover. When ice is melting, more land and sea is exposed,

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absorbing more of the Sun's energy, causing the surface to warm, causing more ice to melt. On the other hand, when the ice is advancing, more of the Sun's energy is reflected, causing the surface to cool, causing more ice to form. What causes the ice to stop advancing before it has covered the Earth? Alternatively, what causes the ice to stop melting before the Earth becomes ice-free?

Answers for these questions were given by models developed independently by Budyko [1] and Sellers [2]. These models consider the annual average surface temperature as a function of latitude. It is assumed that the ice covers the Earth for latitudes higher than a certain value, while no ice occurs at lower latitudes. The annual average incoming solar radiation is a function of latitude. The reflection of the Sun's energy back into space depends on whether ice occurs at that latitude. The outgoing radiation from the Earth's surface is a function of surface temperature. Transport of energy from latitudes with higher temperatures to latitudes with lower temperatures is included in the model.

From a dynamical systems perspective, the state space consists of some set of functions describing the temperature as a function of latitude. Without further restrictions, this space would be infinite dimensional. The model as classically posed did not contain dynamics of the ice boundary. Widiasih has proposed adding a simple equation modelling the advance and retreat of the ice boundary [3]. Although the model remained infinite dimensional, Widiasih was able to show the existence of an attracting one-dimensional invariant manifold containing the dynamics of the ice boundary, thus reducing the long-term dynamics of the system to a single ordinary differential equation.

In this paper we examine an approximation of the model and give a simple explanation for the invariant manifold. We also compare the model with the data to give an estimate for one of the parameters left unspecified in Widiasih's analysis.

The Budyko Model

We use a version of the ice-albedo feedback model described in Tung's book [4], where the reader can find a more detailed description of the model. The basic variable is the annual average surface temperature T as a function of latitude. The dynamical equation can be written

$$R\frac{\partial T}{\partial t} = Qs\left(y\right)\left(1-\alpha\left(y,\eta\right)\right)-\left(A+BT\right)+C\left(\overline{T}-T\right), \qquad (1)$$

where y is the sine of latitude and T is the annual average surface temperature as a function of y and time t. The equation and its solutions are assumed to be symmetric across the equator, so we take $y \in [0, 1]$. The quantity Q is the annual global average insolation (incoming solar radiation) for the entire Earth, while the function s is the distribution of the insolation over latitude, normalized so that

$$\int_0^1 s(y)\,dy = 1.$$

We assume that there is a single ice line at $y = \eta$, with ice covering the hemispheric cap for $y > \eta$, while the equatorial region $y < \eta$ is ice free. The albedo, $\alpha(y, \eta)$, has one value, α_1, where the surface is ice free and another value, α_2, where the surface is ice covered. Thus,

$$\alpha(y, \eta) = \begin{cases} \alpha_1, & y < \eta, \\ \alpha_2, & y > \eta. \end{cases}$$

Since the albedo is the proportion of energy reflected back into space, the average annual rate at which solar energy is absorbed by the Earth's surface is $Qs(y)(1 - \alpha(y, \eta))$.

The Earth radiates energy into space at longer wavelengths than the insolation. The existence of greenhouse gases in the atmosphere causes this outgoing radiation to be a complicated function of the surface temperature. For the Budyko model, this function is approximated linearly, as $A + BT$, where A and B are constants determined by satellite data.

Heat is transported between latitudes by a complex process involving winds and ocean currents. The Budyko model assumes that this transport, averaged over a year, can be approximated by a simple linear relaxation to the mean, $C(\overline{T} - T)$, where C can be determined from data and where $\overline{T}$ is the annual global mean temperature given by

$$\overline{T} = \int_0^1 T(y)\,dy.$$

Note that it is the choice of y to be the sine of the latitude gives that gives us this simple expression for the mean temperature.

It is useful to think about units. Both sides of equation (1) have units of Watts per square meter ($\mathrm{W/m^2}$), which is the same as Joules per second per square meter ($\mathrm{J/s/m^2}$). Hence Q and A have those units, while B and C have units of Watts per square meter per degree Kelvin ($\mathrm{W/m^2/K}$). We use the following values, as given by Tung [4]: $Q = 343\,\mathrm{W/m^2}$, $A = 202\,\mathrm{W/m^2}$,

$B = 1.9\,\mathrm{W/m^2/K}$, and $C = 3.04\,\mathrm{W/m^2/K}$. These values and others are repeated in the table in Appendix 1 for the convenience of the reader.

The function $s(y)$, the distribution of insolation across latitudes, is dimensionless, as is the albedo function $\alpha(y, \eta)$. Here are the dimensionless albedo constants, also as given by Tung [4]: $\alpha_1 = 0.32$ and $\alpha_2 = 0.62$

The variables y and η are also dimensionless. We can think of y as the proportion of the Earth's surface between latitudes $-\arcsin(y)$ and $+\arcsin(y)$, while η is the proportion of the Earth's surface which is ice-free. If σ is the total surface area of the Earth, in square meters (approximately $5.1 \times 10^{14}\mathrm{m}^2$), then σy is the total surface area in square meters between corresponding latitudes, while $\sigma(1-\eta)$ is the area in square meters of the ice-covered surface.

