Math 5490
Topics in Applied Mathematics
Introduction to the Mathematics of Climate

Fall 2023
1:25 - 3:20 Tuesdays and Thursdays
Amundson Hall 162

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course website
www-users.cse.umn.edu/~mcgehee/teaching/Math5490/

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Welander's Model

Reference

Pierre Welander, A simple heat-salt oscillator, *Dynamics of Atmospheres and Oceans* **6** (1982) 233-242.

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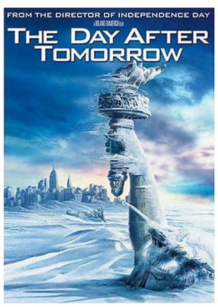
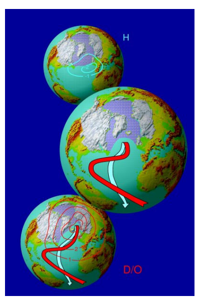
Heinrich and Dansgaard-Oeschger events



What did the film get right scientifically?



http://www.pik-potsdam.de/~stefan/sampleimages.html

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INDEPENDENT

April 12, 2018

Gulf Stream current at 'record low' with potentially devastating consequences for weather, warn scientists

The Atlantic meridional overturning circulation (AMOC), the system of currents that transports warm water from the tropics via the Gulf Stream to the North Atlantic, plays a major role in regulating the world's climate.

A fictional depiction of AMOC's collapse was portrayed in ***The Day After Tomorrow***, and while the film's events were exaggerated, scientists say severe weather events are likely to result from the ongoing changes.

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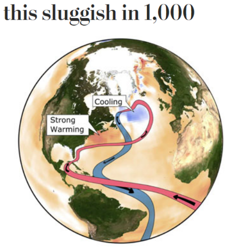
The Washington Post

April 11, 2018

The oceans' circulation hasn't been this sluggish in 1,000 years. That's bad news.

The Atlantic Ocean circulation that carries warmth into the Northern Hemisphere's high latitudes is slowing down because of climate change, a team of scientists asserted Wednesday, suggesting one of the most feared consequences is already coming to pass.



Nature volume 556, pages191–196 (2018)

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Current Atlantic Meridional Overturning Circulation weakest in last millennium

The Atlantic Meridional Overturning Circulation (AMOC)—one of Earth's major ocean circulation systems—redistributes heat on our planet and has a major impact on climate. Here, we compare a variety of published proxy records to reconstruct the evolution of the AMOC since about AD 400. A fairly consistent picture of the AMOC emerges: **after a long and relatively stable period, there was an initial weakening starting in the nineteenth century, followed by a second, more rapid, decline in the mid-twentieth century, leading to the weakest state of the AMOC occurring in recent decades.**

NATURE GEOSCIENCE | VOL 14 | MARCH 2021 | 118–120 | www.nature.com/naturegeoscience118

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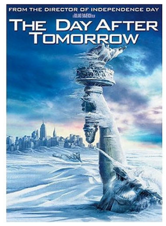
The Day After Tomorrow/Film synopsis

After climatologist Jack Hall (Dennis Quaid) is largely ignored by U.N. officials when presenting his environmental concerns, his research proves true when an enormous "superstorm" develops, setting off catastrophic natural disasters throughout the world. Trying to get to his son, Sam (Jake Gyllenhaal), who is trapped in New York with his friend Laura (Emmy Rossum) and others, Jack and his crew must travel by foot from Philadelphia, braving the elements, to get to Sam before it's too late.

`Google Search`



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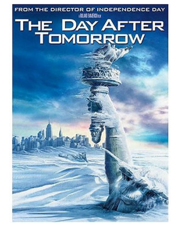
Dansgaard-Oeschger Events

"Global warming" can cause the Northern Hemisphere to cool.

Melting ice can lower the salinity of the North Atlantic, causing a decrease in the flow of the Atlantic Meridional Overturning Circulation (AMOC), slowing the heat transfer to the Northern Hemisphere.

This phenomenon is believed to have caused the Younger Dryas.



