Midterm Exam Solutions

November 21, 2008

(24) **1.** Define each of the following statements or notation. For parts (a) through (d), assume that ν and λ are signed measures and μ is a positive measure on a measurable space $(X, \mathcal{M})$.

(4) **a.** $\nu \perp \lambda$

Solution. There exist $E \in \mathcal{M}$ and $F \in \mathcal{M}$, with $E \cap F = \varnothing$ and $E \cup F = X$, such that E is null for ν and F is null for λ.

(4) **b.** $|\nu|$

Solution. $|\nu| = \nu^+ + \nu^-$, where $\nu = \nu^+ \nu^-$ is the Jordan decomposition of ν.

(4) **c.** $\nu \ll \mu$.

Solution. For every $E \in \mathcal{M}$, $\mu(E) = 0 \Rightarrow \nu(E) = 0.$

(4) **d.** $\dfrac{d\nu}{d\mu}$

Solution. $\nu \ll \mu$ and $\dfrac{d\nu}{d\mu} = f : X \to \mathbb{R}$ is an extended μ-integrable function satisfying

$$d\nu = f\, d\mu$$

(4) **e.** $F : \mathbb{R} \to \mathbb{R}$ is of bounded variation.

Solution.
$$\sup\left\{ \sum_{k=1}^{n} \left| F(x_k) - F(x_{k-1}) \right| : n \in \mathbb{N},\ -\infty < x_0 < \cdots x_n < \infty \right\} < \infty$$

(4) **f.** $F : \mathbb{R} \to \mathbb{R}$ is absolutely continuous.

Solution. For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any finite set of disjoint intervals $(a_1, b_1), \ldots, (a_n, b_n)$,

$$\sum_{k=1}^{n} (b_k - a_k) < \delta \quad \Rightarrow \quad \sum_{k=1}^{n} \left| F(b_k) - F(a_k) \right| < \varepsilon.$$

(20) **2.**

(10) **a.** State Fubini's Theorem for L^1 functions.

Solution. Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be σ-finite measure spaces. If $f\in L^1(\mu\times\nu)$, then $f_x\in L^1(\nu)$ for a.e. $x\in X$, $f^y\in L^1(\mu)$ for a.e. $y\in Y$, $\int f_x d\nu\in L^1(\mu)$, $\int f^y d\mu\in L^1(\nu)$, and

$$\int f\,d(\mu\times\nu)=\iint f\,d\mu d\nu=\iint f\,d\nu d\mu.$$

(10) **b.** You may use the following formula:

$$\int_0^1\left(\int_0^1\frac{x^2-y^2}{\left(x^2+y^2\right)^2}dx\right)dy=-\frac{\pi}{4}\neq\frac{\pi}{4}=\int_0^1\left(\int_0^1\frac{x^2-y^2}{\left(x^2+y^2\right)^2}dy\right)dx$$

Explain why this formula does not contradict Fubini's Theorem.

Solution. The formula does not contradict Fubini's Theorem because the function

$$f(x,y)=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

is not in $L^1([0,1]\times[0,1])$. To see this, note that $|f|$ is symmetric about the line $x=y$. Therefore, by Tonelli's Theorem,

$$\iint_{[0,1]\times[0,1]}|f|\,dm^2=2\int_0^1\left(\int_0^y\frac{y^2-x^2}{\left(x^2+y^2\right)^2}dx\right)dy=2\int_0^1\left[\frac{x}{x^2+y^2}\right]_{x=0}^{x=y}dy=2\int_0^1\frac{1}{2y}dy=\infty$$

(20) **3.**

(10) **a.** State the Fundamental Theorem of Calculus for Lebesgue integrals.

Solution. If $F:[a,b]\to\mathbb{R}$, then the following statements are equivalent.

a. F is absolutely continuous on $[a,b]$.

b. $F(x)-F(a)=\int_a^x f(t)\,dt$ for some $f\in L^1([a,b],m)$.

c. F is differentiable a.e. on $[a,b]$, $F'\in L^1([a,b],m)$, and $F(x)-F(a)=\int_a^x F'(t)\,dt$.

