Approximating Planetary Mean Annual Insolation

Alice Nadeau* Richard McGehee†

July 29, 2022

Abstract

Simple energy balance models of planetary climate use the incoming solar radiation (*insolation*) as input. For these models, the mean annual insolation as a function of latitude is often the appropriate variable. This function also depends on the obliquity, or axial tilt, of the planet, which may change enough over long time spans to affect the climate. The insolation as a function of obliquity and latitude can be expressed as a definite integral, which can be approximated by a polynomial, cutting computing time significantly.

1 Introduction

Interest in planetary climates has increased recently with the explosion of knowledge about extrasolar planets [10], the New Horizons fly-by of Pluto [16], the growing knowledge about Mars [11] and some of the moons Saturn [8, 9]. The classical models of Budyko [5] and Sellers [15] enable one to obtain crude estimates of climate constraints in terms of the orbital parameters. A main component of these models is insolation (from *in*coming *sol*ar radi*ation*) which give the amount of energy reaching the planet's surface.

The insolation at any point on a planet is a function of the latitude and longitude of the point, the orbital parameters (semi-major axis, obliquity[1], and precession angle), the position of the planet along its orbit, and the solar energy output. Using Kepler's laws and integrating over an entire year, one can show that the global annual average power flux (Watts per square meter) is given by

$$Q(a,e) = \frac{K\sqrt{a}}{\sqrt{1-e^2}},$$

where K is proportional to the solar output, a is the semi-major axis of the elliptical orbit, and e is its eccentricity [12].

For a rapidly rotating planet, the orbital parameters and the position of the planet do not change substantially during a day, leading to a simplification of distribution by latitude of the annual average insolation. In this case, annual average insolation distributionreduces to a function only of the obliquity and latitude [6, 12, 19] and is given by

$$s(\alpha,\beta) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-(\cos\alpha\sin\beta\sin\gamma-\sin\alpha\cos\beta)^2}d\gamma, \tag{1}$$

where α is the latitude, β is the obliquity, and γ is the longitude. Since the latitude is measured up and down from the equator, we have $-\pi/2 \le \alpha \le \pi/2$, while, since obliquity is the angle between

*nadea093@umn.edu; School of Mathematics, University of Minnesota
†School of Mathematics, University of Minnesota
[1]tilt of spin axis with respect to the orbital plane

the angular momentum vector of the planetary orbit and the angular momentum vector of the planetary spin, we have $0 \leq \beta \leq \pi$.

For each fixed obliquity β, $s(\alpha, \beta)$ is the distribution of insolation across the surface of the planet, so the annual average insolation at latitude α is given by

$$Q(a, e)s(\alpha, \beta).$$

In this paper, we derive an infinite series representation of the function $s(\alpha, \beta)$ in terms of Legendre polynomials (Theorem 1). Truncating this series gives a polynomial approximation for the insolation function, allowing for faster computation of the values by avoiding the numerical approximation of the integral. A quadratic approximation of s for the Earth's obliquity has been used extensively [1, 2, 12, 13, 18, 21]. However, for other planets, a quadratic approximation fails to capture the qualitative behavior of the insolation as a function of latitude. In a previous paper [14], the authors introduced a sixth order polynomial approximation and showed that it captures the characteristics of Pluto's insolation. Here we generalize that approximation and place it on a firm mathematical foundation using classical results about spherical harmonics.

In modeling studies, it is usually most appropriate to take sine of the latitude instead of latitude. Sine of latitude is used so that the infinitesimal $dy = \cos\theta d\theta$ is proportional to the area of the latitudinal strip parallel to y. Taking cosine of obliquity makes s symmetric in sine of the latitude (η) and cosine of obliquity (ζ):

$$s(\eta, \zeta) = \frac{2}{\pi^2} \int_0^{2\pi} \sqrt{1 - \left(\sqrt{1-\eta^2}\sqrt{1-\zeta^2}\sin\gamma - \eta\zeta\right)^2}\, d\gamma.$$

