Math 5490
Topics in Applied Mathematics
Introduction to the Mathematics of Climate

Fall 2023
1:25 - 3:20 Tuesdays and Thursdays
Amundson Hall 162

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course website
www-users.cse.umn.edu/~mcgehee/teaching/Math5490/

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Glacial Cycles

Glacial Cycles and Time Series Analysis

References

H. Kaper & H. Engler, *Mathematics & Climate*, SIAM Philadelphia 2013, Chapter 11.

R. McGehee & C. Lehman, *SIAM Journal on Applied Dynamical Systems* **11** (2012).

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Glacial Cycles and Time Series Analysis

Example

During the last million years, Earth has undergone extensive glaciations, interspersed with interglacial warm periods, occurring about every 100,000 years.

During the same period, the eccentricity of Earth's orbit has oscillated with a period of about 100,000 years, leading scientists to conclude that the two are related.

Can we quantify this relationship?

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Example:

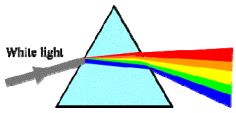
The ice ages occur about every 100,000 year, and the eccentricity of the Earth's orbit cycles through changes every 100,000 years.

Can we quantify this observation?

Fourier Transform (Power Spectrum)

Refraction through a prism

time series



power spectrum

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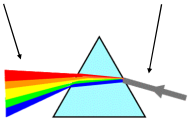
Fourier Transform (Power Spectrum)

$$\widehat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi st}\, dt$$

spectrum — time series t = time



For explanations, see Kaper & Engler, Chapter 11.

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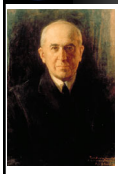
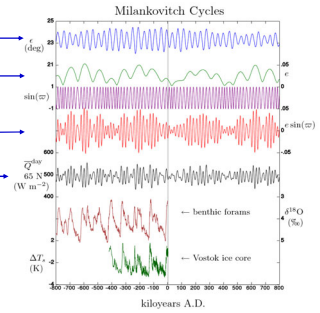


obliquity
eccentricity
precession index
daily average insolation at summer solstice at 65° N



http://en.wikipedia.org/wiki/Milankovitch_cycles

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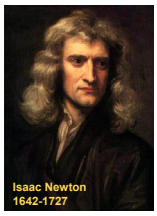
Planetary Motion

$$m_i \frac{d^2 x_i}{dt^2} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_i m_j \left(x_j - x_i\right)}{\left|x_j - x_i\right|^3}$$



Jacques Laskar (1955-)

The orbits of all the planets can be computed (both forward and backward in time) for billions of years.

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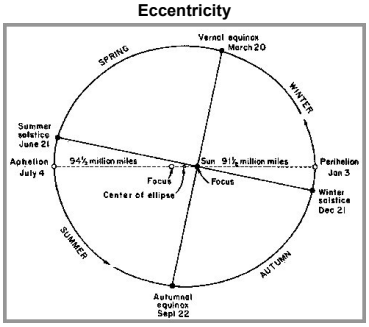
Eccentricity



John Imbrie & Katherine Palmer Imbrie, *Ice Ages: Solving the Mystery,* Harvard Univ. Press, 1979.

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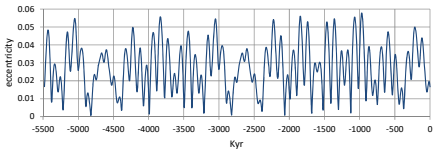
Eccentricity



Note the cycles of about 100 Kyr and 400 kyr.

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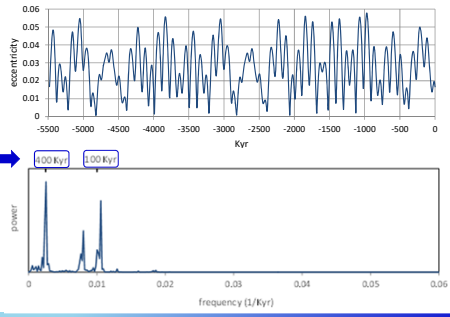
Eccentricity



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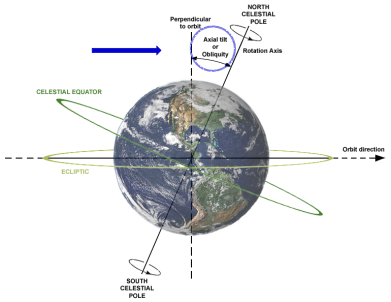
Earth's Obliquity



