Midterm Exam Solutions

October 10, 2008

(36) **1.** Define each of the following statements or notation.

(4) **a.** $\mathcal{M}$ is a σ-algebra on the set X.

If $\mathcal{M} \subset \mathcal{P}(X)$ is nonempty and closed under complements and countable unions, then $\mathcal{M}$ is a σ-algebra.

(4) **b.** μ is a measure on the measurable space $(X, \mathcal{M})$.

If $\mu : \mathcal{M} \to \overline{\mathbb{R}}$ is countably additive and satisfies $\mu(\varnothing) = 0$, then μ is a measure.

(4) **c.** $f : X \to \mathbb{R}$ is measurable.

If the inverse image of every Borel set in $\mathbb{R}$ is a measurable set in X, then is f is measurable.

(4) **d.** $\varphi : X \to \mathbb{R}$ is simple.

If the image of $\varphi : X \to \mathbb{R}$ is finite, then φ is simple.

(4) **e.** $\int \varphi \, d\mu$, where φ is a nonnegative measurable simple function.

If $\varphi = \sum_{k=1}^{n} a_n \chi_{E_n}$ is the standard representation of φ, then $\int \varphi d\mu = \sum_{k=1}^{n} a_k \mu(E_k)$

(4) **f.** $\int f \, d\mu$, where f is a nonnegative measurable function.

$\int f \, d\mu = \sup\left\{ \int \varphi \, d\mu : 0 \le \varphi \le f \text{ and } \varphi \text{ is a measurable simple function} \right\}$.

(4) **g.** $f \in L^1$.

If f is a measurable function satisfying $\int |f| \, d\mu < \infty$, then $f \in L^1$.

(4) **h.** $f_n \to f$ almost everywhere.

If $f_n(x) \to f(x)$ for all $x \in E$, where $\mu(E^c) = 0$, then $f_n \to f$ almost everywhere.

(4) **i.** $f_n \to f$ in L^1.

If $f_n \in L^1$, for every n, if $f \in L^1$, and if $\int |f_n - f| \, d\mu \to 0$, then $f_n \to f$ in L^1.

(20) **2.**

(10) **a.** State the Monotone Convergence Theorem, and give an example to show that monotonicity is a necessary hypothesis.

Theorem. If $\{f_n\}$ is a sequence in L^+ such that $f_j \leq f_{j+1}$ for all j, and if $f = \lim_{n\to\infty} f_n$, then

$$\int f\, d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

Example. Let $f_n = n\chi_{(0,1/n)}$. Then $f = \lim_{n\to\infty} f_n = 0$, and $\int f_n d\mu = 1$, for all n. $\{f_n\}$ is not monotone, and

$$\int f\, d\mu = 0 \neq 1 = \lim_{n\to\infty} \int f_n d\mu.$$

(10) **b.** State Fatou's Lemma, and give an example to show that the inequality cannot be replace with equality.

Lemma. If $\{f_n\}$ is a sequence in L^+, then

$$\int \left(\liminf_{n\to\infty} f_n \right) d\mu = \liminf_{n\to\infty} \int f_n d\mu.$$

Example. Let $f_n = n\chi_{(0,1/n)}$. Then $f = \liminf_{n\to\infty} f_n = 0$, and $\int f_n d\mu = 1$, for all $f_n = n\chi_{(0,1/n)}$.

$$\int f\, d\mu = 0 \neq 1 = \liminf_{n\to\infty} \int f_n d\mu.$$

(20) **3.** If $(X, \mathcal{M}, \mu)$ is a measure space, and if $\{f_n\}$ is a sequence of measurable functions on X, then $\{x : \lim f_n(x) \text{ exists}\}$ is a measurable set.

Solution. Let $E = \{x : \lim f_n(x) \text{ exists}\}$, and let

$$g_1(x) = \liminf_{n\to\infty} f_n(x),$$
$$g_2(x) = \limsup_{n\to\infty} f_n(x).$$

Then g_1 and g_2 are $\overline{\mathbb{R}}$-valued measurable functions, and

$$E = \{x : g_1(x) = g_2(x)\}.$$

Let

$$A = \{x : g_1(x) > -\infty\} \cap \{x : g_2(x) < \infty\}.$$

Note that A is measurable. Since $g_1 \le g_2$, we can also write

$$A = \{x : -\infty < g_1(x) \le g_2(x) < \infty\}.$$

Therefore $g_1 : A \to \mathbb{R}$ and $g_2 : A \to \mathbb{R}$ are both measurable functions and hence so is $g_2 - g_1$. It follows that

$$E_1 = \{x \in A : (g_1 - g_2)(x) = 0\} = \{x \in A : g_1(x) = g_2(x)\}$$

is measurable. Now let

$$E_2 = \{x : g_1(x) = g_2(x) = \infty\} = \{x : g_1(x) = \infty\},$$
$$E_3 = \{x : g_1(x) = g_2(x) = -\infty\} = \{x : g_2(x) = -\infty\}.$$

Both E_2 and E_3 are measurable. Since

$$E = E_1 \cup E_2 \cup E_3,$$

it follows that E is measurable.

(24) **4.** Suppose that $E \subset \mathbb{R}$ has finite Lebesgue measure. Show that $m\left(E \cap [x,\infty)\right) \to 0$ as $x \to \infty$.

Solution. Let

$$E_n = E \cap [n,\infty), \text{ for } n \in \mathbb{N}.$$

Then $E_n \supset E \cap [x,\infty)$ whenever $x \ge n$, so it suffices to show that

$$m(E_n) \to 0 \text{ as } n \to \infty$$

Note that $E_{n+1} \subset E_n$ and that $\bigcap_{n=1}^{\infty} E_n = \varnothing$. Note also that $E_1 \subset E$, and hence that $m(E_1) < m(E) < \infty$. Therefore, continuity from below implies that

$$\lim_{n\to\infty} m(E_n) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = m(\varnothing) = 0.$$