Math 5490
Topics in Applied Mathematics
Introduction to the Mathematics of Climate

Fall 2023
1:25 - 3:20 Tuesdays and Thursdays
Amundson Hall 162

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Math 5490
Energy Balance

What determines the Earth's surface temperature?

	Model	Equilibrium
Perfect Black Body	$R\frac{dT}{dt}=Q-\sigma T^4$	$T=\left(Q/\sigma\right)^{1/4}$
Plus Albedo	$R\frac{dT}{dt}=Q(1-\alpha)-\sigma T^4$	$T=\left((1-\alpha)Q/\sigma\right)^{1/4}$
Switch to Surface Temperature	$R\frac{dT}{dt}=Q(1-\alpha)-(A+BT)$	$T=\left((1-\alpha)Q-A\right)/B$
Dependence on Latitude	$R\frac{\partial T(y,t)}{\partial t}=Qs(y)(1-\alpha)-\left(A+BT(y,t)\right)$	$T(y)=\left((1-\alpha)Qs(y)-A\right)/B$

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Energy Balance

What determines the Earth's surface temperature?

Add Heat Transport $R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha)-(A+BT)+C(\bar{T}-T)$

global mean temperature $\bar{T}(t)=\int_0^1 T(y,t)dt$

Second Law of Thermodynamics:
Energy travels from hot places to cold places.

Why is the integral from 0 to 1?

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Symmetry Assumption

We assume that Earth is symmetric between the northern and southern hemispheres.

Is that true?

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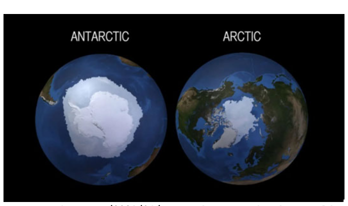
Symmetry Assumption

We assume that Earth is symmetry between the northern and southern hemispheres.

Is that true?

No, but it is good enough for now.

The south pole is in the middle of a continent surrounded by an ocean and covered in an ice sheet two miles thick.



The north pole is in the middle of an ocean surrounded by continents and covered (usually) with sea ice.

https://yaleclimateconnections.org/2021/04/researchers-examine-how-world-apart-ice-sheets-influence-each-other/

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Symmetry Assumption

We assume that Earth is symmetry between the northern and southern hemispheres.

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha)-(A+BT)+C(\bar{T}-T)$$

We consider only northern latitudes and reflect through the equator for the southern hemispheres.

$$0\le y=\text{ sine(latitude) }\le 1$$

Recall that s is a distribution:

$$\int_0^1 s(y)dy=1$$

We use the Chylek & Coakley quadratic approximation:

$$s(y)\approx 1-0.241\left(3y^2-1\right)$$

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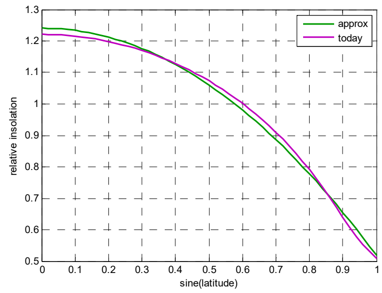
Energy Balance
Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = integral formula using obliquity of 23.5°



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Energy Balance
Budyko's Equilibrium

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + C(\bar{T}-T)$$

As before, the equilibrium solution is the temperature as a function of latitude.

$$T = T^*(y)$$

$$Qs(y)(1-\alpha) - \left(A + BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right) = 0,$$

where $\bar{T}^*$ is the global mean temperature at equilibrium, *i.e.*,

$$\bar{T}^* = \int_0^1 T^*(y)dy. \qquad \text{constant (doesn't depend on } y)$$

How can we compute the constant?

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Energy Balance
Budyko's Equilibrium

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + C(\bar{T}-T)$$

As before, the equilibrium solution is the temperature as a function of latitude.

$$T = T^*(y)$$

Solve for $T^*(y)$

$$Qs(y)(1-\alpha) - \left(A + BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right) = 0,$$

where $\bar{T}^*$ is the global mean temperature at equilibrium, *i.e.*,

$$\bar{T}^* = \int_0^1 T^*(y)dy. \qquad \text{constant (doesn't depend on } y)$$

***Assume** we can somehow compute this constant.*

$$Qs(y)(1-\alpha) - A + C\bar{T}^* - \left(BT^*(y)\right) + C\left(-T^*(y)\right) = 0,$$

$$(B+C)T^*(y) = Qs(y)(1-\alpha) - A + C\bar{T}^*$$

$$\boxed{T^*(y) = \frac{Qs(y)(1-\alpha) - A + C\bar{T}^*}{B+C}}$$

Now we can compute the constant!