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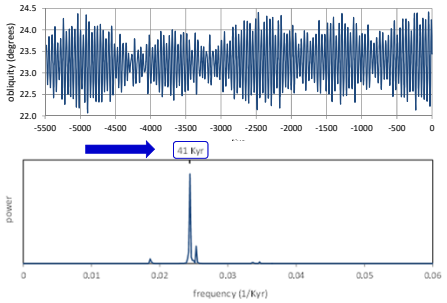
Obliquity



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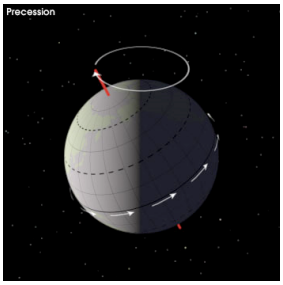
Precession



http://earthobservatory.nasa.gov/Library/Giants/Milankovitch/milankovitch_2.html

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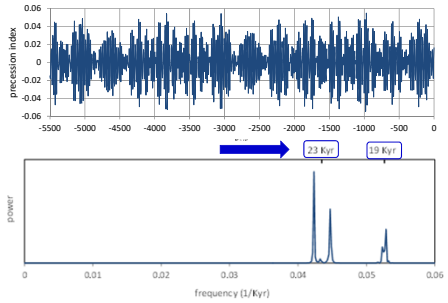
Precession Index



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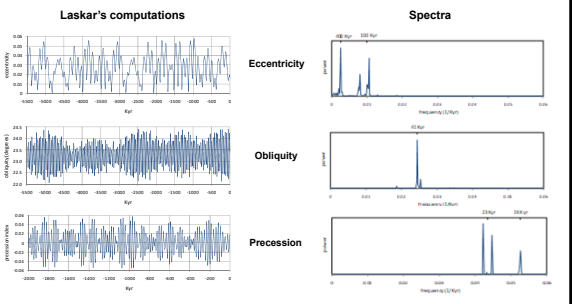
Spectral Analysis of the Milankovitch cycles.



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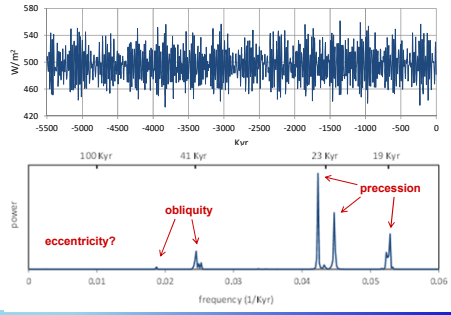
Summer Solstice 65°N



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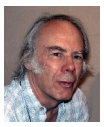
Remember these geologists?

James D. Hays

Hays, Imbrie, and Shackleton, "Variations in the Earth's Orbit: Pacemaker of the Ice Ages," *Science* **194**, 10 December 1976.

John Imbrie

"It is concluded that changes in the earth's orbital geometry are the fundamental cause of the succession of Quaternary ice ages."



Nicholas Shackleton

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Hays, et al, Summary

2) ... climatic variance of these records is concentrated in three discrete spectral peaks at periods of 23,000, 42,000, and approximately 100,000 years. These peaks correspond to the dominant periods of the earth's solar orbit, and contain respectively about 10, 25, and 50 percent of the climatic variance.

Hays, *et al*, *Science* **194** (1976), p. 1125

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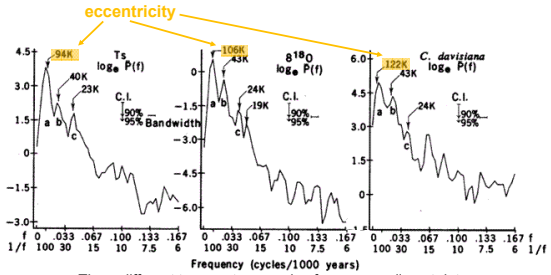
Climate Response, Hays, et al



Three different temperature proxies from sea sediment data.

Hays, *et al*, *Science* **194** (1976), p. 1125

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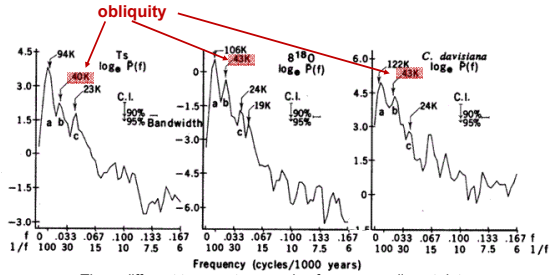
Climate Response, Hays, et al



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Hays, *et al*, *Science* **194** (1976), p. 1125

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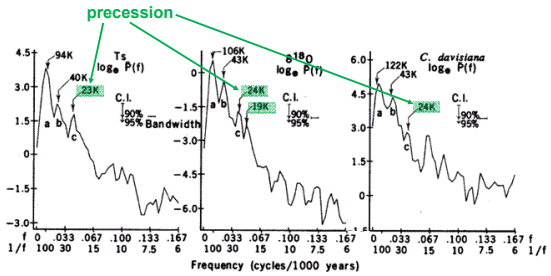
Climate Response, Hays, et al



Three different temperature proxies from sea sediment data.