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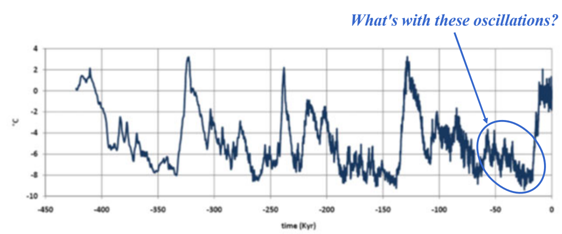
Recent (last 400 Kyr) Temperature Cycles

Vostok Ice Core Data



J.R. Petit, *et al* (1999) Climate and atmospheric history of the past 420,000 years from the Vostok ice core, Antarctica, *Nature* 399, 429-436.

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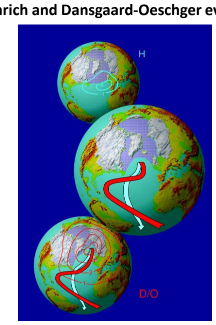
Heinrich and Dansgaard-Oeschger events



`http://www.pik-potsdam.de/~stefan/sampleimages.html`

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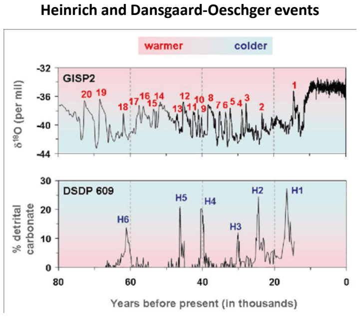
Heinrich and Dansgaard-Oeschger events



`http://www.ncdc.noaa.gov/paleo/abrupt/data3.html`

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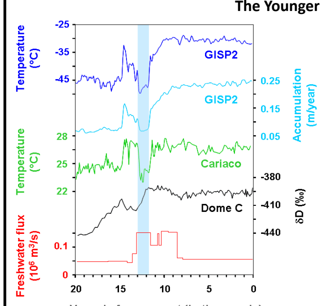
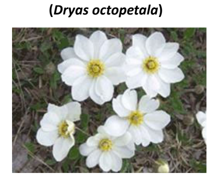
The Younger Dryas



`http://www.ncdc.noaa.gov/paleo/abrupt/data4.html`

Mountain Avens
(*Dryas octopetala*)



`https://www.ncdc.noaa.gov/abrupt-climate-change/The%20Younger%20Dryas`

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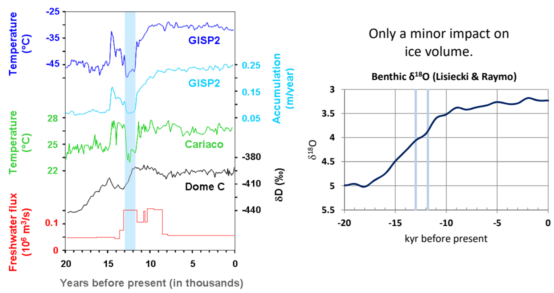
The Younger Dryas



Only a minor impact on ice volume.

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What caused the Dansgaard-Oeschger Oscillations?

For glacial cyles, there is overwhelming evidence that they are "paced" by the Milankovitch cycles.

None of the Milankovitch cycles has periods short enough to trigger the Dansgaard-Oeschger events, and we can't think of any other external triggers.

Could they occur spontaneously?

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What caused the Dansgaard-Oeschger oscillations?

They could be self-oscillations in the natural dynamics of ocean circulation.

Pierre Welander, A simple heat-salt oscillator, *Dynamics of Atmospheres and Oceans* **6** (1982) 233-242.

R/V WeeLander is a 23-foot-long Beach Master work boat, informally named in honor of Professor Pierre Welander (1925—1996).

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Ocean Circulation

What caused the Dansgaard-Oeschger oscillations?

They could be self-oscillations in the natural dynamics of ocean circulation.

Welander constructed a simple (***conceptual!***) box model of ocean circulation and showed that the interactions of temperature and salinity with the atmosphere, the surface ocean, and the deep ocean could create self-oscillations.

