Math 5490
Topics in Applied Mathematics
Introduction to the Mathematics of Climate

Fall 2023
1:25 - 3:20 Tuesdays and Thursdays
Amundson Hall 162

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Can We Predict the Future?

If we know the state of a system now, do we know its state in the future?

For models based on differential equations, the answer is 'yes".

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n, \quad x = x_0 \text{ when } t = 0$$

If f is sufficiently smooth (e.g., continuously differentiable) then there is a unique solution of the differential equations satisfying the initial condition.

Interpretation:

If we know the state of the system now, we can compute its state in the future.

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Can We Predict the Future?

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n, \quad x = x_0 \text{ when } t = 0$$

If we know the state of the system now, we can compute its state in the future.

Yes, but how accurately?

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Can We Predict the Future?

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n, \quad x = x_0 \text{ when } t = 0$$

If we know the state of the system now, we can compute its state in the future.

Yes, but how accurately?

Last Time

Deterministic systems can behave randomly.

It can be impossible to tell whether a time series results from a stochastic process or a deterministic system.

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Example

$$\varphi(x) = 2x \bmod 1$$
$$x(t+1) = \varphi(x(t))$$

If all we can tell is whether $x(t)$ is less than or greater than ½, then we cannot distinguish solutions from the result of flipping a fair coin.

More precisely, let:

$$F(t) = \begin{cases} H & \text{if } x(t) > 1/2 \\ T & \text{if } x(t) < 1/2 \end{cases}$$

Then the sequence

$$F(0), F(1), F(2), \ldots$$

cannot be distinguished from a sequence produced by flipping a fair coin (for almost all starting points $x(0)$).

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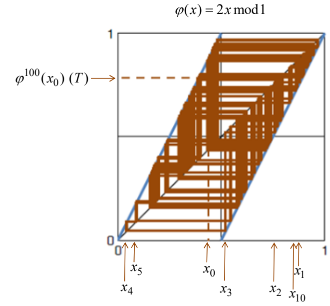
Coin Flips

$$\varphi(x) = 2x \bmod 1$$



$x_0 = 0.440301\ldots\ (T)$
$x_1 = 0.880602\ldots\ (H)$
$x_2 = 0.761204\ldots\ (H)$
$x_3 = 0.522409\ldots\ (H)$
$x_4 = 0.044818\ldots\ (T)$
$x_5 = 0.089637\ldots\ (T)$
$\vdots$
$x_{10} = 0.868400\ldots\ (H)$
$\vdots$
$x_{20} = 0.241922\ldots\ (T)$
$\vdots$
$x_{100} = 0.782505\ldots\ (H)$

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Example

$$\varphi(x) = 2x \bmod 1$$
$$x(t+1) = \varphi(x(t))$$

If all we can tell is whether $x(t)$ is less than or greater than ½, then we cannot distinguish solutions from the result of flipping a fair coin.

However …

If we know $x(0)$ precisely, then we know $x(t)$ precisely, for all time.

What can go wrong?

Suppose we know $x(0)$ only to some accuracy ε. Then, as time increases, we know $x(t)$ to less and less accuracy. Eventually, we know nothing.

For example, if $\varepsilon = ½$, we know nothing after only one time unit.

What if $\varepsilon = 10^{-6}$?

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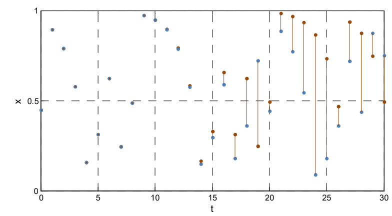
Example

$$\varphi(x) = 2x \bmod 1 \qquad x(t+1) = \varphi(x(t))$$



blue: $x_{\text{blue}}(0) = \sqrt{1/5}$

brown: $x_{\text{brown}}(0) = \sqrt{1/5} + 10^{-6}$

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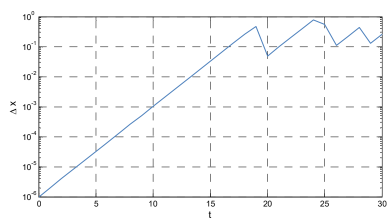
Example

$$\varphi(x) = 2x \bmod 1 \qquad x(t+1) = \varphi(x(t))$$



$$\Delta x(t) = \left| x_{\text{blue}}(t) - x_{\text{brown}}(t) \right|$$

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Lyapunov Multipliers (infinitesimal growth)