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Energy Balance
Budyko's Equilibrium

$$T^*(y) = \frac{Qs(y)(1-\alpha) - A + C\bar{T}^*}{B+C}$$

$$(B+C)T^*(y) = Qs(y)(1-\alpha) - A + C\bar{T}^*$$

Integrate:

$$\int_0^1 (B+C)T^*(y)dy = \int_0^1 \left(Qs(y)(1-\alpha) - A + C\bar{T}^*\right)dy$$

$$(B+C)\underbrace{\int_0^1 T^*(y)dy}_{\bar{T}^*} = Q(1-\alpha)\underbrace{\int_0^1 s(y)dy}_{1} - A\underbrace{\int_0^1 dy}_{1} + C\bar{T}^*\underbrace{\int_0^1 dy}_{1}$$

$$(B+\cancel{C})\bar{T}^* = Q(1-\alpha) - A + \cancel{C\bar{T}^*}$$

$$\bar{T}^* = \frac{Q(1-\alpha) - A}{B}$$

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Energy Balance
Budyko's Equilibrium

$$T^*(y) = \frac{Qs(y)(1-\alpha) - A + C\bar{T}^*}{B+C}$$

$$(B+C)T^*(y) = Qs(y)(1-\alpha) - A + C\bar{T}^*$$

Integrate:

$$\int_0^1 (B+C)T^*(y)dy = \int_0^1 \left(Qs(y)(1-\alpha) - A + C\bar{T}^*\right)dy$$

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$$(B+\cancel{C})\bar{T}^* = Q(1-\alpha) - A + \cancel{C\bar{T}^*}$$

$$\boxed{\bar{T}^* = \frac{Q(1-\alpha) - A}{B}}$$

We computed the constant!

$$\boxed{T^*(y) = \frac{Qs(y)(1-\alpha) - A + C\bar{T}^*}{B+C}, \quad \text{where } \bar{T}^* = \frac{Q(1-\alpha) - A}{B}}$$

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Energy Balance
Budyko's Equilibrium

$$Qs(y)(1-\alpha) - \left(A + BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right) = 0$$

Equilibrium temperature profile: $T^*(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha) - A + C\bar{T}^*\right)$

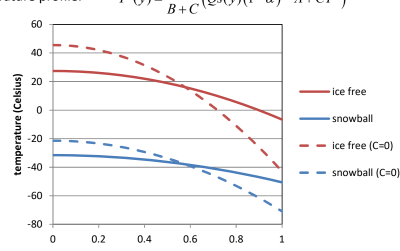
Tung*

$C = 3.04$

$\alpha = 0.32$: ice free

$\alpha = 0.62$: snowball

* K.K. Tung, *Topics in Mathematical Modeling*, Princeton U. Press, 2007



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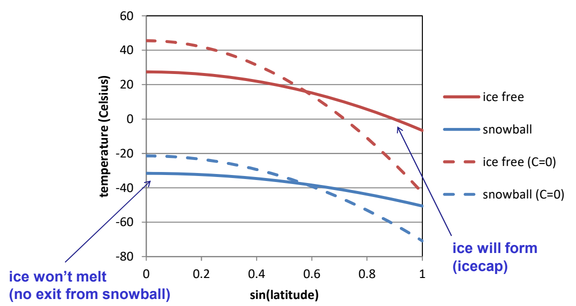
Energy Balance
Budyko's Equilibrium



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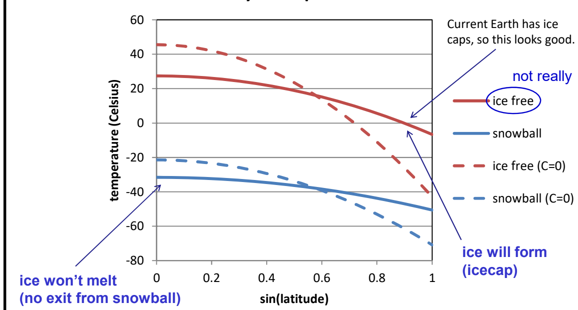
Energy Balance
Budyko's Equilibrium



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Budyko's Equation

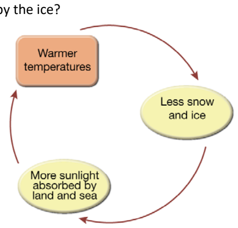
$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + C(\bar{T} - T)$$

There is still something missing.
What role is played by the ice?