(10) **b.** Give an example of a continuous increasing function $F:[0,1]\to\mathbb{R}$ such that

$$F(1)-F(0)\neq\int_0^1 F'(t)\,dt$$

Solution. Let F be the middle third Cantor function. That is, $F:[0,1]\to[0,1]$ is a continuous increasing function such that F is constant on every interval in C^c, where C is the middle third Cantor set. Since $m(C)=0$, $F'=0$ a.e. Therefore,

$$F(1)-F(0)=1-0=1\neq 0=\int_0^1 F'(t)\,dt.$$

(20) **4.** Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, $\mu = \nu =$ counting measure. Define

$$f(m,n) = \begin{cases} 1 & \text{if } m = n, \\ -1 & \text{if } m = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\int |f| d(\mu \times \nu) = \infty$ and that $\iint f\, d\mu d\nu$ and $\iint f\, d\nu d\mu$ exist and are unequal.

Solution. Since μ and ν are σ-finite and since all functions are measurable for $\mathcal{M} \otimes \mathcal{N} = \mathcal{P}(\mathbb{N} \times \mathbb{N})$, Tonelli's Theorem implies that

$$\int |f| d(\mu \times \nu) = \int \left(\int |f| d\mu \right) d\nu = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} |f(m,n)| \right) = \sum_{n=1}^{\infty} \left(\sum_{m=n}^{n+1} 1 \right) = \sum_{n=1}^{\infty} 2 = \infty.$$

On the other hand,

$$\int \left(\int f\, d\mu \right) d\nu = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} f(m,n) \right) = \sum_{n=1}^{\infty} \left(\sum_{m=n}^{n} 1 + \sum_{m=n+1}^{n+1} (-1) \right) = \sum_{n=1}^{\infty} 0 = 0,$$

while

$$\int \left(\int f\, d\nu \right) d\mu = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} f(m,n) \right) = 1 + \sum_{m=2}^{\infty} \left(\sum_{n=m-1}^{m-1} (-1) + \sum_{n=m}^{m} 1 \right) = 1 + \sum_{m=2}^{\infty} 0 = 1.$$

(16) **5.** Suppose that $F:[a,b]\to\mathbb{R}$ and $G:[a,b]\to\mathbb{R}$ are absolutely continuous. Show that FG is absolutely continuous.

Solution. First note that

$$\begin{aligned}\left|F(x)G(x)-F(y)G(y)\right| &= \left|F(x)G(x)-F(x)G(y)+F(x)G(y)-F(y)G(y)\right| \\ &\le \left|F(x)G(x)-F(x)G(y)\right|+\left|F(x)G(y)-F(y)G(y)\right| \\ &\le \left|F(x)\right|\left|G(x)-G(y)\right|+\left|G(y)\right|\left|F(x)-F(y)\right|.\end{aligned}$$

Since F and G are absolutely continuous, they are continuous. Since continuous functions on compact sets are bounded, F and G are bounded on $[a,b]$. Therefore, there exists an M such that

$$F(x)\le M \text{ and } G(x)\le M \text{ for all } x\in[a,b],$$

which implies that

$$\left|FG(x)-FG(y)\right|\le M\left|G(x)-G(y)\right|+M\left|F(x)-F(y)\right|.$$

for all $x,y\in[a,b]$. Given $\varepsilon>0$, choose $\delta>0$ so that

$$\sum_{k=1}^{n}(b_k-a_k)<\delta \quad\Rightarrow\quad \sum_{k=1}^{n}\left|F(b_k)-F(a_k)\right|<\frac{\varepsilon}{2M}$$

and

$$\sum_{k=1}^{n}(b_k-a_k)<\delta \quad\Rightarrow\quad \sum_{k=1}^{n}\left|G(b_k)-G(a_k)\right|<\frac{\varepsilon}{2M}$$

for any finite set of disjoint intervals $(a_1,b_1),\ldots,(a_n,b_n)$. Therefore, $\sum_{k=1}^{n}(b_k-a_k)<\delta$ implies that

$$\begin{aligned}\sum_{k=1}^{n}\left|FG(b_k)-FG(a_k)\right| &\le \sum_{k=1}^{n}\left(M\left|G(b_k)-G(a_k)\right|+M\left|F(b_k)-F(a_k)\right|\right) \\ &\le M\sum_{k=1}^{n}\left|G(b_k)-G(a_k)\right|+M\sum_{k=1}^{n}\left|F(b_k)-F(a_k)\right| \\ &< M\,\frac{\varepsilon}{2M}+M\,\frac{\varepsilon}{2M}=\varepsilon,\end{aligned}$$

which shows that FG is absolutely continuous.