Consider: $x(t+1) = f(x(t))$

orbit: $x_{n+1} = f(x_n) = f^n(x_0), \quad x_0 = x(0)$

nearby orbit: $y_{n+1} = f(y_n) = f^n(y_0), \quad y_0 = x_0 + \xi$

$$|y_1 - x_1| = |f(y_0) - f(x_0)| = |f(x_0 + \xi) - f(x_0)| \approx |f'(x_0)\xi| = |f'(x_0)||\xi|$$
$$|y_2 - x_2| = |f(y_1) - f(x_1)| = |f(x_1 + y_1 - x_1) - f(x_1)| \approx |f'(x_1)(y_1 - x_1)| \approx |f'(x_1)||f'(x_0)||\xi|$$
$$\cdots$$
$$|y_n - x_n| \approx \prod_{k=0}^{n-1} |f'(x_k)||\xi|$$

average multiplier

$$\mu(x_0, n) = \left(\prod_{k=0}^{n-1} |f'(x_k)| \right)^{1/n}$$

average exponent

$$\lambda(x_0, n) = \log(\mu(x_0, n)) = \frac{1}{n} \sum_{k=0}^{n-1} \log |f'(x_k)|$$

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Lyapunov Exponents and Multipliers

orbit: $x_{n+1} = f(x_n) = f^{n+1}(x_0), \quad x_0 = x(0)$

Lyapunov multiplier

$$\mu(x_0) = \lim_{n\to\infty} \mu(x_0, n) = \lim_{n\to\infty} \left(\prod_{k=0}^{n-1} |f'(x_k)| \right)^{1/n}$$

Lyapunov exponent

$$\lambda(x_0) = \lim_{n\to\infty} \lambda(x_0, n) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |f'(x_k)|$$

Interpretation

If the Lyapunov exponent is greater than zero $(\lambda(x_0) > 0)$ or, equivalently, the Lyapunov multiplier is greater than one $(\mu(x_0) > 1)$, then nearby orbits diverge exponentially.

If the Lyapunov exponent is less than zero $(\lambda(x_0) < 0)$ or, equivalently, the Lyapunov multiplier is less than one $(\mu(x_0) < 1)$, then nearby orbits converge exponentially.

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Example

$$\varphi(x) = 2x \bmod 1$$
$$\varphi'(x) = 2$$

Lyapunov multiplier

$$\mu(x_0, n) = \left(\prod_{k=0}^{n-1} |\varphi'(x_k)| \right)^{1/n} = \left(\prod_{k=0}^{n-1} 2 \right)^{1/n} = \left(2^n\right)^{1/n} = 2$$
$$\mu(x_0) = \lim_{n\to\infty} 2 = 2$$

Lyapunov exponent

$$\lambda(x_0, n) = \frac{1}{n} \sum_{k=0}^{n-1} \log |\varphi'(x_k)| = \frac{1}{n} \sum_{k=0}^{n-1} \log 2 = \frac{1}{n} n \log 2 = \log 2$$
$$\lambda(x_0) = \lim_{n\to\infty} \log 2$$

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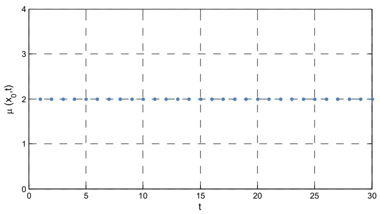
Example

$\varphi(x) = 2x \bmod 1 \qquad x(t+1) = \varphi(x(t))$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t} = 2$$

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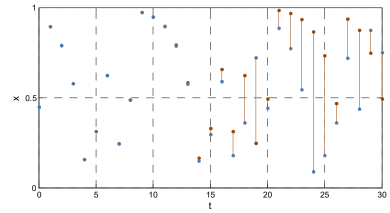
Example

$\varphi(x) = 2x \bmod 1 \qquad x(t+1) = \varphi(x(t))$



blue: $x_{blue}(0) = \sqrt{1/5}$

brown: $x_{brown}(0) = \sqrt{1/5} + 10^{-6}$

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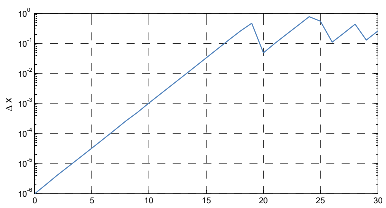
Example