Finally, the parameter R is the heat capacity of the Earth's surface. From equation (1) and from the units described above, we see that $R\frac{\partial T}{\partial t}$ has units of $\mathrm{W/m^2}$. Therefore R has units of $\mathrm{J/m^2/K}$ (Joules per square meter per degree Kelvin). The heat capacity of liquid water is about $4\,\mathrm{J/g/K}$. Since a gram of water is a cubic centimeter, the heat capacity is $4 \times 10^6\,\mathrm{J/m^3/K}$. Assuming that the entire surface of the Earth is water that must be heated to a depth of 100 meters, we find that

$$R = 4 \times 10^8\,\mathrm{J/m^2/K}\,. \tag{2}$$

Dynamics of the Ice Line

Widiasih [3] introduced the following equation for the movement of the ice line.

$$\frac{d\eta}{dt} = \varepsilon\left(T_b - T_c\right), \tag{3}$$

where T_b is the average temperature across the ice boundary, i.e.,

$$T_b = \frac{T(\eta-) + T(\eta+)}{2},$$

and where T_c is the critical annual mean temperature above which ice retreats and below which ice advances. This equation simply states that the movement of the ice line is proportional to the difference between the current temperature at the ice line and the critical temperature. Following Tung [4], we take

$$T_c = -10\,^\circ\mathrm{C}.$$

As we shall see below, we can think of ε as small, reflecting the slow advance and retreat of the glaciers.

It is easier to think about the dimensions of the ice line dynamics if we multiply both sides of the equation by σ:

$$\frac{d\left(\sigma\eta\right)}{dt} = \left(\sigma\varepsilon\right)\left(T_b - T_c\right).$$

We see that $\sigma\varepsilon$ has units square meters per second per degree Kelvin ($\mathrm{m}^2/\mathrm{s}/\mathrm{K}$).

Equation (1) takes into account the energy to heat the surface, but it does not take into account the energy to melt the ice. From the above discussion of the constant R, we see that the amount of energy required to raise the surface temperature by ΔT degrees is $R\Delta T\,\mathrm{J/m}^2$. The heat of fusion of ice is $334\,\mathrm{J/g}$, or $3.34 \times 10^8\,\mathrm{J/m}^3$. Assuming that the ice is on average 450 meters thick, we have that the amount of energy required to melt a square meter of ice is 1.5×10^{11} Joules. If we let

$$\Omega = 1.5 \times 10^{11}\,\mathrm{J/m}^2, \tag{4}$$

then the amount of energy required to move the ice line from η to $\eta + \Delta\eta$ is $\Omega\sigma\Delta\eta$ Joules, or $\Omega\Delta\eta$ Joules per square meter. Including the energy to move the ice line into equation (1), we have

$$R\frac{\partial T}{\partial t} + \Omega\frac{d\eta}{dt} = Qs\left(y\right)\left(1 - \alpha\left(y,\eta\right)\right) - \left(A + BT\right) + C\left(\overline{T} - T\right).$$

Combining this equation with equation (3), we have

$$R\frac{\partial T}{\partial t} + \Omega\varepsilon\left(T_b - T_c\right) = Qs\left(y\right)\left(1 - \alpha\left(y,\eta\right)\right) - \left(A + BT\right) + C\left(\overline{T} - T\right),$$

yielding the system

$$\begin{aligned} \frac{d\eta}{dt} &= \varepsilon(T_b - T_c), \\ \frac{\partial T}{\partial t} &= \frac{1}{R}\big(Qs(y)(1 - \alpha(y,\eta)) - (A + BT) + C(\overline{T} - T) - \varepsilon\Omega(T_b - T_c)\big). \end{aligned} \tag{5}$$

Legendre Expansion

From the dynamical systems viewpoint, the state space is the product of the interval $[0, 1]$, where the variable η lives, with the space of functions giving the temperature T as a function of y, the sine of the latitude. The question of what function space is most appropriate was explored by Widiasih [3]. Here we adopt the perspective that a finite dimensional function space captures the essential behavior of the system.

The precedent was set by North [5], who pointed out that the insolation distribution function s is well-approximated by a quadratic equation. As shown by McGehee & Lehman [6], the equilibrium solutions of equation (1) are piecewise continuous, with a discontinuity at $y = \eta$, and with each continuous piece depending linearly on the function s. This property implies that, if s is quadratic, the equilibrium solutions are piecewise quadratic. We therefore assume that the state space consists of piecewise quadratic functions with a single discontinuity at the ice line $y = \eta$.

We write

$$T(y) = \begin{cases} U(y), & y < \eta, \\ V(y), & y > \eta, \\ (U(\eta) + V(\eta))/2\ , & y = \eta, \end{cases} \tag{6}$$

where U and V are assumed to be quadratic on $[0, 1]$. Our choice of the indicated value of $T(\eta)$ allows us to write

$$T_b = \left(T(\eta-) + T(\eta+)\right)/2 = T(\eta).$$

System (5) then becomes

$$\begin{aligned} \frac{d\eta}{dt} &= \varepsilon(T_b - T_c), \\ \frac{\partial U}{\partial t} &= \frac{1}{R}\left(Qs(y)(1-\alpha_1) - (A + BU) + C(\overline{T} - U) - \varepsilon\Omega(T_b - T_c)\right), \\ \frac{\partial V}{\partial t} &= \frac{1}{R}\left(Qs(y)(1-\alpha_2) - (A + BV) + C(\overline{T} - V) - \varepsilon\Omega(T_b - T_c)\right), \end{aligned} \tag{7}$$

while the global mean temperature becomes

$$\overline{T} = \int_0^\eta U(y,t)dy + \int_\eta^1 V(y,t)dy.$$

Since the function s is even, we assume that U and V are also even. It is convenient to use the first two even Legendre polynomials:

$$\begin{aligned} p_0(y) &= 1, \\ p_2(y) &= \frac{1}{2}(3y^2 - 1), \end{aligned}$$

and to write

$$\begin{aligned} U(y,t) &= u_0(t)p_0(y) + u_2(t)p_2(y), \\ V(y,t) &= v_0(t)p_0(y) + v_2(t)p_2(y), \\ s(y) &= s_0p_0(y) + s_2p_2(y). \end{aligned} \tag{8}$$

