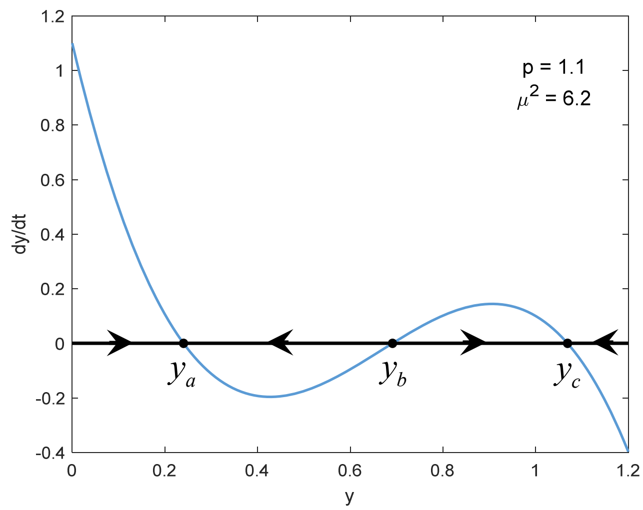
Set 5 Solutions

Please refer to the slides from the lecture on October 24 discussing Cessi's model. The equation and one of the figures are repeated here for your convenience. For the exercises, assume that $\mu^2 = 6.2$.

$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + p$$



1. Compute (to four significant figures) the value of p_1 satisfying the property:

$$1.1 < p < p_1 \Rightarrow \text{ the differential equation has exactly two stable rest points}$$
$$p_1 < p \Rightarrow \text{ the differential equation has exactly one stable rest point}$$

Hint: p_1 occurs when the two rest points y_a and y_b in the figure have collided to become one rest point.

Solution. Let $f(y) = -\left(1+\mu^2\left(y-1\right)^2\right)y + p$, and let η_1 be the point between y_a and y_b where $f'(y) = -\mu^2\left(3y^2 - 4y + \left(1+\frac{1}{\mu^2}\right)\right) = 0$. The quadratic formula yields $y = \frac{2 \pm \sqrt{4-3\left(1+1/\mu^2\right)}}{3}$ and hence that $\eta_1 = \frac{2-\sqrt{1-3/\mu^2}}{3} \approx 0.42719$. Since $f(\eta_1) = 0$, we have that $p_1 = \left(1+\mu^2\left(\eta_1 - 1\right)^2\right)\eta_1 \approx 1.29621$, which, to four significant figures is $\boxed{p_1 \approx 1.296}$.

2. Compute (to four significant figures) the value of p_2 satisfying the property:

$$p_2 < p < 1.1 \Rightarrow \text{ the differential equation has exactly two stable rest points}$$
$$p < p_2 \Rightarrow \text{ the differential equation has exactly one stable rest point}$$

Solution. Let η_2 be the point between y_b and y_c where $f'(y)=0$. Following the use of the quadratic formula in problem 1 yields $\eta_2 = \dfrac{2+\sqrt{1-3/\mu^2}}{3} \approx 0.90614$. Since $f(\eta_2)=0$, we have that $p_2 = \left(1+\mu^2(\eta_2-1)^2\right)\eta_2 \approx 0.95563$, which, to four significant figures is $\boxed{p_2 \approx 0.9556}$.

3. Suppose the system starts at the equilibrium y_a when $p=1.1$. The value of p is raised to the value 1.4 and left there long enough for y to become close to the only stable equilibrium. Discuss how to change the value of p to return the system to a value of y close to y_a.

Solution. One would have to reduce the value of p to below p_2 and leave it then long enough for the system to come close to the only stable equilibrium associated with y_a. Then one could return the value of p to 1.1, and the system would return to value of y close to y_a.