$\varphi(x) = 2x \bmod 1 \qquad x(t+1) = \varphi(x(t))$



$$\Delta x(t) = |x_{blue}(t) - x_{brown}(t)|$$

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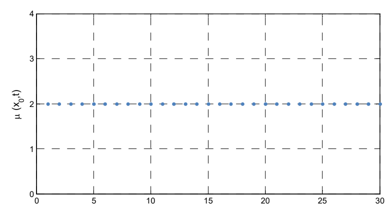
Example

$\varphi(x) = 2x \bmod 1 \qquad x(t+1) = \varphi(x(t))$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t} = 2$$

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Example

$\varphi(x) = 2x \bmod 1$

Lyapunov multiplier

$$\mu(x_0,n) = \left(\prod_{k=0}^{n-1} |\varphi'(x_k)|\right)^{1/n} = \left(\prod_{k=0}^{n-1} 2\right)^{1/n} = \left(2^n\right)^{1/n} = 2$$

$$\mu(x_0) = \lim_{n\to\infty} 2 = 2$$

Interpretation

At each step, the error multiplies by a factor of 2. After n steps, an error of ε becomes $2^n\varepsilon$. Since the diameter of the state space is 1, to have any knowledge of the state of the system after n steps requires an initial error of $\varepsilon < 2^{-n}$.

30 steps: $2^{-30} \approx 9\times 10^{-10}$

100 steps: $2^{-100} \approx 8\times 10^{-31}$

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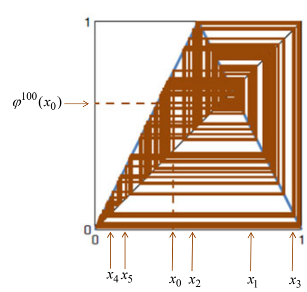
$\varphi(x) = 1 - 2|x - 1/2|$



$x_0 = 0.379750\ldots$
$x_1 = 0.759500\ldots$
$x_2 = 0.480998\ldots$
$x_3 = 0.961997\ldots$
$x_4 = 0.076004\ldots$
$x_5 = 0.152008\ldots$
$\vdots$
$x_{10} = 0.864271\ldots$
$\vdots$
$x_{20} = 0.985695\ldots$
$\vdots$
$x_{100} = 0.602054\ldots$

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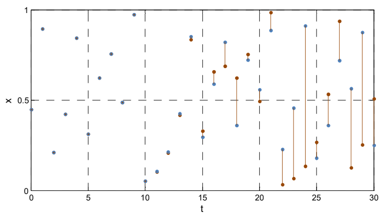
Example

$$\varphi(x)=\begin{cases}2x, & x<1/2\\ 2(1-x), & x\geq 1/2\end{cases}$$



blue: $x_{\text{blue}}(0)=\sqrt{1/5}$

brown: $x_{\text{brown}}(0)=\sqrt{1/5}+10^{-6}$

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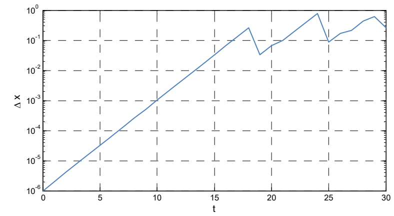

$$\Delta x(t)=\left|x_{\text{blue}}(t)-x_{\text{brown}}(t)\right|$$

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Example

$$\varphi(x)=\begin{cases}2x, & x<1/2\\ 2(1-x), & x\geq 1/2\end{cases}$$

$$\varphi'(x)=\begin{cases}2, & x<1/2\\ -2, & x\geq 1/2\end{cases}$$

Lyapunov multiplier

$$\mu(x_0,n)=\left(\prod_{k=0}^{n-1}\left|\varphi'(x_k)\right|\right)^{1/n}=\left(\prod_{k=0}^{n-1}2\right)^{1/n}=\left(2^n\right)^{1/n}=2$$

$$\mu(x_0)=\lim_{n\to\infty}2=2$$

Interpretation

Same as previous example:

At each step, the error multiplies by a factor of 2.