R/V WeeLander is a 23-foot-long Beach Master work boat, informally named in honor of Professor Pierre Welander (1925—1996).

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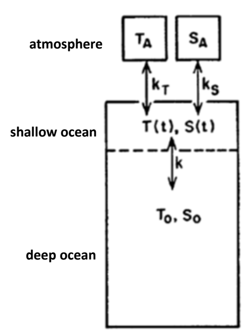
Welander's model divides the Earth into three components: the "atmosphere", the "shallow ocean", and the "deep ocean".

The "shallow ocean" is represented by its temperature T and its salinity S, both of which vary with time.

The "atmosphere" is a ficticious entity encompassing and simplifying all the interactions of the shallow ocean with the rest of the planet. It is assumed to have an "effective" temperature of T_A and an "effective" salinity of S_A.



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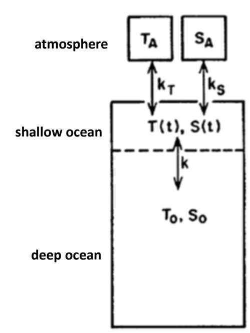
The "deep ocean" is assumed to have a constant temperature T_0 and a constant salinity S_0.

The interaction between the shallow ocean and the atmosphere is modeled as a dynamic transfer of relaxation to equilibrium:

$$\frac{dT}{dt} = k_T(T_A - T),$$
$$\frac{dS}{dt} = k_S(S_A - S),$$

where k_T and k_S are constants.



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Discussion

How do we solve this differential equation?

$$\frac{dT}{dt} = k_T(T_A - T)$$

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Ocean Circulation

Discussion

How do we solve this differential equation?

$$\frac{dT}{dt} = k_T(T_A - T)$$

"separate variables"

$$\frac{dT}{T_A - T} = k_T dt \qquad \frac{dT}{T - T_A} = -k_T dt \qquad \int \frac{dT}{T - T_A} = \int -k_T dt$$

$$\ln(T - T_A) = -k_T t + c$$

$$T - T_A = e^{-k_T t + c} = e^c e^{-k_T t} = Ce^{-k_T t}$$

arbitrary constants

$$\boxed{T(t) = T_A + Ce^{-k_T t}}$$

general solution

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Ocean Circulation

Discussion

How do we solve this differential equation?

$$\frac{dT}{dt} = k_T(T_A - T)$$

$$\boxed{T(t) = T_A + Ce^{-k_T t}}$$

general solution

check

$$\frac{dT}{dt} = \frac{d}{dt}\left(T_A + Ce^{-k_T t}\right) = 0 + Ce^{-k_T t}(-k_T) = -k_T Ce^{-k_T t}$$

$$k_T(T_A - T) = k_T\left(T_A - \left(T_A + Ce^{-k_T t}\right)\right) = -k_T Ce^{-k_T t}$$

checks!

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Discussion

$$\frac{dT}{dt} = k_T(T_A - T)$$

Find an equilibrium solution.

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Discussion

$$\frac{dT}{dt} = k_T(T_A - T)$$

Find an equilibrium solution.

Answer

$$0 = \frac{dT}{dt} = k_T(T_A - T)$$

$$\boxed{T(t) = T_A}$$

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Discussion

$$\frac{dT}{dt} = k_T(T_A - T)$$

equilibrium solution: $T(t) = T_A$

Is it stable?

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Discussion

$$\frac{dT}{dt} = k_T(T_A - T)$$

equilibrium solution: $T(t) = T_A$

Is it stable?

Yes!

general solution: $T(t) = T_A + Ce^{-k_T t}$

$$\lim_{t\to\infty} T(t) = T_A$$

Note that we are assuming that k_T is positive.

