Final Exam

December 15, 2008

Closed book exam. Books, notes, and electronic devices may not be used. Answers should be complete, concise, and mathematically rigorous.

(48) **1.** Define each of the following statements or notation.

(4) **a.** $\mathscr{M}$ is a σ-algebra on the set X.

(4) **b.** $\mathscr{T}$ is a topology on the set X.

(4) **c.** $f : X \to \mathbb{R}$ is measurable, where $(X, \mathscr{M})$ is a measurable space.

(4) **d.** $f : X \to \mathbb{R}$ is continuous, where $(X, \mathscr{T})$ is a topological space.

(4) **e.** $f \in L^1(X, \mathbb{R})$.

(4) **f.** $f \in BC(X,\mathbb{R})$.

(4) **g.** $f_n \to f$ in $L^1(X,\mathbb{R})$.

(4) **h** $f_n \to f$ in $BC(X,\mathbb{R})$.

(4) **i.** $\nu \ll \mu$.

(4) **j.** $F:\mathbb{R} \to \mathbb{R}$ is absolutely continuous.

(4) **k.** $(X,\mathcal{T})$ is a Hausdorff space.

(4) **l.** $(X,\mathcal{T})$ is a separable topological space.

(16) **2.** Suppose that $f_1, f_2 \ldots$ and f are in $L^1(\mathbb{R},\mathbb{R}) \cap BC(\mathbb{R},\mathbb{R})$. Discuss the relation between the statements "$f_n \to f$ in $L^1(\mathbb{R},\mathbb{R})$" and "$f_n \to f$ in $BC(\mathbb{R},\mathbb{R})$". In particular, does either one imply the other?

(20) **3.** State the Dominated Convergence Theorem, and give an example to show that the conclusion is false without the hypothesis of a dominating function.

(15) **4.** Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of measurable functions on X. Show that $\{x : \lim f_n(x) \text{ exists}\}$ is a measurable set.

(15) **5.** Let $f:\mathbb{R}\to\mathbb{R}$ be an L^1 function. Show that $\int_x^\infty f(t)\,dt \to 0$ as $x\to\infty$.

(15) **6.** Give an example of a continuous increasing function $F:[0,1]\to\mathbb{R}$ such that

$$F(1)-F(0)\neq\int_0^1 F'(t)\,dt$$

(15) **7.** Suppose that $F:[a,b]\to\mathbb{R}$ and $G:[a,b]\to\mathbb{R}$ are absolutely continuous. Show that FG is absolutely continuous.

(20) **8.**

(10) **a.** State Fubini's Theorem for L^1 functions.

(10) **b.** Let $X = Y = \mathbb{N}$, $\mathscr{M} = \mathscr{N} = \mathscr{P}(\mathbb{N})$, $\mu = \nu =$ counting measure. Define

$$f(m,n) = \begin{cases} 1 & \text{if } m = n, \\ -1 & \text{if } m = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f satisfies neither the hypotheses nor the conclusions of Fubini's Theorem.

(16) **9.** Let

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x > 0, \\ 0, & \text{if } x \le 0, \end{cases}$$

and let $\nu(E) = \int_E f(x)\,dx$ for Lebesgue measurable E .

(8) **a.** Is ν absolutely continuous with respect to Lebesgue measure?

(8) **b.** Is f absolutely continuous on the open interval $(0,1)$?

(20) **10.** Let $X = \{1, \ldots N\}$, and let $\mathcal{T} = \{U_n : n = 0, \ldots, N+1\}$, where $U_n = \{k \in X : k > n\}$

(5) **a.** Show that $(X, \mathcal{T})$ is a topological space.

(5) **b.** Is there a metric on X that generates the topology $\mathcal{T}$?

(5) **c.** Is X connected?

(5) **d.** Is the function $f : X \to \mathbb{R} : f(n) = n$ continuous?