The Legendre polynomials $P_i(x), i = 0, 1, 2, \ldots$, form a complete orthogonal set in the space $L^2([-1, 1])$ with the properties P_i has degree i and $P_i(1) = 1$ [3]. Therefore, the products $P_{i,j}(x, y) = P_i(x)P_j(y)$ form a complete orthogonal set in the space $L^2([-1, 1] \times [-1, 1])$. Thus we can write

$$s(\eta, \zeta) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} P_i(\zeta) P_j(\eta). \tag{2}$$

The series naturally converges in L^2. One can show that the convergence is also pointwise, but, since we will not use that fact here, we leave the proof to the reader. Instead, we simply abuse notation and interpret the equal sign in equation (2) as equality in L^2. Surprisingly, c_{ij} is diagonal, in particular:

Theorem 1. *The annual average insolation distribution function can be written*

$$s(\eta, \zeta) = \sum_{n=0}^{\infty} A_{2n} P_{2n}(\zeta) P_{2n}(\eta),$$

where P_{2n} is the Legendre polynomial of degree $2n$, and where

$$A_{2n} = \frac{(-1)^n (4n+1)}{2^{2n-1}} \sum_{k=0}^{n} \binom{2n}{n-k} \binom{2n+2k}{2k} \binom{1/2}{k+1}.$$

Here we are using the standard notation

$$\binom{r}{j} = \frac{r(r-1)\cdots(r-j+1)}{j!}.$$

The proof of this theorem relies on two main lemmas which are stated and proved in the following two sections. The proof of the theorem is given in Section 4. The discussion in Section 5 elaborates on the main point of this study—approximating a planet's insolation distribution with polynomials.

2 Rotational symmetries of square integrable functions on S^2

The proof of Lemma 1 relies on rotational symmetries of the spherical harmonics; however some ambiguities can arise when discussing rotations. For this reason, we first lay out definitions that will be used in the proof.

In $\mathbb{R}^3$, any orientation can be achieved by composing three elemental rotations, starting from a known standard direction. Let the standard direction be (x, y, z) and the elemental matrices be

$$R_1(\cdot) = \begin{bmatrix} \cos(\cdot) & 0 & \sin(\cdot) \\ 0 & 1 & 0 \\ -\sin(\cdot) & 0 & \cos(\cdot) \end{bmatrix} \quad \text{and} \quad R_2(\cdot) = \begin{bmatrix} \cos(\cdot) & -\sin(\cdot) & \\ \sin(\cdot) & \cos(\cdot) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The rotation $R_1(\cdot)$ rotates the (y, z)-plane around the x-axis using the right hand rule while $R_2(\cdot)$ rotates the (x, y)-plane around the z-axis using the right hand rule. The rotation matrix $R(\rho, \beta, \alpha)$ defined by

$$R(\rho, \beta, \alpha) = R_2(\rho)R_1(\beta)R_2(\alpha)$$

is intended to operate by pre-multiplying the column vector (x, y, z) and represents an active rotation.[2] Each matrix is meant to represent the composition of intrinsic rotations.[3] We have

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = R(\rho, \beta, \alpha) \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Furthermore, write each set of coordinates in spherical coordinates then

$$\begin{aligned} x &= \cos\theta \sin\phi \\ y &= \sin\theta \sin\phi \\ z &= \cos\phi \end{aligned} \qquad \text{and} \qquad \begin{aligned} \hat{x} &= \cos\hat{\theta} \sin\hat{\phi} \\ \hat{y} &= \sin\hat{\theta} \sin\hat{\phi} \\ \hat{z} &= \cos\hat{\phi} \end{aligned}$$

where θ and $\hat{\theta}$ are the azimuth angles as measured counterclockwise from the x- and $\hat{x}$-axes, respectively and ϕ and $\hat{\phi}$ are the usual polar angles. Let $R_{\rho,\beta,\alpha}$ denote the same rotation described by $R(\rho, \beta, \alpha)$ but which relates $(\hat{\theta}, \hat{\phi})$ to (θ, ϕ) so that

$$(\hat{\theta}, \hat{\phi}) = R_{\rho,\beta,\alpha}(\theta, \phi), \quad \text{and} \quad (\theta, \phi) = R^{-1}_{\rho,\beta,\alpha}(\hat{\theta}, \hat{\phi}).$$