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Discussion

How do we solve this "initial value problem"?

$$\frac{dT}{dt} = k_T(T_A - T), \quad \textit{equation}$$
$$T = T_0 \text{ when } t = 0. \quad \textit{initial value}$$

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Discussion

How do we solve this "initial value problem"?

$$\frac{dT}{dt} = k_T(T_A - T),$$
$$T = T_0 \text{ when } t = 0.$$

general solution: $T(t) = T_A + Ce^{-k_T t}$

$$T(0) = T_A + Ce^{-k_T 0} = T_A + C = T_0$$
$$C = T_0 - T_A$$
$$T(t) = T_A + Ce^{-k_T t}$$
$$\boxed{T(t) = T_A + (T_0 - T_A)e^{-k_T t}}$$

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Ocean Circulation

Discussion

How do we solve this "initial value problem"?

$$\frac{dT}{dt} = k_T(T_A - T),$$
$$T = T_0 \text{ when } t = 0.$$

solution: $T(t) = T_A + (T_0 - T_A)e^{-k_T t}$

check:

$$T'(t) = -k_T(T_0 - T_A)e^{-k_T t}$$
$$k_T(T_A - T) = k_T\left(-(T_0 - T_A)e^{-k_T t}\right)$$

equal

$$T(0) = T_A + (T_0 - T_A)e^{-k_T 0} = T_A + (T_0 - T_A) = T_0$$ *checks!*

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Ocean Circulation

Discussion

How do we solve this "initial value problem"?

$$\frac{dT}{dt} = k_T(T_A - T), \quad \text{equation}$$
$$T = T_0 \text{ when } t = 0. \quad \text{initial value}$$

alternative approach:
Introduce the departure from equilibrium

$$y = T - T_A, \qquad T = T_A + y$$

$$\frac{dy}{dt} = \frac{dT}{dt} = k_T(T_A - T) = k_T(-y) = -k_T y$$

new initial value problem

$$\boxed{\begin{array}{l}\frac{dy}{dt} = -k_T y, \\ y = T_0 - T_A \text{ when } t = 0.\end{array}}$$

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Discussion

How do we solve this "initial value problem"?

$$\boxed{\begin{array}{l}\frac{dy}{dt} = -k_T y, \\ y = T_0 - T_A \text{ when } t = 0.\end{array}}$$

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Discussion

How do we solve this "initial value problem"?

$$\frac{dy}{dt} = -k_T y, \quad y = T_0 - T_A \text{ when } t = 0.$$

"separate variables"

$$\frac{dy}{y} = -k_T dt \qquad \frac{dy}{y} = -k_T dt \qquad \int_{T_0 - T_A}^{y} \frac{dy}{y} = \int_0^t -k_T dt$$

$$\ln y\Big|_{T_0-T_A}^{y} = -k_T t\Big|_0^t \qquad \ln y - \ln(T_0 - T_A) = -k_T t \qquad \frac{y}{T_0 - T_A} = e^{-k_T t}$$

$$y(t) = (T_0 - T_A)e^{-k_T t}$$

$$y \to 0 \text{ as } t \to 0$$

stable

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Summary

$$\frac{dT}{dt} = k_T(T_A - T)$$

This differential equation has a stable equilibrium solution $T = T_A$.

Remember Welander?

T is the Welander temperature.

Salinity:

$$\frac{dS}{dt} = k_S(S_A - S)$$

This differential equation also has a stable equilibrium solution $S = S_A$.

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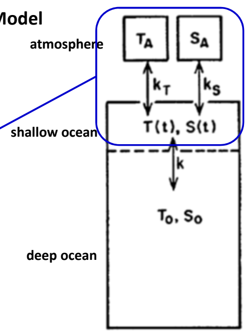
Welander's Model

The interaction between the shallow ocean and the atmosphere is modeled as a dynamic transfer of relaxation to equilibrium:

$$\frac{dT}{dt} = k_T(T_A - T), \qquad \frac{dS}{dt} = k_S(S_A - S),$$

where k_T and k_S are positive constants. This system has a stable equilibrium point at

$$(T, S) = (T_A, S_A).$$



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Welander's Model

Atmosphere - Ocean Surface Interaction

system of differential equations:

$$\frac{dT}{dt} = k_T(T_A - T) = -k_T(T - T_A), \quad T(0) = T_0,$$
$$\frac{dS}{dt} = k_S(S_A - S) = -k_S(S - S_A), \quad S(0) = S_0.$$

initial condition

Matrix Notation

differential equation:

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} -k_T & 0 \\ 0 & -k_S \end{bmatrix}\begin{bmatrix} T - T_A \\ S - S_A \end{bmatrix}, \qquad \begin{bmatrix} T \\ S \end{bmatrix}(0) = \begin{bmatrix} T_0 \\ S_0 \end{bmatrix}.$$

initial condition

singular! *ONE* vector equation

Solution

$$\begin{bmatrix} T \\ S \end{bmatrix}(t) = \begin{bmatrix} T(t) \\ S(t) \end{bmatrix} = \begin{bmatrix} T_A + (T_0 - T_A)e^{-k_T t} \\ S_A + (S_0 - S_A)e^{-k_S t} \end{bmatrix}$$

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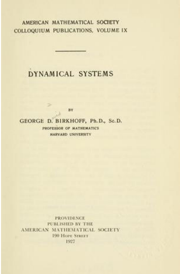
Dynamical Systems



AMERICAN MATHEMATICAL SOCIETY
COLLOQUIUM PUBLICATIONS, VOLUME IX

DYNAMICAL SYSTEMS

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PUBLISHED BY THE
AMERICAN MATHEMATICAL SOCIETY
190 HOPE STREET
1927

https://openlibrary.org/works/OL86546W/Dynamical_systems

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Dynamical Systems

Basic idea:
The dependence of the solution on ***initial conditions*** is just as important as its dependence on ***time***.

Inital value problem

differential equation:

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} -k_T & 0 \\ 0 & -k_S \end{bmatrix}\begin{bmatrix} T - T_A \\ S - S_A \end{bmatrix}, \qquad \begin{bmatrix} T \\ S \end{bmatrix}(0) = \begin{bmatrix} T_0 \\ S_0 \end{bmatrix}.$$

solution:

$$\begin{bmatrix} T \\ S \end{bmatrix}(t) = \begin{bmatrix} T(t) \\ S(t) \end{bmatrix} = \begin{bmatrix} T_A + (T_0 - T_A)e^{-k_T t} \\ S_A + (S_0 - S_A)e^{-k_S t} \end{bmatrix}$$

flow:

$$\varphi\left(\begin{bmatrix} T_0 \\ S_0 \end{bmatrix}, t\right) = \begin{bmatrix} T_A + (T_0 - T_A)e^{-k_T t} \\ S_A + (S_0 - S_A)e^{-k_S t} \end{bmatrix}$$

initial condition — time

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Dynamical Systems

Basic idea:
The dependence of the solution on ***initial conditions*** is just as important as its dependence on ***time***.

General notation:

$$x=\begin{bmatrix}T\\S\end{bmatrix} \qquad f(x)=\begin{bmatrix}-k_T & 0\\0 & -k_S\end{bmatrix}\begin{bmatrix}T-T_A\\S-S_A\end{bmatrix} \qquad \xi=\begin{bmatrix}T_0\\S_0\end{bmatrix} \qquad \dot{x}=\frac{dx}{dt} \qquad \dot{x}=f(x)$$

state variable — vector field — initial condition — dot notation — differential equation

flow

$$\varphi(\xi,t)=\varphi\left(\begin{bmatrix}T_0\\S_0\end{bmatrix},t\right)=\begin{bmatrix}T_A+(T_0-T_A)e^{-k_T t}\\S_A+(S_0-S_A)e^{-k_S t}\end{bmatrix}$$

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Dynamical Systems

The dependence of the solution on ***initial conditions*** is just as important as its dependence on ***time***.

$$x\in\mathbb{R}^n, \quad \xi\in\mathbb{R}^n, \quad f:\mathbb{R}^n\to\mathbb{R}^n$$

intital value problem

$$\dot{x}=f(x)$$
$$x(0)=\xi$$

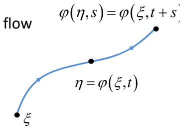
The initial value problem generates a flow

$$\varphi:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$$

with properties

intital condition $\varphi(\xi,0)=\xi$

"group property" $\varphi(\varphi(\xi,t),s)=\varphi(\xi,t+s)$



If we start the system at state ξ and follow the solution for time t, then restart the system at the new state and follow the solution for time s, we end up at the same state as starting at ξ and following for time $t+s$.