Ice-albedo Feedback

temperature warms
ice melts
albedo decreases
more sunlight absorbed
temperature warms
REPEAT

Why would it stop?



http://www.i-fink.com/melting-polar-ice/

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Energy Balance
Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + C(\bar{T} - T)$$

There is still something missing.
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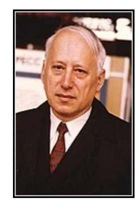

M. I. Budyko, "The effect of solar radiation variations on the climate of the Earth," **Tellus XXI**, 611-619 , 1969.

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Budyko's Model
Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + C(\bar{T} - T)$$

There is still something missing. Where is the ice?

Ice-albedo Feedback

albedo of ice: 0.62
albedo of land and water: 0.32

If the boundary between ice and no ice is at $y = \eta$, then

$$\alpha = \alpha(y,\eta) = \begin{cases} \alpha_1 = 0.32, & y < \eta \\ \alpha_2 = 0.62, & y > \eta \end{cases}$$

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Budyko's Model
Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + C(\bar{T} - T)$$

There is still something missing. Where is the ice?

Ice-albedo Feedback

albedo of ice: 0.62
albedo of land and water: 0.32

Assumption: there is a single boundary between ice and no ice occurring at $y = \eta$.

$$\alpha = \alpha(y,\eta) = \begin{cases} \alpha_1 = 0.32, & y < \eta \\ \alpha_2 = 0.62, & y > \eta \end{cases}$$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\bar{T} - T)$$

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Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)\big(1-\alpha(y,\eta)\big) - \big(A+BT(y,t)\big) + C\big(\bar{T}(t)-T(y,t)\big)$$

Equilibrium:

$$Qs(y)\big(1-\alpha(y,\eta)\big) - \big(A+BT^*(y)\big) + C\big(\bar{T}^*-T^*(y)\big) = 0$$

This looks worse than before!

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Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)\big(1-\alpha(y,\eta)\big) - \big(A+BT(y,t)\big) + C\big(\bar{T}(t)-T(y,t)\big)$$

Equilibrium:

$$Qs(y)\big(1-\alpha(y,\eta)\big) - \big(A+BT^*(y)\big) + C\big(\bar{T}^*-T^*(y)\big) = 0$$

This looks worse than before!

Fortunately, the same technique works:
Assume we know the global mean temperature.

Solve for $T^*(y)$

$$Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}^* - \big(BT^*(y)\big) + C\big(-T^*(y)\big) = 0,$$

$$(B+C)T^*(y) = Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}^*$$

$$T^*(y) = \frac{Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}^*}{B+C}$$

Now we can compute the global mean temperature!

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Budyko's Model

Budyko's Equilibrium

$$T^*(y) = \frac{Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}^*}{B+C}$$

$$(B+C)T^*(y) = Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}^*$$

Integrate:

$$\int_0^1 (B+C)T^*(y)dy = \int_0^1 \Big(Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}^*\Big)dy$$

$$(B+C)\underbrace{\int_0^1 T^*(y)dy}_{\bar{T}^*} = Q\Big(\underbrace{\int_0^1 s(y)dy}_{1} - \underbrace{\int_0^1 \alpha(y,\eta)s(y)dy}_{\bar{\alpha}(\eta)}\Big) - A\underbrace{\int_0^1 dy}_{1} + C\bar{T}^*\underbrace{\int_0^1 dy}_{1}$$

$$(B+C)\bar{T}^* = Q\big(1-\bar{\alpha}(\eta)\big) - A + C\bar{T}^*$$

$$\bar{T}^* = \frac{Q\big(1-\bar{\alpha}(\eta)\big) - A}{B}$$

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Budyko's Model

Budyko's Equilibrium

$$T^*(y) = \frac{Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}^*}{B+C}$$

$$(B+C)T^*(y) = Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}^*$$

Integrate:

$$\int_0^1 (B+C)T^*(y)dy = \int_0^1 \Big(Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}^*\Big)dy$$

$$(B+C)\underbrace{\int_0^1 T^*(y)dy}_{\bar{T}^*} = Q\Big(\underbrace{\int_0^1 s(y)dy}_{1} - \underbrace{\int_0^1 \alpha(y,\eta)s(y)dy}_{\bar{\alpha}(\eta)}\Big) - A\underbrace{\int_0^1 dy}_{1} + C\bar{T}^*\underbrace{\int_0^1 dy}_{1}$$

$$(B+C)\bar{T}^* = Q\big(1-\bar{\alpha}(\eta)\big) - A + C\bar{T}^*$$

We computed the constant!