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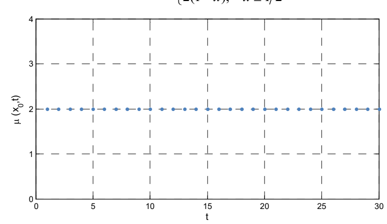
Example

$$\varphi(x)=\begin{cases}2x, & x<1/2\\ 2(1-x), & x\geq 1/2\end{cases}$$



$$\mu(x_0,t)=\left(\prod_{k=0}^{t-1}\left|\varphi'(x_k)\right|\right)^{1/t}=2$$

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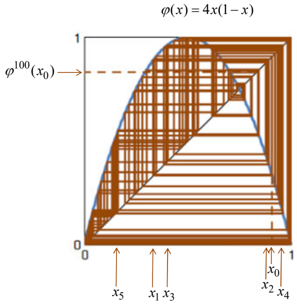
Logistic Growth

$$\varphi(x)=4x(1-x)$$



$x_0=0.908778\ldots$
$x_1=0.331601\ldots$
$x_2=0.886567\ldots$
$x_3=0.402260\ldots$
$x_4=0.961788\ldots$
$x_5=0.147006\ldots$
$\vdots$
$x_{10}=0.000639\ldots$
$\vdots$
$x_{20}=0.489364\ldots$
$\vdots$
$x_{100}=0.828471\ldots$

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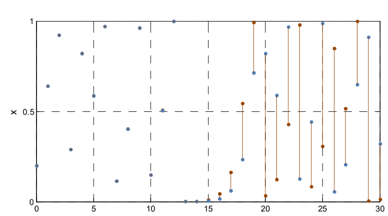
Example

$$\varphi(x)=4x(1-x)$$



blue: $x_{\text{blue}}(0)=1/5$

brown: $x_{\text{brown}}(0)=1/5+10^{-6}$

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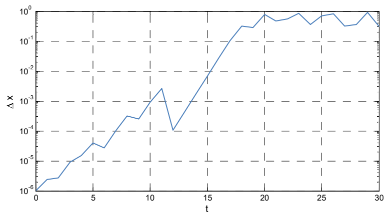
Example

$$\varphi(x) = 4x(1-x)$$



$$\Delta x(t) = |x_{\text{blue}}(t) - x_{\text{brown}}(t)|$$

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Example

$$\varphi(x) = 4x(1-x)$$
$$\varphi'(x) = 4 - 8x$$

Lyapunov multiplier

$$\mu(x_0, n) = \left(\prod_{k=0}^{n-1} |\varphi'(x_k)| \right)^{1/n} = ?$$

$$\mu(x_0) = \lim_{n\to\infty} \mu(x_0, n) = ?$$

Let $x_0 = 0.2$, so $\varphi'(x_0) = 4 - 8x_0 = 2.4$,

$$\mu(x_0, 1) = 2.4$$

$x_1 = 4x_0(1-x_0) = 0.64$, so $\varphi'(x_1) = 4 - 8x_1 = -1.12$,

$$\mu(x_1, 1) = (|\varphi'(x_0)| \cdot |\varphi'(x_1)|)^{1/2} \approx 1.64$$

$x_2 = 4x_1(1-x_1) = 0.9216$, so $\varphi'(x_2) = 4 - 8x_2 = -3.3728$,

$$\mu(x_2, 1) = (|\varphi'(x_0)| \cdot |\varphi'(x_1)| \cdot |\varphi'(x_2)|)^{1/3} \approx 2.085$$

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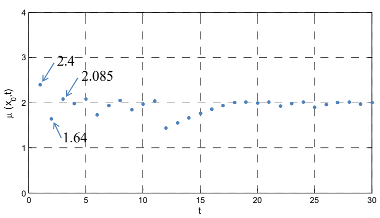
Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 1/5$$



$$\mu(x_0, t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t}$$

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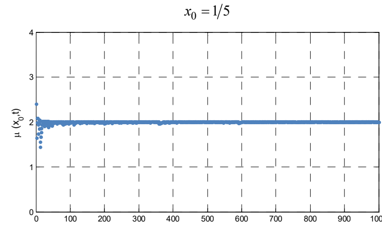
Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 1/5$$



Looks like $\mu(x_0) = 2$

$$\mu(x_0, t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t}$$

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Example

$$\varphi(x) = 4x(1-x)$$

Lyapunov multiplier

$$\mu(x_0, n) = \left(\prod_{k=0}^{n-1} |\varphi'(x_k)| \right)^{1/n}$$

Evidence indicates that

$$\mu(x_0) = \lim_{n\to\infty} \mu(x_0, n) = 2$$

Interpretation

On average, the error multiplies by a factor of 2 at each step.

After n steps, an error of ε becomes about $2^n \varepsilon$.

