Midterm Exam Solutions

October 16, 2009

(16) **1.** Define each of the following statements or notation.

(4) **a.** $\mathcal{M}$ is a σ-algebra on the set X.

If $\mathcal{M} \subset \mathcal{P}(X)$ is nonempty and closed under complements and countable unions, then $\mathcal{M}$ is a σ-algebra.

(4) **b.** $(X, \mathcal{M})$ is a measurable space.

If X is a set and if $\mathcal{M}$ is a σ-algebra on X, then $(X, \mathcal{M})$ is a measurable space.

(4) **c.** μ is a measure on the measurable space $(X, \mathcal{M})$.

If $\mu : \mathcal{M} \to [0, \infty]$ is countably additive and satisfies $\mu(\varnothing) = 0$, then μ is a measure.

(4) **d.** $f : X \to \mathbb{R}$ is measurable.

If the inverse image of every Borel set in $\mathbb{R}$ is a measurable set in X, then is f is measurable.

(15) **2.**

(5) **a.** Let $(X, \mathcal{M}, \mu)$ be a measure space. State the fundamental theorem about "continuity from above."

Theorem. If $\{E_n\}$ is a sequence in $\mathcal{M}$ such that $E_1 \supset E_2 \supset \cdots$, and if $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

(10) **b.** Let $E \subset \mathbb{R}$ have finite Lebesgue measure, and let x_n be an increasing sequence of real numbers such that $x_n \to \infty$ as $n \to \infty$. Show that $m(E \cap [x_n, \infty)) \to 0$ as $n \to \infty$.

Solution. Since x_n is increasing, $E_{n+1} \subset E_n$ Note that $x_n \to \infty$ implies that $\bigcap_{n=1}^{\infty} E_n = \varnothing$. Note also that $E_1 \subset E$, and hence that $m(E_1) < m(E) < \infty$. Therefore, continuity from below implies that

$$\lim_{n\to\infty} m(E_n) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = m(\varnothing) = 0.$$

(16) **3.** Let $(X,\mathcal{M},\mu)$ be a measure space and let f_n be a sequence of real-valued measurable functions on X.

(4) **a.** Define " f_n converges almost everywhere."

If $E=\{x\in X : f_n(x) \text{ converges}\}$ and if $\mu(E^c)$=0, then f_n converges almost everywhere.

(4) **b.** Define " f_n converges in L^1 ".

If $f_n \in L^1$ for all n and if there exists a function $f\in L^1$ such that $\int |f_n - f| \to 0$ as $n\to\infty$, then f_n converges in L^1.

(8) **c.** Give an example of a sequence converging almost everywhere but not in L^1.

Trivial Solution. Let $(X,\mathcal{M},\mu)=(\mathbb{R},\mathcal{L},m)$ and let $f_n=1$ for all n. Then $f_n\to 1$ pointwise, but $f_n\notin L^1$, so f_n does not converge in L^1.

Better Solution. Let $(X,\mathcal{M},\mu)=(\mathbb{R},\mathcal{L},m)$ and let $f_n=\chi_{(n-1,n)}$ for $n\in\mathbb{N}$. Then $f_n\to 0$ pointwise, and hence a.e.

Claim: f_n does not converge in L^1.

Proof: Suppose that $f_n\to f$ in L^1. Then, for $n>N$,

$$\int_{(-\infty,N)}|f| = \int_{(-\infty,N)}|f_n - f| \le \int_{\mathbb{R}}|f_n - f| \to 0 .$$

Therefore, $\int_{(-\infty,N)}|f|=0$, which implies that $f=0$ a.e. on $(-\infty,N)$, and hence that $f=0$ a.e. on $\mathbb{R}$. We have shown that, if $f_n\to f$ in L^1, then $f=0$ a.e. But $\int |f_n - 0| = 1$ for all n, so f_n does not converge to f in L^1, so f_n does not converge in L^1.

(15) 4.

(5) **a.** State Fatou's Lemma.

Lemma. If $\{f_n\}$ is a sequence in L^+, then

$$\int\left(\liminf_{n\to\infty} f_n\right) \le \liminf_{n\to\infty}\int f_n.$$

(10) **b.** Let $(X,\mathcal{M},\mu)$ be a measure space, let f_n be a sequence of measurable real-valued functions on X converging pointwise to f, and suppose that $\liminf \int |f_n| = 0$. Show that $f=0$ almost everywhere.

Solution. Since $f_n\to f$ pointwise, $|f_n|\to|f|$ pointwise, so Fatou's Lemma implies that

$$\int |f| = \int\left(\lim_{n\to\infty}|f_n|\right) = \int\left(\liminf_{n\to\infty}|f_n|\right) \le \liminf_{n\to\infty}\int |f_n| = 0.$$

Therefore, $\int |f| = 0$, which implies that $f = 0$ a.e. (either by Proposition 2.16 or 2.23).

(20) **5.**

(5) **a.** State the Dominated Convergence Theorem.

Theorem. If $\{f_n\}$ is a sequence in L^1 such that

(a) $f_n \to f$ a.e., and

(b) there exists a nonnegative $g \in L^1$ such that $|f_n| \le g \in L^1$ a.e. for all n,

then $f \in L^1$ and $\int f = \lim \int f_n$.

(15) **b.** Let $E \subset \mathbb{R}$ be Lebesgue measurable, let $m(E) < \infty$, and let $f(t) = \int_E \sin xt \, dm(x)$. Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Solution. Write $h(x,t) = \chi_E(x) \sin xt$, and note that $f(t) = \int h(\cdot,t)$. For each $t \in \mathbb{R}$, $h(\cdot,t)$ is measurable, since χ_E is measurable and since $\sin xt$ is measurable in x for fixed t. Also, $|h(\cdot,t)| \le \chi_E \in L^1$, since $m(E) < \infty$. Hence $h(\cdot,t)$ is integrable, which implies that $f(t)$ exists and is finite for all $t \in \mathbb{R}$. Therefore, $f : \mathbb{R} \to \mathbb{R}$.

We will show that f is continuous at t_0, for any $t_0 \in \mathbb{R}$. Let $t_n \to t_0$, let $\varphi_n(x) = h(x,t_n)$, and let $\varphi(x) = h(x,t_0)$. Since $h(x,\cdot)$ is continuous for each fixed x, $\varphi_n \to \varphi$ pointwise. Since $|\varphi_n| = |h(\cdot,t_n)| \le \chi_E \in L^1$, the Dominated Convergence Theorem implies that

$$f(t_0) = \int h(\cdot,t_0) = \int \varphi = \lim \int \varphi_n = \lim \int h(\cdot,t_n) = \lim f(t_n),$$

which implies that f is continuous at t_0.

(18) **6.** Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Show that μ_E is a measure on $(X, \mathcal{M})$.

Solution. First note that $\mu_E : \mathcal{M} \to [0,\infty]$.

Next, note that $\mu_E(\varnothing) = \mu(\varnothing \cap E) = \mu(\varnothing) = 0$.

Finally, let A_n be a sequence of disjoint measurable sets. It follows that $A_n \cap E$ is also a sequence of disjoint measurable sets. Therefore,

$$\mu_E\left(\bigcup A_n\right) = \mu\left(\left(\bigcup A_n\right) \cap E\right) = \mu\left(\bigcup (A_n \cap E)\right) = \sum \mu(A_n \cap E) = \sum \mu_E(A_n),$$

which establishes the countable additivity property for μ_E.