$$\bar{T}^* = \frac{Q\big(1-\bar{\alpha}(\eta)\big) - A}{B}$$

Note the dependence on the constant η.

$$T^*(y,\eta) = \frac{Qs(y)\big(1-\bar{\alpha}(\eta)\big) - A + C\bar{T}^*}{B+C}, \quad \text{where } \bar{T}^*(\eta) = \frac{Q\big(1-\bar{\alpha}(\eta)\big) - A}{B}$$

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Budyko's Model

Budyko's Equilibrium

$$T^*(y,\eta) = \frac{Qs(y)\big(1-\bar{\alpha}(\eta)\big) - A + C\bar{T}^*}{B+C}, \quad \text{where } \bar{T}^*(\eta) = \frac{Q\big(1-\bar{\alpha}(\eta)\big) - A}{B}$$

Can we compute the global albedo?

recall: $$\alpha(y,\eta) = \begin{cases} \alpha_1 = 0.32, & y<\eta \\ \alpha_2 = 0.62, & y>\eta \end{cases}$$

global albedo $$\bar{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)dy = \int_0^\eta \alpha_1 s(y)dy + \int_\eta^1 \alpha_2 s(y)dy = \alpha_1 S(\eta) + \alpha_2\big(1-S(\eta)\big)$$

where $$S(\eta) = \int_0^\eta s(y)dy = \int_0^\eta \underbrace{\Big(1-0.241\big(3y^2-1\big)\Big)}_{\text{Chylek \& Coakley}}dy = \eta - 0.241\big(\eta^3-\eta\big)$$

Time to summarize.

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Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)\big(1-\alpha(y,\eta)\big) - \big(A+BT(y,t)\big) + C\big(\bar{T}(t)-T(y,t)\big)$$

Equilibrium temperature distribution:

$$T^*(y,\eta) = \frac{Qs(y)\big(1-\bar{\alpha}(\eta)\big) - A + C\bar{T}^*(\eta)}{B+C}$$

$$\bar{T}^*(\eta) = \frac{Q\big(1-\bar{\alpha}(\eta)\big) - A}{B}$$

$$\bar{\alpha}(\eta) = \alpha_1 S(\eta) + \alpha_2\big(1-S(\eta)\big)$$

$$s(y) = 1 - 0.241\big(3y^2-1\big)$$

$$S(\eta) = \eta - 0.241\big(\eta^3-\eta\big)$$

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Budyko's Model
Budyko's Equilibrium

For each fixed η, there is an equilibrium solution for Budyko's equation.

$$T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\overline{T}_\eta^*\right)$$

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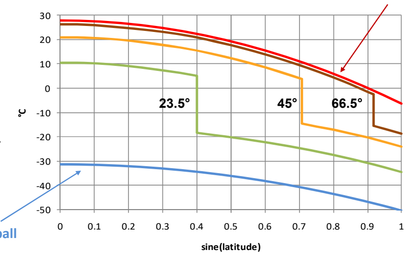
Math 5490
Budyko's Model
Equilibria

$$T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\overline{T}_\eta^*\right)$$

For each fixed η, there is an equilibrium solution for Budyko's equation.



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Budyko's Model
Stability of Equilibria

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T} - T)$$

Let X be the space of functions where T lives. (*e.g.* $L^1([0,1])$)
Let

$$L : X \to X : LT = C\overline{T} - (B+C)T,$$
$$f(y) = Qs(y)\left(1-\alpha(y)\right) - A$$

Budyko's equation can be written as a linear vector field on X.

$$R\frac{dT}{dt} = f + LT$$

The operator L has only point spectrum, with all eigenvalues negative.
Therefore, ***all solutions are stable***.
True for any albedo function.

experts only

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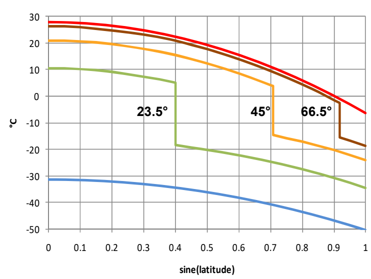
Math 5490
Budyko's Model
Stablity

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT(y,t)\right) + C\left(\overline{T}(t) - T(y,t)\right)$$

For each fixed η, there is a **globally stable** equilibrium solution for Budyko's equation.