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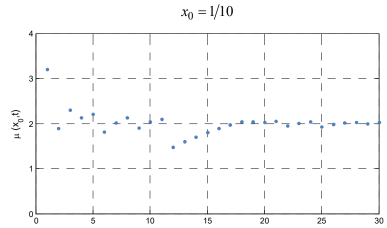
Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 1/10$$



$$\mu(x_0, t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t}$$

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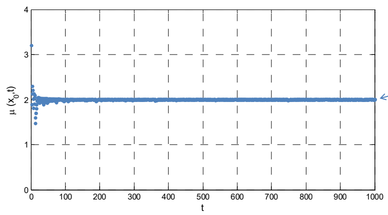
Example

$$\varphi(x)=4x(1-x)$$
$$x_0 = 1/10$$



Looks like $\mu(x_0)=2$

$$\mu(x_0,t)=\left(\prod_{k=0}^{t-1}\left|\varphi'(x_k)\right|\right)^{1/t}$$

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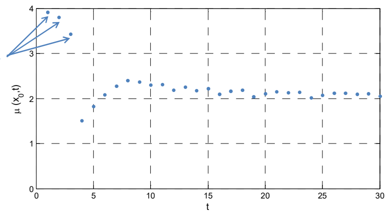
Example

$$\varphi(x)=4x(1-x)$$
$$x_0 = 1/100$$

Close to 4



$$\mu(x_0,t)=\left(\prod_{k=0}^{t-1}\left|\varphi'(x_k)\right|\right)^{1/t}$$

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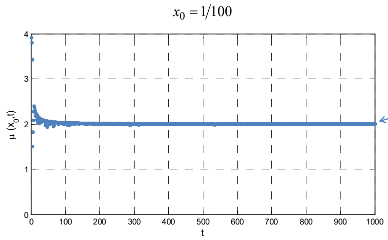
Example

$$\varphi(x)=4x(1-x)$$
$$x_0 = 1/100$$



Looks like $\mu(x_0)=2$

$$\mu(x_0,t)=\left(\prod_{k=0}^{t-1}\left|\varphi'(x_k)\right|\right)^{1/t}$$

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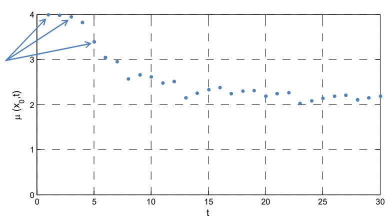
Example

$$\varphi(x)=4x(1-x)$$
$$x_0 = 1/1000$$

Close to 4



$$\mu(x_0,t)=\left(\prod_{k=0}^{t-1}\left|\varphi'(x_k)\right|\right)^{1/t}$$

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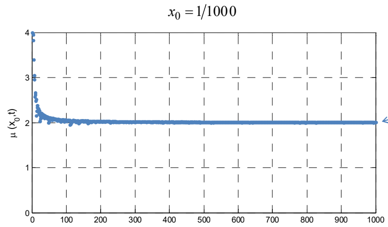
Example

$$\varphi(x)=4x(1-x)$$
$$x_0 = 1/1000$$



Looks like $\mu(x_0)=2$

$$\mu(x_0,t)=\left(\prod_{k=0}^{t-1}\left|\varphi'(x_k)\right|\right)^{1/t}$$

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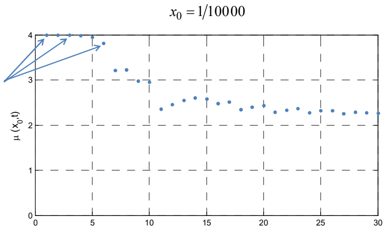
Example

$$\varphi(x)=4x(1-x)$$
$$x_0 = 1/10000$$

Close to 4



$$\mu(x_0,t)=\left(\prod_{k=0}^{t-1}\left|\varphi'(x_k)\right|\right)^{1/t}$$

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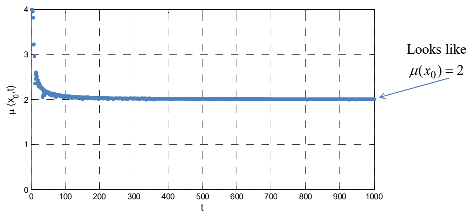
Example

$\varphi(x) = 4x(1-x)$

$x_0 = 1/10000$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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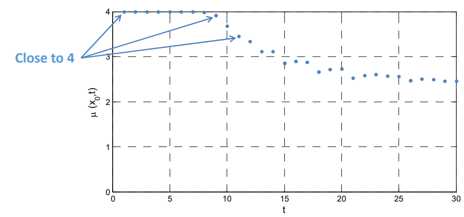
Example