Note that

$$\cos\hat{\phi} = \hat{z} = \hat{z}(\alpha, \beta, \theta, \phi) = \cos\beta\cos\phi - \sin\beta\sin\phi\cos(\alpha + \theta). \tag{3}$$

Lemma 1. *Suppose $f(\theta, \phi)$ is a square integrable function on $S^2 \subset \mathbb{R}^3$ where θ is the azimuthal angle and ϕ is the polar angle. Suppose also that $R_{\rho,\beta,\alpha}$ is a proper Euler rotation as described above so that*

$$(\hat{\theta}, \hat{\phi}) = R_{\rho,\beta,\alpha}(\theta, \phi)$$

and there exists a coordinate system where f depends only on the cosine of the polar angle (i.e. $f(R^{-1}_{\rho,\beta,\alpha}(\hat{\theta}, \hat{\phi})) = \tilde{f}(\cos\hat{\phi})$). Then we have

$$\tilde{f}(\cos\hat{\phi}) = \tilde{f}(\alpha, \beta, \theta, \phi) = \sum_{n=0}^{\infty} \tilde{c}_n \left(\sum_{k=-n}^{n} Y_n^k(\alpha, \beta) Y_n^k(\theta, \phi) \right)$$

[2]The matrices act on the coordinates of vectors defined in the initial fixed reference frame and give, as a result, the coordinates of a rotated vector defined in the same reference frame.

[3]Rotations around the axes of the rotating reference frame.

where $Y_n^k(\theta,\phi) = N_{n,k}e^{ik\theta}P_n^k(\cos\phi)$ *is the* $n-k$ *spherical harmonic with normalizing factor*

$$N_{n,k} = (-1)^k\sqrt{\frac{2n+1}{4\pi}\frac{(n-k)!}{(n+k)!}},$$

associated Legendre polynomial P_n^k*, and*

$$\tilde{c}_n = 2\pi\int_{-1}^{1}\tilde{f}(\hat{z})P_n(\hat{z})d\hat{z}$$

where P_n *is the* n*-th Legendre polynomial. Furthermore, for* β *and* θ *fixed we have*

$$\int_0^{2\pi}\tilde{f}(\alpha,\beta,\theta,\phi)d\theta = \int_0^{2\pi}\tilde{f}(\alpha,\beta,\theta,\phi)d\alpha = \sum_{n=0}^{\infty}\tilde{b}_nP_n(\cos\beta)P_n(\cos\phi)$$

where

$$\tilde{b}_n = \pi(2n+1)\int_{-1}^{1}\tilde{f}(\hat{z})P_n(\hat{z})d\hat{z}.$$

Proof. Notice that $\tilde{f}(\cos\hat{\phi}) = \tilde{f}(\hat{z})$. As stated above, the Legendre polynomials $P_i(x), i = 0,1,2,\ldots,$ form a complete orthogonal set in the space $L^2([-1,1])$ with the properties P_i has degree i and $P_i(1) = 1$ [3].

Expanding $\tilde{f}(\hat{z})$ into its Legendre series gives

$$\tilde{f}(\hat{z}) = \sum_{n=0}^{\infty}c_nP_n(\hat{z}), \tag{4}$$

where

$$c_n = \frac{\int_{-1}^{1}\tilde{f}(\hat{z})P_n(\hat{z})d\hat{z}}{\int_{-1}^{1}P_n(\hat{z})^2d\hat{z}} = \frac{2n+1}{2}\int_{-1}^{1}\tilde{f}(\hat{z})P_n(\hat{z})d\hat{z}$$

and P_n is the n-th Legendre polynomial. The series naturally converges in L^2. One can show that the convergence is also pointwise, but, since we will not use that fact here, we leave the proof to the reader. Instead, we simply abuse notation and interpret the equal sign in Equation (4) as equality in L^2. Changing back to spherical coordinates yields

$$\tilde{f}(\cos\hat{\phi}) = \sum_{n=0}^{\infty}c_nP_n(\cos\hat{\phi}). \tag{5}$$