How to pick one?



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Math 5490
Budyko's Model
Ice Albedo Feedback

Summary

If we artificially hold the ice line at a fixed latitude, then the surface temperature will come to an equilibrium.

However,
if the temperature is high, we would expect ice to melt and the ice line to retreat to higher latitudes. If the temperature is low, we would expect ice to form and the ice line to advance to lower latitudes.

How to model this expectation?

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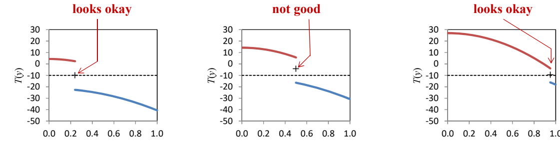
Math 5490
Budyko's Model
Ice Albedo Feedback

For each fixed η, there is a stable equilibrium solution for Budyko's equation.

Standard assumption: Permanent ice forms if the annual average temperature is below T_c=-10 °C and melts if the annual average temperature is above T_c.

Additional condition: The average temperature across the ice boundary must be the critical temperature T_c.

$$\frac{1}{2}\left(T_\eta^*(\eta+) + T_\eta^*(\eta-)\right) = T_c = -10$$



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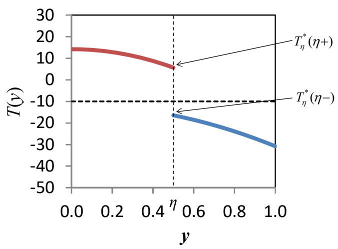
Budyko's Model
Ice Albedo Feedback

ice line condition: $\frac{1}{2}\left(T_\eta^*(\eta+)+T_\eta^*(\eta-)\right)=T_c=-10$



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Budyko's Model
Ice Albedo Feedback

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta))-(A+BT)+C(\overline{T}-T)$$

Equilibrium: $T_\eta^*(y)=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right)-A+C\overline{T}_\eta^*\right)$

Ice line condition: $\frac{1}{2}\left(T_\eta^*(\eta+)+T_\eta^*(\eta-)\right)=T_c=-10$

Albedo: $\alpha(y,\eta)=\begin{cases}\alpha_1=0.32, & y<\eta\\ \alpha_2=0.62, & y>\eta\end{cases}$ $\quad \alpha(\eta-,\eta)=\alpha_1,\quad \alpha(\eta+,\eta)=\alpha_2$

$$T_\eta^*(\eta+)=\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_2\right)-A+C\overline{T}_\eta^*\right)\qquad T_\eta^*(\eta-)=\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_1\right)-A+C\overline{T}_\eta^*\right)$$

Ice line condition:

$$\frac{1}{2}\left(T_\eta^*(\eta+)+T_\eta^*(\eta-)\right)=\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right)-A+C\overline{T}_\eta^*\right)=T_c=-10$$

where: $\alpha_0=\frac{\alpha_1+\alpha_2}{2}=0.47$

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Budyko's Model
Ice Albedo Feedback

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta))-(A+BT)+C(\overline{T}-T)$$

Ice line condition: $\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right)-A+C\overline{T}_\eta^*\right)=T_c=-10$

Rewrite: $h(\eta)=\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right)-A+C\overline{T}_\eta^*\right)-T_c=0$

Recall equilibrium GMT: $\overline{T}_\eta^*=\frac{1}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right)-A\right)$

Recall average albedo: $\overline{\alpha}(\eta)=\int_0^1\alpha(y,\eta)s(y)dy=\alpha_2-(\alpha_2-\alpha_1)S(\eta)=0.62-0.3S(\eta)$

where: $S(\eta)=\int_0^\eta s(y)dy=\eta-0.241\left(\eta^3-\eta\right)$

$$h(\eta)\equiv\frac{Q}{B+C}\left(s(\eta)\left(1-\alpha_0\right)+\frac{C}{B}\left(1-\alpha_2+\left(\alpha_2-\alpha_1\right)S(\eta)\right)\right)-\frac{A}{B}-T_c=0$$