Note that the normalization of the distribution function s implies that $s_0 = 1$. Let

$$\begin{aligned} P_0(\eta) &= \int_0^\eta p_0(y)dy = \eta, \\ P_2(\eta) &= \int_0^\eta p_2(y)dy = \frac{1}{2}(\eta^3 - \eta). \end{aligned}$$

and note that

$$\begin{aligned} &\int_\eta^1 p_0(y)dy = 1 - P_0(\eta) = 1 - \eta, \\ &\int_\eta^1 p_2(y)dy = -P_2(\eta). \end{aligned}$$

Therefore,

$$\begin{aligned} \overline{T} &= \int_0^\eta U(y,t)dy + \int_\eta^1 V(y,t)dy \\ &= \int_0^\eta \left(u_0(t)p_0(y) + u_2(t)p_2(y)\right) dy + \int_\eta^1 \left(v_0(t)p_0(y) + v_2(t)p_2(y)\right) dy \\ &= u_0(t)P_0(\eta) + u_2(t)P_2(\eta) + v_0(t)\left(1 - P_0(\eta)\right) - v_2(t)P_2(\eta) \\ &= \eta u_0(t) + (1-\eta)v_0(t) + P_2(\eta)\left(u_2(t) - v_2(t)\right). \end{aligned}$$

System (7) becomes

$$\begin{aligned} \dot{\eta} &= \varepsilon(T_b - T_c), \\ \dot{u}_0 p_0 + \dot{u}_2 p_2 &= \frac{1}{R}\big(Q(p_0 + s_2 p_2)(1-\alpha_1) - Ap_0 \\ &\qquad - (B+C)(u_0 p_0 + u_2 p_2) + C\overline{T}p_0 - \varepsilon\Omega(T_b - T_c)p_0\big), \\ \dot{v}_0 p_0 + \dot{v}_2 p_2 &= \frac{1}{R}\big(Q(p_0 + s_2 p_2)(1-\alpha_2) - Ap_0 \\ &\qquad - (B+C)(v_0 p_0 + v_2 p_2) + C\overline{T}p_0 - \varepsilon\Omega(T_b - T_c)p_0\big), \end{aligned}$$

where the dot over the variable indicates differentiation with respect to time t. The temperature T_b along the ice line becomes

$$\begin{aligned} T_b &= \frac{1}{2}(u_0 p_0 + u_2 p_2 + v_0 p_0 + v_2 p_2) \\ &= \frac{u_0 + v_0}{2} + \frac{u_2 + v_2}{2} p_2. \end{aligned}$$

Equating coefficients of the Legendre polynomials, we have

$$\begin{aligned}
\dot{\eta} &= \varepsilon\left(T_b - T_c\right), \\
\dot{u}_0 &= \frac{1}{R}\big(Q(1-\alpha_1) - A - (B+C)u_0 + C\overline{T}(\eta) - \varepsilon\Omega(T_b - T_c)\big), \\
\dot{v}_0 &= \frac{1}{R}\big(Q(1-\alpha_2) - A - (B+C)v_0 + C\overline{T}(\eta) - \varepsilon\Omega(T_b - T_c)\big), \\
\dot{u}_2 &= \frac{1}{R}\big(Qs_2(1-\alpha_1) - (B+C)u_2\big), \\
\dot{v}_2 &= \frac{1}{R}\big(Qs_2(1-\alpha_2) - (B+C)v_2\big),
\end{aligned}$$

where the temperature along the ice line is given by

$$T_b = \frac{u_0 + v_0}{2} + \frac{u_2 + v_2}{2} p_2(\eta),$$

and where the global mean temperature is given by

$$\overline{T} = \eta u_0 + (1-\eta)v_0 + P_2(\eta)\left(u_2 - v_2\right). \tag{9}$$

The system becomes slightly more tractable with the introduction of two new variables

$$w = \frac{u_0 + v_0}{2}, \qquad z = u_0 - v_0. \tag{10}$$

The system then becomes:

$$\begin{aligned}
\dot{\eta} &= \varepsilon(T_b - T_c), \\
\dot{w} &= \frac{1}{R}\big(Q(1-\alpha_0) - A - (B+C)w + C\overline{T} - \varepsilon\Omega(T_b - T_c)\big), \\
\dot{z} &= \frac{1}{R}\big(Q(\alpha_2 - \alpha_1) - (B+C)z\big), \\
\dot{u}_2 &= \frac{1}{R}\left(Qs_2(1-\alpha_1) - (B+C)u_2\right), \\
\dot{v}_2 &= \frac{1}{R}\left(Qs_2(1-\alpha_2) - (B+C)v_2\right),
\end{aligned}$$

