Midterm Exam

November 23, 2009

Closed book exam. Books, notes, and electronic devices may not be used.

(20) **1.**

(4) **a.** Let ν and λ be signed measures on a measurable space $(X,\mathcal{M})$. Define $\nu \perp \lambda$.

(4) **b.** State the Jordan Decomposition Theorem for signed measures.

(4) **c.** Let ν be a signed measure on a measurable space $(X,\mathcal{M})$. Define the total variation $|\nu|$ of ν.

(4) **d.** Let ν be signed a measure and μ a positive measure on a measurable space $(X,\mathcal{M})$. Define $\nu \ll \mu$.

(4) **e.** Show that $|\nu| \ll \mu \quad \Rightarrow \quad \nu \ll \mu$.

(20) **2.**

(10) **a.** State the Lebesgue-Radon-Nikodym Theorem.

(10) **b.** Let μ be the Lebesgue-Stieltjes measure associated with $F(x) = x + \chi_{[0,\infty)}(x)$. Find the Lebesgue decomposition of μ with respect to Lebesgue measure on $\mathbb{R}$.

(20) **3.** Let $f_n:[0,1]\to\mathbb{R}$ be a sequence of measurable functions, and let $f:[0,1]\to\mathbb{R}$ be a measurable function. Prove or disprove:

(10) **a.** If $f_n \to f$ a.e., then $f_n \to f$ in L^1.

(10) **b.** If $f_n \to f$ in L^1, then $f_n \to f$ a.e.

(20) **4.**

(5) **a.** State Tonelli's Theorem for functions in L^+.

For parts (b), (c), and (d), let $X = Y = [0,1]$ with the σ-algebra of Borel measurable sets. Let m be Lebesgue measure on X, and let ν be counting measure on Y. Let $D = \{(x,x) : x \in [0,1]\}$.

(5) **b.** Show that $\iint \chi_D \, dm \, d\nu = 0$.

(5) **c.** Show that $\iint \chi_D \, d\nu \, dm = 1$.

(5) **d.** Explain why parts (b) and (c) do not contradict Tonelli's Theorem.

(20) 5. Let $F:[a,b]\to\mathbb{R}$, where $-\infty<a<b<\infty$.

(5) a. Define "F is of bounded variation on $[a,b]$."

(5) b. Define "F is absolutely continuous on $[a,b]$."

(5) c. Prove or disprove: If F is absolutely continuous on $[a,b]$, then F is uniformly continuous on $[a,b]$.

(5) d. Prove or disprove: If F is continuous and of bounded variation on $[a,b]$, then F is absolutely continuous on $[a,b]$.