$\varphi(x) = 4x(1-x)$

$x_0 = 10^{-6}$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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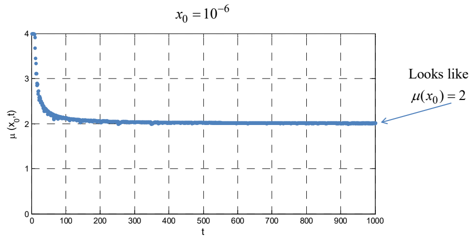
$x_0 = 10^{-6}$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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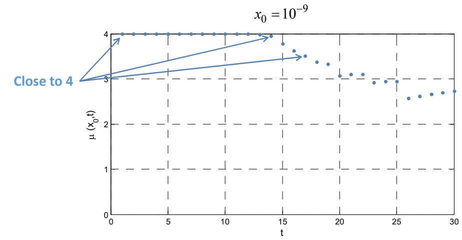
Dynamical Systems

Example

$\varphi(x) = 4x(1-x)$

$x_0 = 10^{-9}$



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Math 5490
Dynamical Systems

Example

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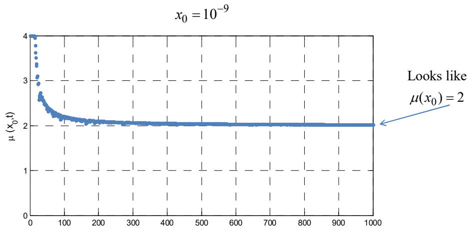


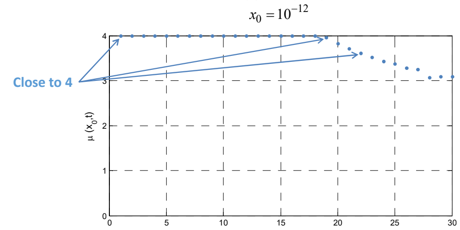
$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

Math 5490
Dynamical Systems

Example

$\varphi(x) = 4x(1-x)$

$x_0 = 10^{-12}$

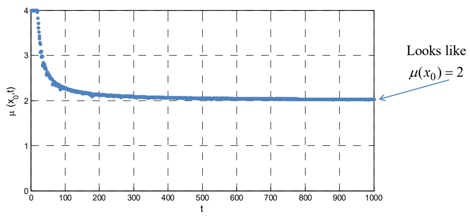


$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

Math 5490
Dynamical Systems

Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 10^{-12}$$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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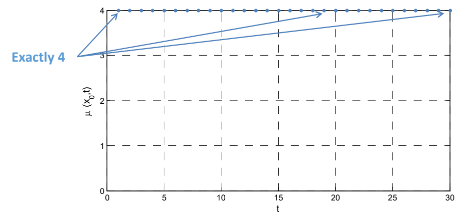
Dynamical Systems

Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 0$$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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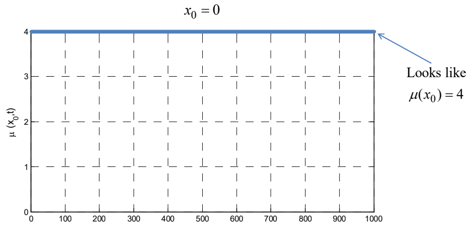
Dynamical Systems

Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 0$$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

Math 5490
Dynamical Systems

Example

$$\varphi(x) = 4x(1-x)$$

$$\varphi(0) = 0$$
$$\varphi'(0) = 4$$

Lyapunov multiplier

$$\mu(0,n) = \left(\prod_{k=0}^{n-1} |\varphi'(0)|\right)^{1/n} = \left(\prod_{k=0}^{n-1} 4\right)^{1/n} = \left(4^n\right)^{1/n} = 4$$

$$\mu(0) = \lim_{n\to\infty} \mu(0,n) = 4$$

Interpretation

The origin is a rest point with eigenvalue 4.
The Lyapunov multiplier is the eigenvalue.

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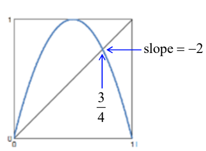
Dynamical Systems

Example

$$\varphi(x) = 4x(1-x)$$

$$\varphi\left(\frac{3}{4}\right) = 4\cdot\frac{3}{4}\left(1-\frac{3}{4}\right) = \frac{3}{4}$$
$$\varphi'\left(\frac{3}{4}\right) = 4 - 8\cdot\frac{3}{4} = -2$$



Lyapunov multiplier

$$\mu\left(\frac{3}{4},n\right) = \left(\prod_{k=0}^{n-1} |\varphi'(0)|\right)^{1/n} = \left(\prod_{k=0}^{n-1} 2\right)^{1/n} = \left(2^n\right)^{1/n} = 2$$

$$\mu\left(\frac{3}{4}\right) = \lim_{n\to\infty} \mu\left(\frac{3}{4},n\right) = 2$$

Interpretation

$\frac{3}{4}$ is a rest point with eigenvalue -2.