The addition formula for spherical harmonics [7, 20] says that

$$P_n(\cos\omega) = \frac{4\pi}{2n+1}\sum_{k=-n}^{n}Y_n^k(\theta,\phi)Y_n^k(\theta',\phi')^*$$

where

$$\cos\omega = \cos\phi\cos\phi' + \sin\phi\sin\phi'\cos(\theta-\theta') \tag{6}$$

and $Y_n^k(\theta,\phi) = N_{n,k}e^{ik\theta}P_n^k(\cos\phi)$ is the $n-k$ spherical harmonic. Recall that

$$\cos\hat{\phi} = \cos\beta\cos\phi - \sin\beta\sin\phi\cos(\alpha+\theta)$$

which can be written in the form of Equation (6) by letting $\beta = -\tilde{\beta}$ and $\alpha = -\tilde{\alpha}$. Then for any n

$$\begin{aligned} P_n(\cos\hat{\phi}) &= \frac{4\pi}{2n+1} \sum_{k=-n}^{n} Y_n^k(\theta,\phi) Y_n^k(\tilde{\alpha},\tilde{\beta})^* \\ &= \frac{4\pi}{2n+1} \sum_{k=-n}^{n} Y_n^k(\theta,\phi) Y_n^k(\alpha,\beta) \end{aligned}$$

because $Y_n^k(\alpha,\cdot) = Y_n^k(-\alpha,\cdot)^*$ and Y_n^k is even in the second argument. Substituting the above into Equation (5) yields

$$\tilde{f}(\cos\hat{\phi}) = \sum_{n=0}^{\infty} \frac{4\pi c_n}{2n+1} \left(\sum_{k=-n}^{n} Y_n^k(\alpha,\beta) Y_n^k(\theta,\phi) \right).$$

Writing $\tilde{f}(\cos\hat{\phi}) = \tilde{f}(\cos\beta\cos\phi - \sin\beta\sin\phi\cos(\alpha+\theta)) = \tilde{f}(\alpha,\beta,\theta,\phi)$ gives the formula from the statement of the theorem.

To prove that

$$\int_0^{2\pi} \tilde{f}(\alpha,\beta,\theta,\phi)d\theta = \int_0^{2\pi} \tilde{f}(\alpha,\beta,\theta,\phi)d\alpha = 2\pi \sum_{n=0}^{\infty} P_n(\cos\beta) P_n(\cos\phi)$$

notice that

$$\int_0^{2\pi} \sum_{n=0}^{\infty} \tilde{c}_n \sum_{k=-n}^{n} Y_n^k(\theta,\phi) Y_n^k(\alpha,\beta) d\theta = \sum_{n=0}^{\infty} \tilde{c}_n \int_0^{2\pi} \sum_{k=-n}^{n} Y_n^k(\theta,\phi) Y_n^k(\alpha,\beta) d\theta$$

because the function is absolutely integrable over a finite interval. We see that

$$\begin{aligned} \int_0^{2\pi} \sum_{k=-n}^{n} Y_n^k(\theta,\phi) Y_n^k(\alpha,\beta) d\theta &= \sum_{k=-n}^{n} Y_n^k(\alpha,\beta) N_{n,k} P_n^k(\cos\theta) \int_0^{2\pi} e^{ik\theta} d\theta \\ &= \sum_{k=-n}^{n} Y_n^k(\alpha,\beta) N_{n,k} P_n^k(\cos\phi)(2\pi\delta_{k,0}) \end{aligned}$$