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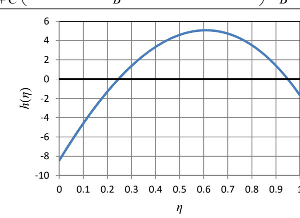
Budyko's Model
Ice Albedo Feedback

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta))-(A+BT)+C(\overline{T}-T)$$

The additional condition: $\frac{1}{2}\left(T_\eta^*(\eta+)+T_\eta^*(\eta-)\right)=T_c=-10$

can be written: $h(\eta)\equiv\frac{Q}{B+C}\left(s(\eta)\left(1-\alpha_0\right)+\frac{C}{B}\left(1-\alpha_2+\left(\alpha_2-\alpha_1\right)S(\eta)\right)\right)-\frac{A}{B}-T_c=0$

Two equilibria (zeros of h) satisfy the additional condition.



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Budyko's Model
Ice Albedo Feedback

Equilibrium temperature profiles $T_\eta^*(y)=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right)-A+C\overline{T}_\eta^*\right)$

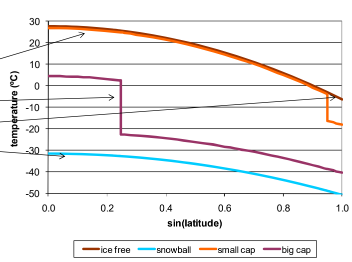
Interesting Solutions:

small cap

large cap

ice free

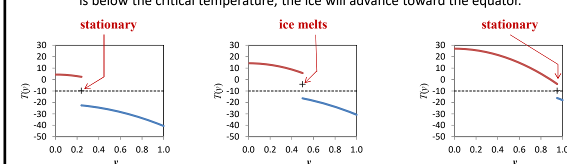
snowball



Math 5490
Budyko's Model
Dynamics of the Ice Line

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta))-(A+BT)+C(\overline{T}-T)$$

Idea:
If the average temperature across the ice line is above the critical temperature, some ice will melt, moving the ice line toward the pole. If it is below the critical temperature, the ice will advance toward the equator.



Widiasih's equation: $\frac{d\eta}{dt}=\varepsilon\left(T(\eta)-T_c\right)$

Math 5490
Budyko's Model

Dynamics of the Ice Line

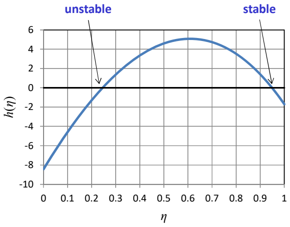
$$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T} - T)$$

Widiasih's Theorem. For sufficiently small ε, the system has an attracting invariant curve given by the graph of a function $\Phi_\varepsilon : [0,1] \to X$. On this curve, the dynamics are approximated by the equation

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$

experts only



Esther R. Widiasih, Dynamics of the Budyko Energy Balance Model, *SIAM J. Appl. Dyn. Syst.*, 12(4), 2068–2092.

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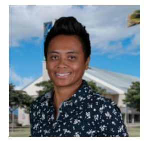
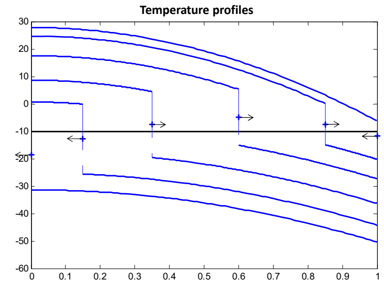
Budyko's Model

Budyko-Widiasih Model



Esther Widiasih

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$



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Budyko's Model

Budyko-Widiasih Model

Esther →  ← Dick

Oahu, February 2020

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Budyko's Model

Summary

surface temperature; sin(latitude); $\overline{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T} - T)$$

heat capacity; insolation; albedo; OLR; heat transport

reduces to

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$

$$\frac{d\eta}{dt} = \varepsilon h(\eta) = \varepsilon\left(\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_2 + (\alpha_2-\alpha_1)S(\eta)\right)\right) - \frac{A}{B} - T_c\right)$$

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Budyko's Model

Budyko-Widiasih Model

surface temperature; sin(latitude); $\overline{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T} - T)$$

heat capacity; insolation; albedo; OLR; heat transport

What about the greenhouse effect?

$A+BT$ is the outgoing long wave radiation. This term decreases if the greenhouse gases increase.
We view A as a parameter.

