Midterm Exam

October 16, 2009

Closed book exam. Books, notes, and electronic devices may not be used.

(16) **1.** Define each of the following statements or notation.

(4) **a.** $\mathscr{M}$ is a σ-algebra on the set X.

(4) **b.** $(X, \mathscr{M})$ is a measurable space.

(4) **c.** μ is a measure on the measurable space $(X, \mathscr{M})$.

(4) **d.** $f : X \to \mathbb{R}$ is measurable.

(15) **2.**

(5) **a.** Let $(X, \mathcal{M}, \mu)$ be a measure space. State the basic theorem about "continuity from above."

(10) b. Let $E \subset \mathbb{R}$ have finite Lebesgue measure, and let x_n be an increasing sequence of real numbers such that $x_n \to \infty$ as $n \to \infty$. Show that $m(E \cap [x_n, \infty)) \to 0$ as $n \to \infty$.

(16) **3.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let f_n be a sequence of real-valued functions on X.

(4) b. Define "f_n converges almost everywhere."

(4) b. Define "f_n converges in L^1.

(8) b. Give an example of a sequence converging almost everywhere but not in L^1.

(15) 4.

(5) **a.** State Fatou's Lemma.

(10) **b.** Let $(X, \mathcal{M}, \mu)$ be a measure space, let f_n be a sequence of real-valued functions on X converging pointwise to f, and suppose that $\liminf \int |f_n| = 0$. Show that $f = 0$ almost everywhere.

(20) 5.

(5) a. State the Dominated Convergence Theorem.

(15) b. Let $E \subset \mathbb{R}$ be Lebesgue measurable, and let $f(t) = \int_E \sin xt \, dm(x)$. Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous.

(18) **6.** Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Show that μ_E is a measure on $(X, \mathcal{M})$.