where $\delta_{k,0}$ is the Kronecker Delta function indicating that the integral is zero except when $k = 0$. Then

$$\begin{aligned} \int_0^{2\pi} f(\alpha,\beta,\theta,\phi)d\theta &= 2\pi \sum_{n=0}^{\infty} \tilde{c}_n \left(N_{n,0}\right)^2 P_n(\cos\beta) P_n(\cos\phi) \\ &= \sum_{n=0}^{\infty} \left(\frac{2n+1}{2}\right) \tilde{c}_n P_n(\cos\beta) P_n(\cos\phi) \\ &= \sum_{n=0}^{\infty} \tilde{b}_n P_n(\cos\beta) P_n(\cos\phi) \end{aligned}$$

Integrating in α yields the same result. $\square$

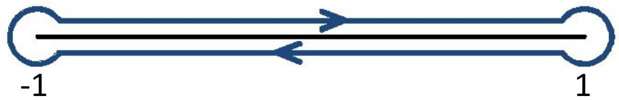

Figure 1: The contour, C, used in the calculation of the integral in Lemma 2.

3 Lemma 2

The second lemma is instrumental for computing the coefficients of the Legendre series for the insolation distribution function.

Lemma 2. *For any non-negative integer k,*

$$\int_{-1}^{1} \sqrt{1-x^2}\, x^{2k}\, dx = (-1)^k \pi \binom{1/2}{k+1}.$$

Proof. Let

$$a_{2k} = \int_{-1}^{1} \sqrt{1-x^2} x^{2k} dx.$$

We compute a_{2k} via the integral around the contour C shown in Figure 1. The integral around C is given by the residue at infinity of the integrand. Namely

$$\begin{aligned}
2a_{2k} &= 2\int_{-1}^{1} \sqrt{1-x^2} x^{2k} dx \\
&= \int_C \sqrt{1-z^2} z^{2k} z^2 dz \\
&= -2\pi i \operatorname{Res}(\sqrt{1-z^2} z^{2k}, \infty) \\
&= 2\pi i \operatorname{Res}\left(\sqrt{1-\frac{1}{z^2}} \frac{1}{z^{2k}} \frac{1}{z^2}, 0\right) \\
&= -2\pi \operatorname{Res}\left(\frac{\sqrt{1-z^2}}{z^{2k+3}}, 0\right)
\end{aligned}$$

The series expansion of $\sqrt{1-z^2}$ is given by

$$\sqrt{1-z^2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} z^{2n}.$$

were $\binom{1/2}{n}$ is a standard generalized binomial coefficient. Multiplying through by $1/(z^{2k+3})$ yields

$$\frac{\sqrt{1-z^2}}{z^{2k+3}} = \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} z^{2n-2k-3}.$$

Then we calculate the residue as

$$\operatorname{Res}\left(\frac{\sqrt{1-z^2}}{z^{2k+3}}, 0\right) = (-1)^{k+1} \binom{1/2}{k+1}$$

which establishes the formula given in the lemma. □

4 Proof of Theorem 1

Proof. (Theorem 1)

McGehee and Lehman [12] showed that the annual average insolation at a point on the surface of a non-rotating planet is proportional to the sine of the co-latitude in the ecliptic coordinates. Normalizing so that the total insolation is 2 gives us the insolation distribution

$$I(\hat{\phi}) = \frac{2}{\pi^2}\sin\hat{\phi} = \frac{2}{\pi^2}\sqrt{1-\hat{z}^2}$$

where $\hat{\phi}$ is the polar angle in the ecliptic coordinates. The factor of two maintains compatibility with the usual normalization of over one hemisphere.