Hays, *et al*, *Science* **194** (1976), p. 1125

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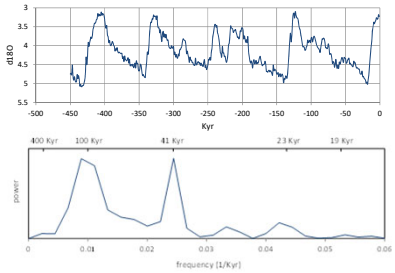
Hays with Modern Data



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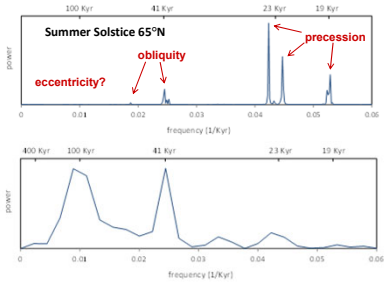
Milankovitch vs. Climate



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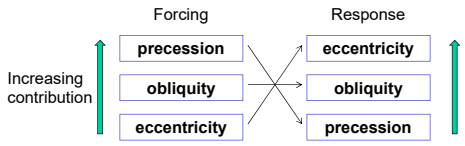
Hays, et al, Summary



Hays' explanation is that there are nonlinear feedbacks.

Are there other explanations?

Hays, *et al*, *Science* **194** (1976), p. 1127

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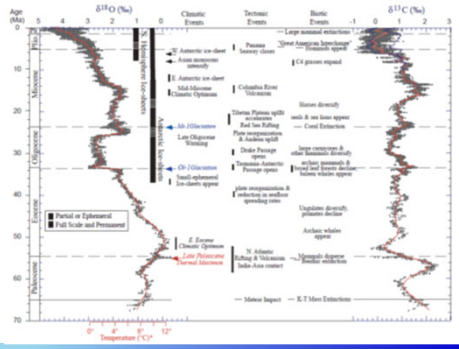

Zachos, *et al*, *Science* **292** (2001), p. 689

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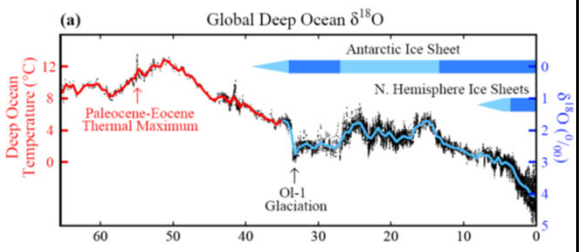
Temperatures in the Cenozoic Era



Hansen, et al, Target atmospheric CO2: Where should humanity aim? *Open Atmos. Sci. J.* 2 (2008)

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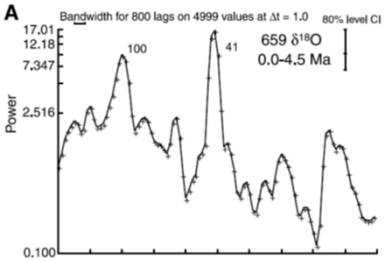
Climate Response (Zachos, et al)

Power spectrum of climate for the last 4.5 Myr. Note the peaks at 41Kyr and 100 Kyr.



Zachos, *et al*, *Science* **292** (2001), p. 689

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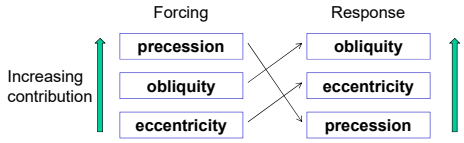
Zachos, et al, Summary



Nonlinear effects?

Other explanations?

Zachos, *et al*, *Science* **292** (2001), p. 689

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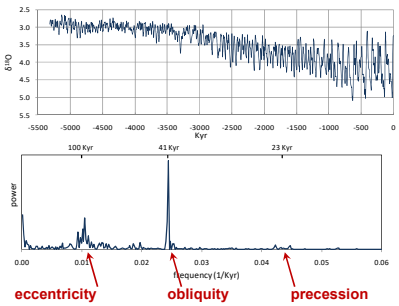
Spectral Analysis of the Climate Data



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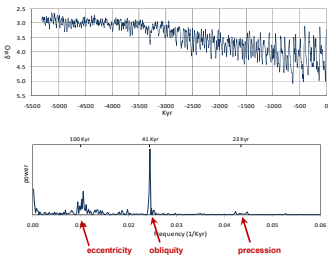
Spectral Analysis of the Climate Data



Conclusion Remains:
The Milankovitch cycles "pace" the Earth's climate.

