Midterm Exam

November 21, 2008

Closed book exam. Books, notes, and electronic devices may not be used.

(24) **1.** Define each of the following statements or notation. For parts (a) through (d), assume that ν and λ are signed measures and μ is a positive measure on a measurable space $(X, \mathcal{M})$.

(4) **a.** $\nu \perp \lambda$

(4) **b.** $|\nu|$

(4) **c.** $\nu \ll \mu$.

(4) **d.** $\dfrac{d\nu}{d\mu}$

(4) **e.** $F : \mathbb{R} \to \mathbb{R}$ is of bounded variation.

(4) **f.** $F : \mathbb{R} \to \mathbb{R}$ is absolutely continuous.

(20) **2.**

(10) **a.** State Fubini's Theorem for L^1 functions.

(10) **b.** You may use the following formula:

$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} \, dx \right) dy = -\frac{\pi}{4} \neq \frac{\pi}{4} = \int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} \, dy \right) dx$$

Explain why this formula does not contradict Fubini's Theorem.

(20) **3.**

(10) **a.** State the Fundamental Theorem of Calculus for Lebesgue integrals.

(10) **b.** Give an example of a continuous increasing function $F:[0,1]\to\mathbb{R}$ such that

$$F(1)-F(0)\neq\int_0^1 F'(t)\,dt$$

(20) **4.** Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, $\mu = \nu =$ counting measure. Define

$$f(m,n) = \begin{cases} 1 & \text{if } m = n, \\ -1 & \text{if } m = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\int |f| \, d(\mu \times \nu) = \infty$ and that $\iint f \, d\mu \, d\nu$ and $\iint f \, d\nu \, d\mu$ exist and are unequal.

(16) **5.** Suppose that $F:[a,b]\to\mathbb{R}$ and $G:[a,b]\to\mathbb{R}$ are absolutely continuous. Show that FG is absolutely continuous.