where

$$\alpha_0 = \frac{\alpha_1 + \alpha_2}{2},$$

where the ice line temperature is

$$T_b = w + \frac{u_2 + v_2}{2} p_2(\eta),$$

and where the global mean temperature is

$$\overline{T} = w + \left(\eta - \frac{1}{2}\right) z + P_2(\eta)(u_2 - v_2).$$

Two Dimensional Invariant Subspace

Note that the equations for z, u_2, and v_2 are simple linear equations completely decoupled from the other variables. Therefore, there is a globally attracting invariant two dimensional subspace given by

$$\begin{aligned} z &= \frac{Q(\alpha_2 - \alpha_1)}{B + C}, \\ u_2 &= \frac{Qs_2(1 - \alpha_1)}{B + C}, \\ v_2 &= \frac{Qs_2(1 - \alpha_2)}{B + C}. \end{aligned} \tag{11}$$

On this subspace, the equations become

$$\begin{aligned} \dot{\eta} &= \varepsilon\big(T_b(w, \eta) - T_c\big) \\ \dot{w} &= \frac{1}{R}\big(Q(1 - \alpha_0) - A - (B + C)w \\ &\qquad + C\overline{T}(w, \eta) - \varepsilon\Omega(T_b(w, \eta) - T_c)\big), \end{aligned} \tag{12}$$

where the ice line temperature is

$$T_b\left(w, \eta\right) = w + \frac{Qs_2\left(1 - \alpha_0\right)}{B + C} p_2\left(\eta\right)$$

and where the global mean temperature is

$$\begin{aligned} \overline{T}(w, \eta) &= w + \left(\eta - \frac{1}{2}\right) \frac{Q(\alpha_2 - \alpha_1)}{B + C} + P_2(\eta) \frac{Qs_2(\alpha_2 - \alpha_1)}{B + C} \\ &= w + \frac{Q(\alpha_2 - \alpha_1)}{B + C} \left(\eta - \frac{1}{2} + s_2 P_2(\eta)\right). \end{aligned}$$

The essential behavior of the system is determined by the two-dimensional system (12).

We introduce the following function, which will be useful in the discussion below:

$$\Phi_0(\eta) = \frac{1}{B} \left(Q(1 - \alpha_0) - A + C \frac{Q(\alpha_2 - \alpha_1)}{B + C} \left(\eta - \frac{1}{2} + s_2 P_2(\eta)\right)\right).$$

System (12) can then be written:

$$\begin{aligned}\dot{\eta} &= \varepsilon\left(w + \frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) - T_c\right), \\ \dot{w} &= \frac{1}{R}\left(B\Phi_0(\eta) - Bw - \varepsilon\Omega\left(w + \frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) - T_c\right)\right).\end{aligned} \tag{13}$$

Rest Points

Setting the right hand sides of system (13) equal to zero, we can solve the following equations for the rest points of the system:

$$\begin{aligned}w &= -\frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) + T_c, \\ w &= \Phi_0(\eta).\end{aligned}$$

Note that, independent of the value of ε, the rest points occur on the curve $w = \Phi_0(\eta)$. Indeed, the rest points themselves are independent of ε, and can be found by solving the following equation for η:

$$h(\eta) = \Phi_0(\eta) + \frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) - T_c = 0. \tag{14}$$

Figure 1 shows a graph of this function h for the current state of the Earth's orbit. Here we are using the value of $s_2 = -0.482$ used by North [5] and based on computations by Chylek and Coakley [7]. Note the existence of two zeros, one at approximately 0.25 and the other at approximately 0.95. The lower one corresponds to an unstable rest point (a saddle), while the upper one corresponds to a stable rest point, as we shall now see.

We label the two fixed points (η_1, w_1) and (η_2, w_2), where $\eta_1 \approx 0.25$ and $\eta_2 \approx 0.95$, and where $w_i = \Phi_0(\eta_i)$. The Jacobian matrix for system (13) at these rest points is

$$\begin{bmatrix} \varepsilon\frac{Qs_2(1-\alpha_0)}{B+C}p_2'(\eta_i) & \varepsilon \\ \frac{B}{R}\Phi_2'(\eta_i) - \frac{\varepsilon\Omega}{R}\frac{Qs_2(1-\alpha_0)}{B+C}p_2'(\eta_i) & -\frac{B}{R}\end{bmatrix} \tag{15}$$

For $\varepsilon = 0$, the matrix becomes triangular, with eigenvalues 0 and $-B/R$. For small $\varepsilon > 0$, the matrix has a negative eigenvalue near $-B/R$ and an eigenvalue near 0. The small eigenvalue can be written $\varepsilon h'(\eta_i) + O(\varepsilon^2)$. We

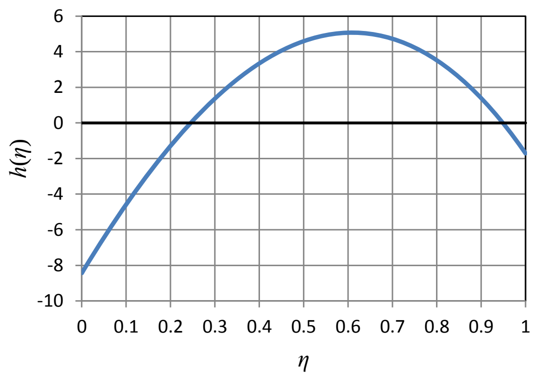

Figure 1: The function h.

see from the graph in Figure 1 that $h'(\eta_1) > 0$ while $h'(\eta_2) < 0$. Therefore, (η_1, w_1) is a saddle, while (η_2, w_2) is a sink.