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

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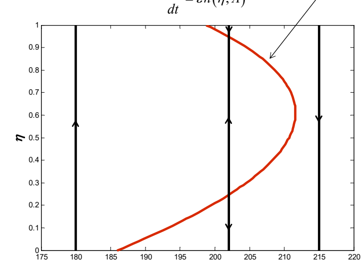
Budyko's Model

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

isocline $h(\eta, A) = 0$



high CO_2 low CO_2

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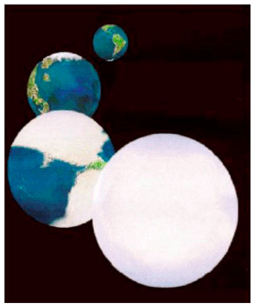
Snowball Earth

Is it possible for Earth to become completely covered in ice? (**Snowball Earth**)

Did it ever happen?



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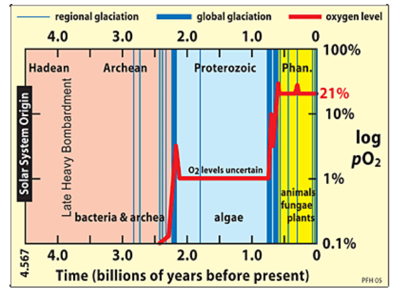
Math 5490
Snowball Earth

There is evidence that Snowball Earth has occurred, the last time about 600 million years ago.



http://www.snowballearth.org/when.html

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Snowball Earth

The continents were clustered near the equator.

Hoffman & Schrag, *Snowball Earth*, Scientific American, January 2000, 68-75

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Snowball Earth

"Ice-rafted debris" occurred in ocean sediments near the equator, indicating large equatorial glaciers calving icebergs.

Hoffman & Schrag, *Snowball Earth*, Scientific American, January 2000, 68-75

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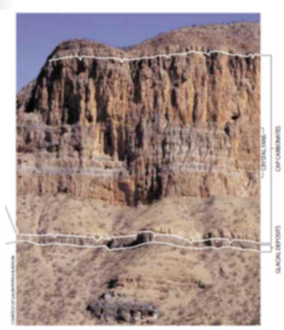
Snowball Earth

Large limestone deposits "cap carbonates" are found immediately above the glacial debris, indicating a rapid warming period following the snowball.



Hoffman & Schrag, *Snowball Earth*, Scientific American, January 2000, 68-75

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Math 5490
Snowball Earth

Idea:

When the Earth is mostly ice-covered, silicate weathering slows down, but volcanic activity stays the same, allowing for a build-up of CO_2 in the atmosphere.

When the Earth is mostly ice-free, silicate weathering speeds up, drawing down the CO_2 in the atmosphere.