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Dynamical Systems

Example

initial value problem

$$\frac{d}{dt}\begin{bmatrix}T\\S\end{bmatrix}=\begin{bmatrix}-k_T & 0\\0 & -k_S\end{bmatrix}\begin{bmatrix}T-T_A\\S-S_A\end{bmatrix}$$
$$\begin{bmatrix}T\\S\end{bmatrix}(0)=\begin{bmatrix}T_0\\S_0\end{bmatrix}$$

flow

$$\varphi\left(\begin{bmatrix}T_0\\S_0\end{bmatrix},t\right)=\begin{bmatrix}T_A+(T_0-T_A)e^{-k_T t}\\S_A+(S_0-S_A)e^{-k_S t}\end{bmatrix}$$

group property

$$\varphi\left(\begin{bmatrix}T_0\\S_0\end{bmatrix},t+s\right)=\begin{bmatrix}T_A+(T_0-T_A)e^{-k_T(t+s)}\\S_A+(S_0-S_A)e^{-k_S(t+s)}\end{bmatrix}=\begin{bmatrix}T_A+(T_0-T_A)e^{-k_T(t+s)}\\S_A+(S_0-S_A)e^{-k_S(t+s)}\end{bmatrix}$$

$$\varphi\left(\varphi\left(\begin{bmatrix}T_0\\S_0\end{bmatrix},t\right),s\right)=\begin{bmatrix}T_A+\left(T_A+(T_0-T_A)e^{-k_T t}-T_A\right)e^{-k_T s}\\S_A+\left(S_A+(S_0-S_A)e^{-k_S t}-S_A\right)e^{-k_S s}\end{bmatrix}=\begin{bmatrix}T_A+(T_0-T_A)e^{-k_T(t+s)}\\S_A+(S_0-S_A)e^{-k_S(t+s)}\end{bmatrix}$$

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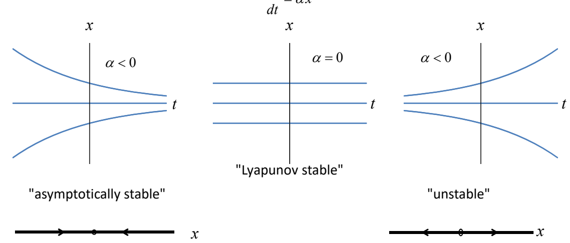
Dynamical Systems

Example

$$\frac{dx}{dt}=\alpha x$$



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Dynamical Systems

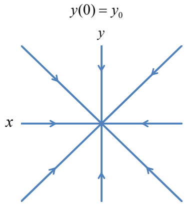
"Phase Plane"

Example

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} \qquad \begin{bmatrix}x\\y\end{bmatrix}(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$$

$$\dot{x}=-x \quad \dot{y}=-y \quad x(0)=x_0 \quad y(0)=y_0$$

$$\varphi\left(\begin{bmatrix}x_0\\y_0\end{bmatrix},t\right)=\begin{bmatrix}x_0e^{-t}\\y_0e^{-t}\end{bmatrix}$$



$$\begin{bmatrix}x_0\\y_0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix} \qquad \begin{bmatrix}x_0\\y_0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \qquad \begin{bmatrix}x_0\\y_0\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$$

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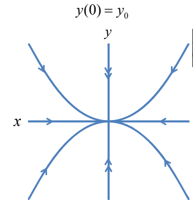
"Phase Plane"