It is useful to note that the function h is actually a cubic polynomial in η. An elementary but tedious exercise shows that, for the parameters given above, h has three distinct real roots, two in the unit interval as shown in Figure 1. Therefore, the conclusions about the eigenvalues in the previous paragraph can be made rigorous.

Invariant Curve

If we let $\varepsilon = 0$, then η becomes constant, so system (13) becomes

$$\dot{w} = \frac{1}{R}\left(Q(1-\alpha_0) - A - Bw + C\frac{Q(\alpha_1-\alpha_2)}{B+C}\left(\eta - \frac{1}{2} + s_2P_2(\eta)\right)\right),$$

and we have a curve of fixed points given by

$$\begin{aligned} w &= \Phi_0(\eta) \\ &= \frac{1}{B}\left(Q(1-\alpha_0) - A + C\frac{Q(\alpha_1-\alpha_2)}{B+C}\left(\eta - \frac{1}{2} + s_2P_2(\eta)\right)\right). \end{aligned}$$

Therefore, for $\varepsilon = 0$, we can write system (13) as

$$\begin{aligned} \dot{\eta} &= 0, \\ \dot{w} &= \frac{B}{R}\left(\Phi_0(\eta) - w\right). \end{aligned}$$

Note that the invariant curve given by $w = \Phi_0(\eta)$ is globally exponentially attracting at a constant exponential rate of $-B/R$. Therefore, for sufficiently small $\varepsilon > 0$, there exists an exponentially attracting invariant curve given by $w = \Phi_\varepsilon(\eta)$, where Φ_ε is a C^r function, close to Φ_0 in the C^r topology, and where r can be made large by choosing ε small. In particular, we can write the invariant curve as

$$w = \Phi_\varepsilon(\eta) = \Phi_0(\eta) + O(\varepsilon). \tag{16}$$

On this invariant curve, system (13) reduces to the single equation

$$\begin{aligned}\dot{\eta} &= \varepsilon\left(\Phi_\varepsilon(\eta) + \frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) - T_c\right)\\ &= \varepsilon\left(\Phi_0(\eta) + \frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) - T_c\right) + O(\varepsilon^2)\\ &= \varepsilon h(\eta) + O(\varepsilon^2),\end{aligned}$$

where $h(\eta)$ is given above in equation (14).

In summary, for sufficiently small ε, the motion of the ice line in system (5) is well approximated by the following single ordinary differential equation:

$$\dot{\eta} = \varepsilon h(\eta). \tag{17}$$

Paleoclimate

Although all of the parameters in system (5) vary over geologic time, the only ones whose changes are well-understood are the average global annual insolation Q and the latitudinal distribution $s(y)$, which are associated with variations in the Earth's orbital parameters, called Milankovitch cycles. It is well-known that Q is a function of the eccentricity e of the Earth's orbit, while $s(y)$ depends only on the obliquity β, i.e., the tilt of the Earth's spin axis with respect to the orbital plane. Indeed, as shown by McGehee and Lehman [6], these dependencies can be written

$$Q(e) = \frac{Q_0}{\sqrt{1-e^2}},$$

where Q_0 is the value that Q would assume if the eccentricity were zero, and

$$s(y,\beta) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\theta - y\cos\beta\right)^2}\,d\theta.$$

Recall that we expanded the insolation distribution function s in Legendre polynomials (see equations (8)). Taking into account the dependence on the obliquity β, we can write

$$s(y,\beta) \approx 1 + s_2(\beta)p_2(y).$$

It is shown in Appendix 2 that

$$s_2(\beta) = \frac{5}{16}\left(-2 + 3\sin^2\beta\right). \tag{18}$$

The Value of Epsilon

The values of all the parameters except ε are given in the text above. This parameter determines how fast the ice line moves in response to changes in the temperature at the ice line. For an estimate of ε, we look to the paleoclimate data. McGehee and Lehman showed that the Budyko model is a reasonable fit for the early Pleiocene [6]. However, they only followed the equilibrium solution of equation (1); their analysis did not have a dynamic ice line. They showed that the data given in the Lisecki-Raymo stack lags the equilibrium ice line by about 2.5 Kyr [6]. To be consistent with the Lisiecki-Raymo stack during the early Pleiocene, we should expect the solutions of equation (17) to lag the running equilibrium with a delay of about 2.5 Kyr. We will estimate the value of ε by adjusting it so that equation (17) exhibits this delay.

McGehee and Lehman showed that the stable equilibrium solution for the Budyko-Sellers equation, when forced by the insolation cycles, exhibits a dominant response with a period of 41 Kyr, corresponding to the Earth's obliquity cycles [6]. Assuming that the reduced equation (17) exhibits the same behavior and noting the equilibrium cycles are small [6], we can approximate equation (17) with its linearization about the stable fixed point at $\eta = \eta_2$.