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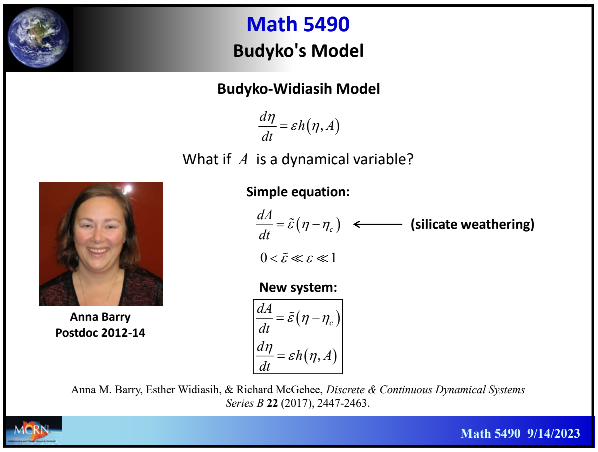
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Budyko's Model
Budyko-Widiasih Model
$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$
What if A is a dynamical variable?
Simple equation:
$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$ (silicate weathering)
$0 < \tilde{\varepsilon} \ll \varepsilon \ll 1$
New system:
$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$
$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$
Anna Barry
Postdoc 2012-14
Anna M. Barry, Esther Widiasih, & Richard McGehee, Discrete & Continuous Dynamical Systems Series B 22 (2017), 2447-2463.
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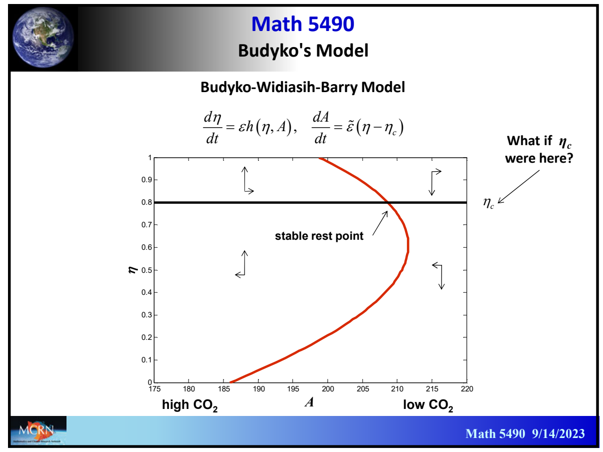
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Budyko's Model
Budyko-Widiasih-Barry Model
$\frac{d\eta}{dt} = \varepsilon h(\eta, A), \quad \frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$
What if η_c were here?
η_c
stable rest point
η
high CO_2
A
low CO_2
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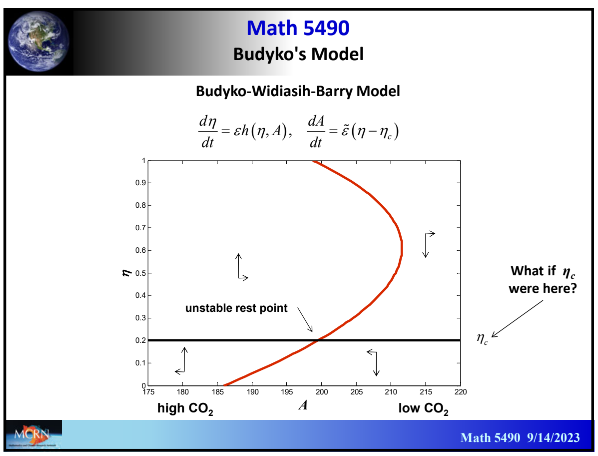
Math 5490
Budyko's Model
Budyko-Widiasih-Barry Model
$\frac{d\eta}{dt} = \varepsilon h(\eta, A), \quad \frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$
What if η_c were here?
η_c
unstable rest point
η
high CO_2
A
low CO_2
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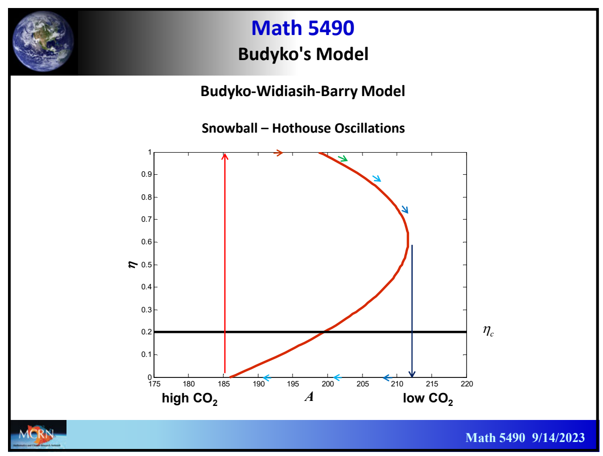
Math 5490
Budyko's Model
Budyko-Widiasih-Barry Model
Snowball – Hothouse Oscillations
η_c
η
high CO_2
A
low CO_2
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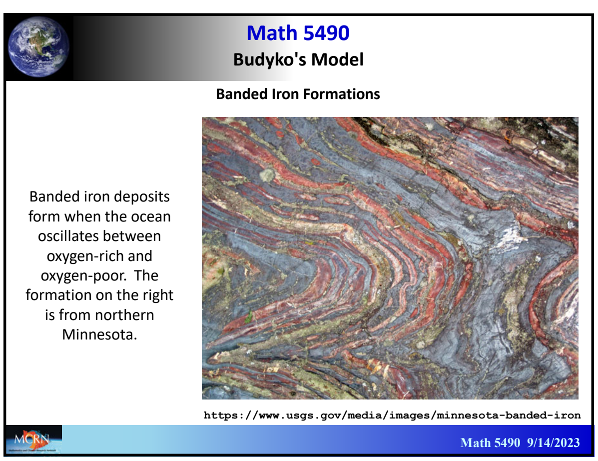
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Budyko's Model
Banded Iron Formations
Banded iron deposits form when the ocean oscillates between oxygen-rich and oxygen-poor. The formation on the right is from northern Minnesota.
https://www.usgs.gov/media/images/minnesota-banded-iron
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