Exactly how is not so clear.

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Why such a small precession contribution?

Incoming **So**lar Radi**ation** (*Insolation*), averaged over the entire globe and over a full year, depends only on eccentricity e, not on either obliquity or precession.

$$Q(e) = \frac{Q_0}{\sqrt{1-e^2}}$$

Insolation as a function of latitude, averaged over a full year, depends on eccentricity e and obliquity β, but not precession.

$$I = Q(e)\, s(y, \beta)$$

where

$$s(\varphi, \beta) = \frac{2}{\pi^2}\int_0^{2\pi} \sqrt{1-(\cos\varphi \sin\beta \cos\theta - \sin\varphi \cos\beta)^2}\, \cos\theta\, d\theta$$

φ = latitude

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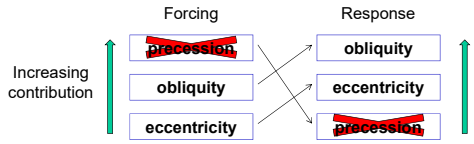
Zachos Summary (Revised)

If we assume that glaciation depends on annual average insolation instead of insolation at summer solstice, then forcing and response are aligned.



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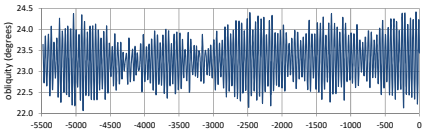
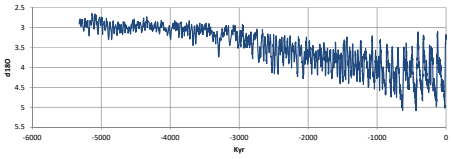
Something's Missing

obliquity



climate



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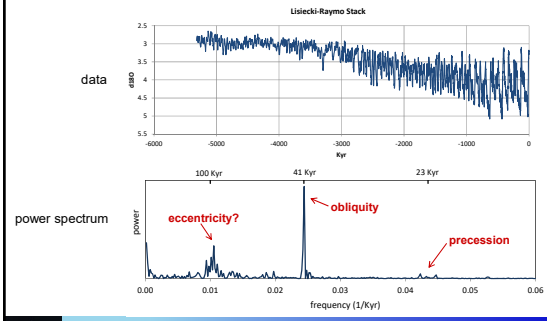
Something's Missing



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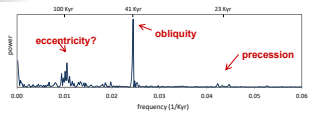
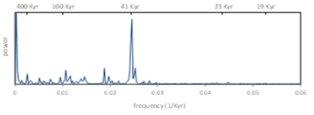
Last Million Years is Different

A transition occurred about one million years ago: the amplitude increased and the dominant period changed from 41 kyr to 100 kyr.

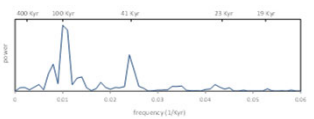
-5 to 0 Myr



-5 to -1 Myr



-1 to 0 Myr



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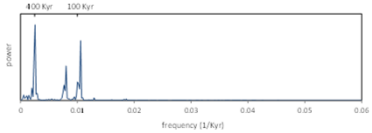
What's up with the Last Million Years?

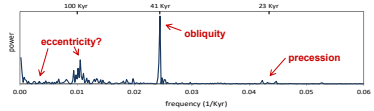
100,000 Year Problem: Why did the eccentricity signal become so dominant during the last million years?

400,000 Year Problem: If the last million years is dominated by eccentricity, what happened to the 400,000 year cycle?

eccentricity



data



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Budyko's Model

surface temperature, sin(latitude), ice line, $\bar{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\bar{T}-T)$$

heat capacity, insolation, albedo, OLR, heat transport

reduces to

$$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right) \equiv h(\eta)$$

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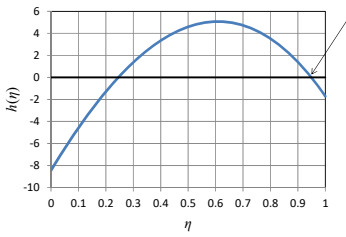
Budyko's Model

$$\frac{d\eta}{dt} = h(\eta)$$

stable equilibrium η^*



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Budyko's Model

$$\frac{d\eta}{dt} = h(\eta)$$

The function h, and hence the equilibrium solution η^*, depends on all the parameters of the Budyko model.