We therefore consider the linearized version of equation (17),

$$\dot{\xi} = -\lambda(\xi - \eta_2),$$

where $\lambda = -\varepsilon\kappa h'(\eta_2)$. We have changed the units of time from seconds to kiloyears; hence the factor $\kappa = 3.16 \times 10^{10}$, the number of seconds in a kiloyear. As a further approximation, we hold λ constant, but force η_2 with a simple cosine function with a period of 41 Kyr to obtain

$$\dot{\xi} = -\lambda(\xi - a\cos\omega t), \tag{19}$$

where $\omega = 2\pi/41$. The steady state solution of this equation can be solved explicitly and is

$$\xi = (a\cos\psi)\cos(\omega t - \psi),$$

where $\psi = \arctan(\omega/\lambda)$ is the phase shift corresponding to the delay by which the response lags the forcing. A delay of 2.5 Kyr implies that $\psi = 2.5\omega = 5\pi/41$. This delay therefore corresponds to a value of $\lambda = \omega\cot\psi = -\varepsilon\kappa h'(\eta_2)$. We thus have derived

$$\varepsilon = -\frac{\omega\cot\psi}{\kappa h'(\eta_2)}$$

as an approximation for the value of ε giving a 2.5 Kyr delay of the climate following the orbital forcing. Wading through the above formulae, one computes that $h'(\eta_2) \approx -30.9$, which implies that

$$\varepsilon \approx 3.9 \times 10^{-13}. \tag{20}$$

It is interesting to note how close the 2.5 Kyr delay is to the time constant of equation (19). Recall that the time constant is just the time τ it takes for the linear equation to decay to $1/e$ of its original value, i.e., $\tau = 1/\lambda$. Working through the numbers, we see that $\lambda \approx 0.38$ and $\tau \approx 2.6$ Kyr.

We can ask whether this value of ε is "small enough" to justify some of the approximations we made above. To examine this question, we return to the Jacobian matrix (15) given above. If we multiply the matrix by κ, so that the time units become Kyr, the numerical value of the matrix is approximately

$$\begin{bmatrix} -0.62 & 0.0123 \\ 3180 & -150 \end{bmatrix},$$

and the eigenvalues of the matrix are $\lambda_1 \approx -0.36$ and $\lambda_2 \approx -150$. The smaller eigenvalue λ_1 corresponds to the rate at which the ice line approaches its equilibrium and agrees quite closely with the approximation $-\lambda \approx -0.38$ we derived above by ignoring the terms higher order in ε. Thus we are reassured that ignoring those higher order terms was appropriate. The larger eigenvalue λ_2 corresponds to the rate at which the surface temperature approaches its equilibrium value. It has a corresponding time constant of 0.0067 Kyr or 6.7 years. So, for this model, the surface temperature lags changes in the insolation by less than a decade, while the ice line lags by two and a half millennia.

Another interesting point to note is that the effect due to the heat of fusion of water can effectively be ignored. If we take $\Omega = 0$ in matrix (15), again multiplying by κ, we get

$$\begin{bmatrix} -0.62 & 0.0123 \\ 2940 & -150 \end{bmatrix}$$

which has eigenvalues $\lambda_1 \approx -0.38$ and $\lambda_2 \approx -150$. The response of the ice line to changes in the insolation is only marginally slower if we take the heat of fusion into account. For most purposes, it would be sufficient to take $\Omega = 0$ in all the computations above.

Discussion

From the dynamical systems perspective, we have reduced an infinite dimensional system down to a simple one dimensional system, a reduction originally accomplished by Widiasih [3]. Our contribution is to use a quadratic approximation first to reduce the system to five dimensions, then to use invariant manifold theory to reduce the ice line dynamics the single ordinary differential equation

$$\dot{\eta} = \varepsilon h(\eta),$$

where h is given by equation (14). Although the actual equation for h looks very complicated, it is in fact a cubic polynomial in the variable η. All of the parameters have been estimated in the literature, or follow from first principles, except for the parameter ε, which was introduced by Widiasih [3]. Here we have found a value for ε (3.9×10^{-13}) so that the equation reproduces the delay found in some of the paleoclimate data. The values of all these parameters are collected in Appendix 1.

A refinement of our work would involve simulating solutions of the equation, using computed Milankovitch cycles from various paleoclimate regimes. As we pointed out above, the function h depends on the eccentricity e and the obliquity β. We can therefore write

$$\dot{\eta} = \varepsilon h(\eta, e(t), \beta(t)),$$

where

$$h(\eta, e, \beta) = \Phi_0(\eta, e, \beta) + \frac{Q(e) s_2 (1-\alpha_0)}{B+C} p_2(\eta) - T_c = 0$$

and where

$$\Phi_0(\eta, e, \beta) = \frac{1}{B}\Big(Q(e)(1-\alpha_0) - A + C\frac{Q(e)(\alpha_1 - \alpha_2)}{B+C}\big(\eta - \frac{1}{2} + s_2(\beta) P_2(\eta)\big)\Big).$$

This simple ordinary differential equation could be used to simulate the effect of the Milankovitch cycles on the ice line. Once the variable $\eta(t)$ is computed, one can find $w(t)$ using equation (16), which, to lowest order, is just

$$w(t) = \Phi_0(\eta(t), e(t), \beta(t)).$$

The three variables defining the two dimensional invariant subspace can be determined by equations (11):

$$\begin{aligned} z(t) &= \frac{Q(e(t))(\alpha_2 - \alpha_1)}{B + C}, \\ u_2(t) &= \frac{Q(e(t))s_2(\beta(t))(1 - \alpha_1)}{B + C}, \\ v_2(t) &= \frac{Q(e(t))s_2(\beta(t))(1 - \alpha_2)}{B + C}. \end{aligned}$$