Applying Lemma 1 tells us that

$$I(\hat{\phi}) = I(0,\beta,\theta,\phi) = \sum_{n=0}^{\infty}\tilde{c}_n\left(\sum_{k=-n}^{n}Y_n^k(0,\beta)Y_n^k(\theta,\phi)\right).$$

Since we want the annual average insolation, we integrate over θ (a day) as was done in McGehee and Lehman (2012) [12]. Lemma 1 then tells us

$$s(\beta,\phi) = \sum_{n=0}^{\infty}A_nP_n(\cos\beta)P_n(\cos\phi)$$

where

$$\begin{aligned} A_n &= \pi(2n+1)\int_{-1}^{1}\frac{2}{\pi^2}\sqrt{1-z^2}P_n(z)dz \\ &= \frac{2(2n+1)}{\pi}\int_{-1}^{1}\sqrt{1-z^2}P_n(z)dz \end{aligned}$$

If n is odd, the integral is zero. Furthermore, the coefficients of the Legendre polynomials are well-known [3]. In particular,

$$P_{2n}(x) = \sum_{k=0}^{n}p_{2n,2k}x^{2k},$$

where

$$p_{2n,2k} = \frac{(-1)^{n-k}}{4^n}\binom{2n}{n-k}\binom{2n+2k}{2n}.$$

Therefore, we can write

$$\begin{aligned} A_{2n} &= \frac{2(4n+1)}{\pi}\int_{-1}^{1}\sqrt{1-z^2}P_{2n}(z)dz \\ &= \frac{2(4n+1)}{\pi}\int_{-1}^{1}\sqrt{1-z^2}\left(\sum_{k=0}^{n}p_{2n,2k}z^{2k}\right)dz \\ &= \frac{2(4n+1)}{\pi}\sum_{k=0}^{n}p_{2n,2k}\int_{-1}^{1}\sqrt{1-z^2}z^{2k}dz \\ &= \frac{2(4n+1)}{\pi}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{4^n}\binom{2n}{n-k}\binom{2n+2k}{2n}\int_{-1}^{1}\sqrt{1-z^2}z^{2k}dz \end{aligned}$$

Applying Lemma 2 gives

$$A_{2n} = \frac{(4n+1)(-1)^n}{2^{2n-1}} \sum_{k=0}^{n} \binom{2n}{n-k} \binom{2n+2k}{2n} \binom{1/2}{k+1}$$

which proves the formula. □

5 Discussion

Our aim is to approximate the function s with polynomials. Truncating the series at $n = N$ produces a polynomial of degree $2N$ in each of the two variables which is the best approximation to s in the L^2 norm by a polynomial of that degree.

More precisely, let $\mathcal{P}_k$ be the space of polynomials of degree k in in each of two variables. That is, let

$$\mathcal{P}_k = \operatorname{span}\{x^i y^j : 0 \leq i \leq k, 0 \leq j \leq k\},$$

and let

$$\sigma_{2N}(\eta, \zeta) = \sum_{n=0}^{N} A_{2n} P_{2n}(\zeta) P_{2n}(\eta).$$

An appeal to standard approximation theory yields

$$\|s - \sigma_{2N}\|_2 \leq \|s - \sigma\|_2, \qquad \text{for all } \sigma \in \mathcal{P}_{2N}.$$

If we fix the obliquity, *i.e.*, we fix ζ, then the function $\sigma_{2N}(\cdot, \zeta)$ is the best approximation to $s(\cdot, \zeta)$ in the space $L^2([-1, 1])$ by a polynomial of degree $2N$. Appealing again to standard approximation theory, we have

$$\|s(\cdot, \zeta) - \sigma_{2N}(\cdot, \zeta)\|_2 \leq \|s(\cdot, \zeta) - \sigma(\cdot, \zeta)\|_2 \text{ for all } \sigma \in \mathcal{P}_{2N}.$$

As described above, the truncation $\sigma_{2N}(\cdot, \zeta)$ is a good L^2 estimation of insolation distribution function $s(\cdot, \zeta)$. In other words, for a fixed obliquity, we have a polynomial function of latitude that is a good least squares estimate for the insolation as a function of latitude.

Acknowledgements

The authors gratefully acknowledge the support of the Mathematics and Climate Research Network (NSF Grants DMS-0940366 and DMS-0940363).

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