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\bar{T}-T)$$

In particular, η^* depends on Q and $s(y)$, which depend on the eccentricity e and the obliquity β.

$$Q(e) = \frac{Q_0}{\sqrt{1-e^2}}$$

$$s(y,\beta) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\theta - y\cos\beta\right)^2}\,d\theta$$

$$\eta^* = \eta^*(e,\beta)$$

McGehee & Lehman, A Paleoclimate Model of Ice-Albedo Feedback Forced by Variations in Earth's Orbit, SIAM J. Applied Dynamical Systems **11** (2012), 684–707.

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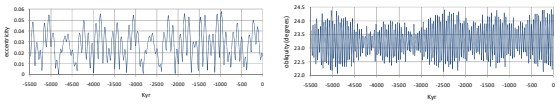
Budyko's Model

$$\frac{d\eta}{dt} = h(\eta, e, \beta)$$

The eccentricity e and the obliquity β are given by Laskar as functions of time.



Therefore, the stable equilibrium ice line is a function of time:

$$\eta^*(t) = \eta^*(e(t),\beta(t))$$

McGehee & Lehman, A Paleoclimate Model of Ice-Albedo Feedback Forced by Variations in Earth's Orbit, SIAM J. Applied Dynamical Systems **11** (2012), 684–707.

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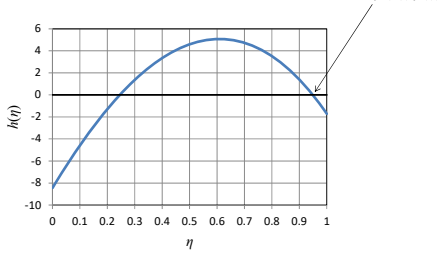
Budyko's Model

$$\frac{d\eta}{dt} = h(\eta, e, \beta)$$

stable equilibrium $\eta^*(e(t),\beta(t))$



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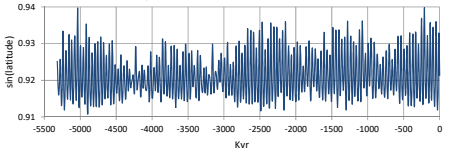
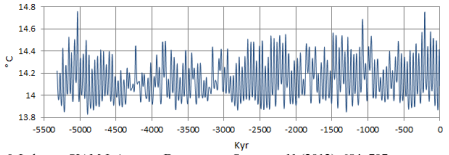
Budyko's Model

ice line



GMT



McGehee & Lehman, SIAM J. Applied Dynamical Systems **11** (2012), 684–707.

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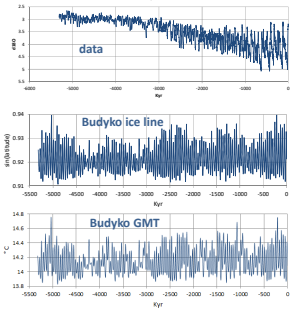
Math 5490
Glacial Cycles

Budyko's Model



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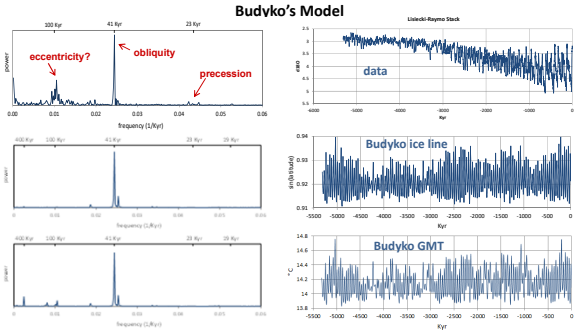

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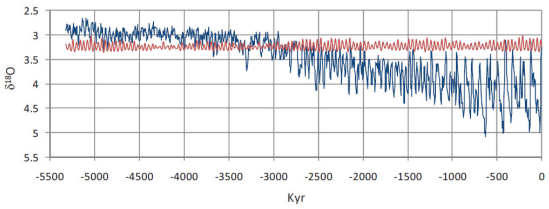

red: model simulation
blue: data

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Glacial Cycles

Budyko's Model

Budyko's model of ice-albedo feedback produces a climate response driven primarily by obliquity cycles, consistent with the dominance of obliquity in the climate data.

The model fails to produce:

1. the amplitude changes over the past 5 million years, and
2. the frequency change 1 million years ago ("mid-Pleistocene transition).

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Glacial Cycles

Budyko's Model

MATLAB Program:
PaleoBudyko

Download from
http://www.math.umn.edu/~mcgehee/Software/