Inverting the transformation given by equations (10), one computes

$$\begin{aligned} u_0(t) &= w(t) + z(t)/2, \\ v_0(t) &= w(t) - z(t)/2. \end{aligned}$$

The global mean temperature as a function of time can be computed using equation (9):

$$\overline{T}(t) = \eta(t)u_0(t) + (1 - \eta(t))v_0(t) + P_2(\eta(t))\left(u_2(t) - v_2(t)\right).$$

Finally, equations (6) and (8) give us the temperature profile as a function of time:

$$T(y, t) = \begin{cases} u_0(t) + u_2(t)p_2(y), & y < \eta(t), \\ v_0(t) + v_2(t)p_2(y), & y > \eta(t), \end{cases}$$

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Appendix 1

The table shows the values of the parameters discussed in this paper. Most of the values were taken from Tung's book [4], which gives references to the original sources. The numbers in the "Source" column indicate equation numbers in this paper. The units of ϵ (indicated by "$*$") are a bit hard to fit into a table, since $\sigma\epsilon$ has units of $m^2/s/K$, where σ is the surface area of the Earth in square meters.

Parameter	Value	Units	Source
Q	343	W/m^2	Tung
A	202	W/m^2	Tung
B	1.9	$W/m^2/K$	Tung
C	3.04	$W/m^2/K$	Tung
α_1	0.32	dimensionless	Tung
α_2	0.62	dimensionless	Tung
T_c	-10	°C	Tung
R	4×10^8	$J/m^2/K$	(2)
Ω	1.5×10^{11}	J/m^2	(4)
ϵ	3.9×10^{-13}	$*$	(20)

Appendix 2

Here we derive equation (18), which is a formula in terms of obliquity of the coefficient of the quadratic Legendre polynomial in the expansion of the distribution of the insolation as a function of latitude and which is reproduced here for the reader's convenience.

$$s_2(\beta) = \frac{5}{16}(-2 + 3\sin^2\beta).$$

We begin with the following expression for the insolation distribution, as derived by McGehee and Lehman [6]:

$$s(y, \beta) = \frac{2}{\pi^2} \int_0^{2\pi} \sqrt{1 - \left(\sqrt{1 - y^2}\sin\beta\cos\theta - y\cos\beta\right)^2} d\theta. \tag{21}$$

Since the Legendre polynomials are orthogonal, the second order coefficient is just

$$s_2(\beta) = \frac{\int_0^1 p_2(y)s(y,\beta)dy}{\int_0^1 p_2(y)^2 dy}.$$

Recall that the second order Legendre polynomials is

$$p_2(y) = \frac{1}{2}(3y^2 - 1).$$

An elementary computation yields

$$\int_0^1 p_2(y)^2 dy = \frac{1}{5},$$

which implies that

$$\begin{aligned} s_2(\beta) &= 5\int_0^1 p_2(y)s(y,\beta)dy \\ &= \frac{15}{2}\int_0^1 y^2 s(y,\beta)dy - \frac{5}{2}\int_0^1 s(y,\beta)dy. \end{aligned}$$

Since the insolation distribution s was chosen so that $\int_0^1 s(y,\beta)dy = 1$, we have

$$s_2(\beta) = \frac{15}{2}\int_0^1 y^2 s(y,\beta)dy - \frac{5}{2}. \qquad (22)$$

We shall show that

$$\int_0^1 y^2 s(y,\beta)dy = \frac{1}{4} + \frac{1}{8}\sin^2\beta, \qquad (23)$$

which, combined with equation (22), yields

$$s_2(\beta) = \frac{15}{2}\left(\frac{1}{4} + \frac{1}{8}\sin^2\beta\right) - \frac{5}{2} = \frac{5}{16}(-2 + 3\sin^2\beta),$$

and establishes equation (18).

We have only left to establish equation (23). We begin by showing that the insolation distribution is an even function of y:

$$\begin{aligned} s(-y,\beta) &= \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1 - \left(\sqrt{1-y^2}\sin\beta\cos\theta + y\cos\beta\right)^2}d\theta \\ &= \frac{2}{\pi^2}\int_{-\pi}^{\pi}\sqrt{1 - \left(\sqrt{1-y^2}\sin\beta\cos\theta + y\cos\beta\right)^2}d\theta \\ &= \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1 - \left(\sqrt{1-y^2}\sin\beta\cos(\gamma-\pi) + y\cos\beta\right)^2}d\gamma \\ &= \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1 - \left(-\sqrt{1-y^2}\sin\beta\cos\gamma + y\cos\beta\right)^2}d\gamma \\ &= \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1 - \left(\sqrt{1-y^2}\sin\beta\cos\gamma - y\cos\beta\right)^2}d\gamma \\ &= s(y,\beta). \end{aligned}$$

Using this property and equation (21), we compute:

$$
\begin{aligned}
&\int_0^1 y^2 s(y,\beta)\,dy \\
&\quad = \frac{1}{2}\int_{-1}^{1} y^2 s(y,\beta)\,dy \\
&\quad = \frac{1}{\pi^2}\int_{-1}^{1}\int_0^{2\pi} y^2 \sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\theta - y\cos\beta\right)^2}\, d\theta dy \\
&\quad = \frac{1}{\pi^2}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} \sin^2\phi \sqrt{1-(\cos\phi\sin\beta\cos\theta-\sin\phi\cos\beta)^2}\cos\phi\, d\theta d\phi.
\end{aligned}
$$