The Lyapunov multiplier is the absolute value of the eigenvalue.

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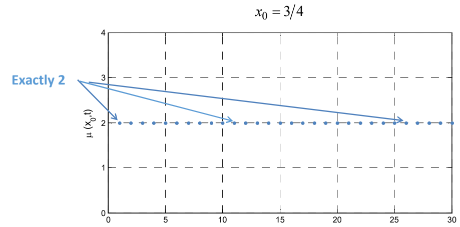
Dynamical Systems

Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 3/4$$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)|\right)^{1/t}$$

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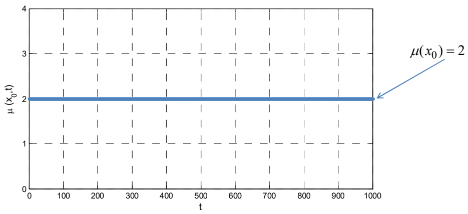
Dynamical Systems

Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 3/4$$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

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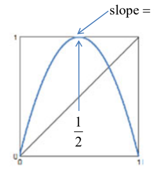
Dynamical Systems

Example

$$\varphi(x) = 4x(1-x)$$



$$\varphi(\tfrac{1}{2}) = 4\tfrac{1}{2}(1-\tfrac{1}{2}) = 1, \quad \varphi^2\left(\tfrac{1}{2}\right) = \varphi(1) = 0,$$
$$\varphi^n\left(\tfrac{1}{2}\right) = 0, \quad n \geq 2$$
$$\varphi'\left(\tfrac{1}{2}\right) = 4 - 8\cdot\tfrac{1}{2} = 0$$

Lyapunov multiplier

$$\mu\left(\tfrac{1}{2},n\right) = \left(\prod_{k=0}^{n-1}\left|\varphi'\left(\varphi^k\left(\tfrac{1}{2}\right)\right)\right|\right)^{1/n} = \left(0\times\prod_{k=1}^{n-1}\left|\varphi'\left(\varphi^k\left(\tfrac{1}{2}\right)\right)\right|\right)^{1/n} = (0)^{1/n} = 0$$
$$\mu\left(\tfrac{1}{2}\right) = \lim_{n\to\infty}\mu\left(\tfrac{1}{2},n\right) = 0$$

Interpretation

If $\varphi'\left(\varphi^n(x_0)\right) = 0$ for any n, then the Lyapunov multiplier of the orbit is 0.

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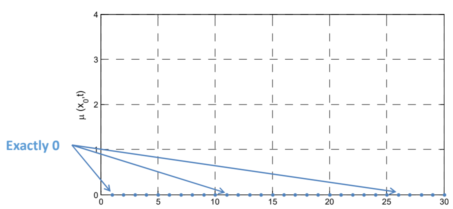
Dynamical Systems

Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = 1/2$$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

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Example

$$\varphi(x) = 4x(1-x)$$
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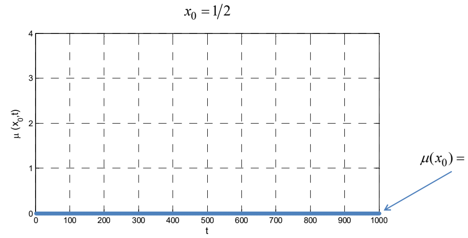

$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

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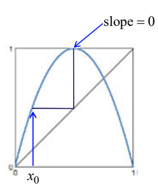
Dynamical Systems

Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = \frac{2-\sqrt{2}}{4}$$



$$x_1 = \varphi(x_0) = 4\cdot\frac{2-\sqrt{2}}{4}\cdot\left(1-\frac{2-\sqrt{2}}{4}\right) = 4\cdot\frac{2-\sqrt{2}}{4}\cdot\left(\frac{2+\sqrt{2}}{4}\right) = 4\cdot\frac{4-2}{16} = \frac{1}{2}$$
$$\varphi'(x_1) = 0$$