Example

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} \qquad \begin{bmatrix}x\\y\end{bmatrix}(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$$

$$\dot{x}=-x \quad \dot{y}=-2y \quad x(0)=x_0 \quad y(0)=y_0$$

$$\varphi\left(\begin{bmatrix}x_0\\y_0\end{bmatrix},t\right)=\begin{bmatrix}x_0e^{-t}\\y_0e^{-2t}\end{bmatrix}$$



$$\begin{bmatrix}x_0\\y_0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix} \qquad \begin{bmatrix}x_0\\y_0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \qquad \begin{bmatrix}x_0\\y_0\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$$

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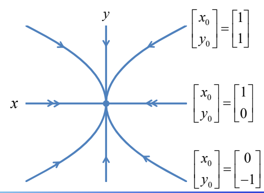
"Phase Plane"

Example

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2&0\\0&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\qquad \begin{bmatrix}x\\y\end{bmatrix}(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$$

$$\dot{x}=-2x,\quad \dot{y}=-y,\quad x(0)=x_0,\quad y(0)=y_0$$

$$\varphi\left(\begin{bmatrix}x_0\\y_0\end{bmatrix},t\right)=\begin{bmatrix}x_0e^{-2t}\\y_0e^{-t}\end{bmatrix}$$



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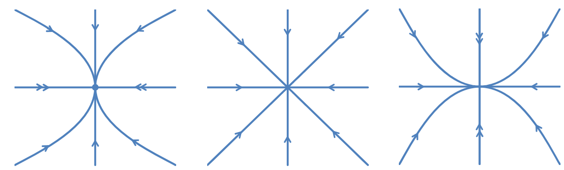
"Phase Plane"

Example: "stable nodes"

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\qquad \begin{bmatrix}x\\y\end{bmatrix}(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$$

$$\dot{x}=ax,\quad \dot{y}=by,\quad x(0)=x_0,\quad y(0)=y_0$$

$$\varphi\left(\begin{bmatrix}x_0\\y_0\end{bmatrix},t\right)=\begin{bmatrix}x_0e^{at}\\y_0e^{bt}\end{bmatrix}$$



$a<b<0$ $\quad a=b<0$ $\quad b<a<0$

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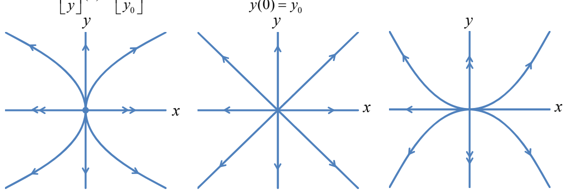
"Phase Plane"

Example: "unstable nodes"

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\qquad \begin{bmatrix}x\\y\end{bmatrix}(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$$

$$\dot{x}=ax,\quad \dot{y}=by,\quad x(0)=x_0,\quad y(0)=y_0$$

$$\varphi\left(\begin{bmatrix}x_0\\y_0\end{bmatrix},t\right)=\begin{bmatrix}x_0e^{at}\\y_0e^{bt}\end{bmatrix}$$



$\alpha>\beta>0$ $\quad \alpha=\beta>0$ $\quad 0<\alpha<\beta$

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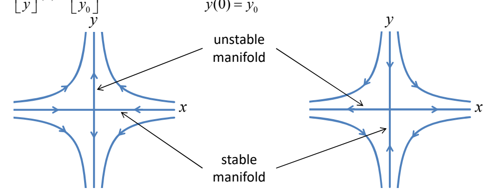
"Phase Plane"

Example: "saddles"

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\qquad \begin{bmatrix}x\\y\end{bmatrix}(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$$

$$\dot{x}=ax,\quad \dot{y}=by,\quad x(0)=x_0,\quad y(0)=y_0$$

$$\varphi\left(\begin{bmatrix}x_0\\y_0\end{bmatrix},t\right)=\begin{bmatrix}x_0e^{at}\\y_0e^{bt}\end{bmatrix}$$



$\alpha<0<\beta$ $\quad \beta<0<\alpha$

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