If we switch to Cartesian coordinates,

$$
\begin{aligned}
\xi &= \cos\phi\cos\theta, \\
\eta &= \cos\phi\sin\theta, \\
\zeta &= \sin\phi,
\end{aligned}
$$

and note that

$$
\cos\phi\, d\theta d\phi = \xi d\eta d\zeta - \eta d\xi d\zeta + \zeta d\xi d\eta,
$$

we see that

$$
\begin{aligned}
&\int_0^1 y^2 s(y,\beta)\,dy \\
&\qquad = \frac{1}{\pi^2}\iint\limits_{S^2} \zeta^2 \sqrt{1-(\xi\sin\beta-\zeta\cos\beta)^2}(\xi d\eta d\zeta - \eta d\xi d\zeta + \zeta d\xi d\eta)\,,
\end{aligned}
$$

where S^2 denotes the unit ball. We now make the orthogonal transformation

$$
\begin{bmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\zeta} \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}.
$$

Note that, since the determinant of this transformation is one, the two-form used above is invariant. Hence,

$$
\begin{aligned}
&\int_0^1 y^2 s(y,\beta)\,dy \\
&\qquad = \frac{1}{\pi^2}\iint\limits_{S^2} \left(\hat{\xi}\sin\beta + \hat{\zeta}\cos\beta\right)^2 \sqrt{1-\hat{\zeta}^2}\left(\hat{\xi} d\hat{\eta} d\hat{\zeta} - \hat{\eta} d\hat{\xi} d\hat{\zeta} + \hat{\zeta} d\hat{\xi} d\hat{\eta}\right) \\
&\qquad = I_1\sin^2\beta + 2I_2\sin\beta\cos\beta + I_3\cos^2\beta\,,
\end{aligned}
$$

where

$$I_1 = \frac{1}{\pi^2} \iint\limits_{S^2} \hat{\xi}^2 \sqrt{1-\hat{\zeta}^2} \left(\hat{\xi} d\hat{\eta} d\hat{\zeta} - \hat{\eta} d\hat{\xi} d\hat{\zeta} + \hat{\zeta} d\hat{\xi} d\hat{\eta} \right),$$

$$I_2 = \frac{1}{\pi^2} \iint\limits_{S^2} \hat{\xi}\hat{\zeta} \sqrt{1-\hat{\zeta}^2} \left(\hat{\xi} d\hat{\eta} d\hat{\zeta} - \hat{\eta} d\hat{\xi} d\hat{\zeta} + \hat{\zeta} d\hat{\xi} d\hat{\eta} \right),$$

$$I_1 = \frac{1}{\pi^2} \iint\limits_{S^2} \hat{\zeta}^2 \sqrt{1-\hat{\zeta}^2} \left(\hat{\xi} d\hat{\eta} d\hat{\zeta} - \hat{\eta} d\hat{\xi} \hat{\zeta} + \hat{\zeta} d\hat{\xi} d\hat{\eta} \right).$$

Switching back to spherical coordinates,

$$\begin{aligned} \hat{\xi} &= \cos\hat{\phi}\cos\hat{\theta}, \\ \hat{\eta} &= \cos\hat{\phi}\sin\hat{\theta}, \\ \hat{\zeta} &= \sin\hat{\phi}, \end{aligned}$$

we have that

$$\begin{aligned} I_1 &= \frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \left(\cos\hat{\phi}\cos\hat{\theta} \right)^2 \cos\hat{\phi}\cos\hat{\phi}\, d\hat{\theta} d\hat{\phi} \\ &= \frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} \left(\int_0^{2\pi} \cos^2\hat{\theta}\, d\hat{\theta} \right) \cos^4\hat{\phi}\, d\hat{\phi} = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos^4\hat{\phi}\, d\hat{\phi} = \frac{3}{8}, \end{aligned}$$

$$\begin{aligned} I_2 &= \frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \left(\cos\hat{\phi}\cos\hat{\theta}\sin\hat{\phi} \right) \cos\hat{\phi}\cos\hat{\phi}\, d\hat{\theta} d\hat{\phi} \\ &= \frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} \left(\int_0^{2\pi} \cos\hat{\theta}\, d\hat{\theta} \right) \sin\hat{\phi}\cos^3\hat{\phi}\, d\hat{\phi} = 0, \end{aligned}$$

$$\begin{aligned} I_3 &= \frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \sin^2\hat{\phi}\cos\hat{\phi}\cos\hat{\phi}\, d\hat{\theta} d\hat{\phi} \\ &= \frac{2}{\pi^2} \int_{-\pi/2}^{\pi/2} \sin^2\hat{\phi}\cos^2\hat{\phi}\, d\hat{\phi} = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \sin^2 2\hat{\phi}\, d\hat{\phi} = \frac{1}{4}. \end{aligned}$$

Therefore,

$$\int_0^1 y^2 s(y,\beta) dy = \frac{3}{8}\sin^2\beta + \frac{1}{4}\cos^2\beta = \frac{1}{4} + \frac{1}{8}\sin^2\beta,$$

which establishes equation (23) and completes the derivation of equation (18).

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