Lyapunov multiplier

$$\mu(x_0,n) = \left(\prod_{k=0}^{n-1}\left|\varphi'(x_k)\right|\right)^{1/n} = \left(\varphi'(x_0)\times 0\times\prod_{k=2}^{n-1}\left|\varphi'(x_k)\right|\right)^{1/n} = (0)^{1/n} = 0$$
$$\mu(x_0) = \lim_{n\to\infty}\mu(x_0,n) = 0$$

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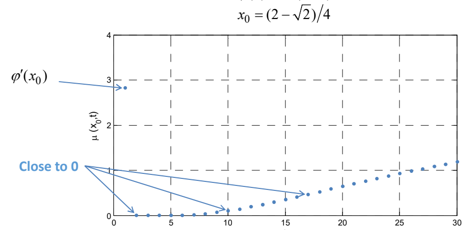
Dynamical Systems

Example

$$\varphi(x) = 4x(1-x)$$
$$x_0 = (2-\sqrt{2})/4$$



$$\mu(x_0,t) = \left(\prod_{k=0}^{t-1} \left|\varphi'(x_k)\right|\right)^{1/t}$$

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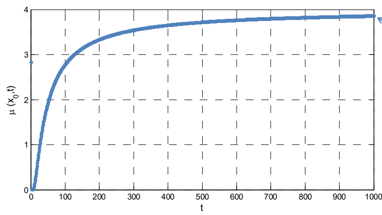
Example

$\varphi(x) = 4x(1-x)$

$x_0 = (2-\sqrt{2})/4$



Looks like $\mu(x_0) = 4$ (but it's not)

$$\mu(x_0, t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t}$$

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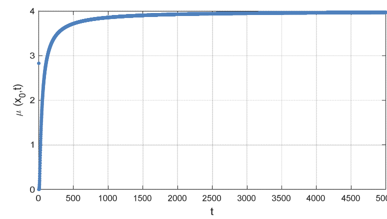
Example

$\varphi(x) = 4x(1-x)$

$x_0 = (2-\sqrt{2})/4$



Looks like $\mu(x_0) = 4$ (but it's not)

$$\mu(x_0, t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t}$$

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Example

$\varphi(x) = 4x(1-x)$

For almost every initial condition, the Lyapunov multiplier is 2.

$\mu(x_0) = 2$, for almost all x_0.

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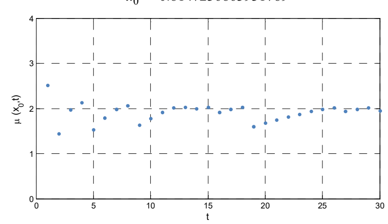
Example

$\varphi(x) = 4x(1-x)$

$x_0 =$ 0.8147236863931789 ← random number generated by MATLAB



$$\mu(x_0, t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t}$$

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Example

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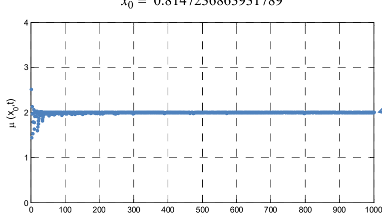
$x_0 =$ 0.8147236863931789 ← random number generated by MATLAB



$\mu(x_0) = 2$

$$\mu(x_0, t) = \left(\prod_{k=0}^{t-1} |\varphi'(x_k)| \right)^{1/t}$$

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Dynamical Systems

Summary

orbit: $x_{n+1} = f(x_n) = f^{n+1}(x_0), \quad x_0 = x(0)$

Lyapunov multiplier

$$\mu(x_0) = \lim_{n\to\infty} \mu(x_0, n) = \lim_{n\to\infty} \left(\prod_{k=0}^{n-1} |f'(x_k)| \right)^{1/n}$$

Lyapunov exponent

$$\lambda(x_0) = \lim_{n\to\infty} \lambda(x_0, n) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log|f'(x_k)|$$

Interpretation

If the Lyapunov exponent is greater than zero $(\lambda(x_0) > 0)$ or, equivalently, the Lyapunov multiplier is greater than one $(\mu(x_0) > 1)$, then nearby orbits diverge exponentially.

If the Lyapunov exponent is less than zero $(\lambda(x_0) < 0)$ or, equivalently, the Lyapunov multiplier is less than one $(\mu(x_0) < 1)$, then nearby orbits converge exponentially.

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Dynamical Systems

Summary

Positive Lyapunov exponents are associated with unpredictability.

Errors propagate exponentially.

Orbits behave randomly.

Simple deterministic systems can have random behavior.