Midterm Exam Solutions

November 23, 2009

(20) **1.**

(4) **a.** Let ν and λ be signed measures on a measurable space $(X,\mathcal{M})$. Define $\nu \perp \lambda$.

Solution. There exist $E \in \mathcal{M}$ and $F \in \mathcal{M}$, with $E \cap F = \varnothing$ and $E \cup F = X$, such that E is null for ν and F is null for λ.

(4) **b.** State the Jordan Decomposition Theorem for signed measures.

The Jordan Decomposition Theorem. If ν is a signed measure, there exist unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

(4) **c.** Let ν be a signed measure on a measurable space $(X,\mathcal{M})$. Define the total variation $|\nu|$ of ν.

Solution. $|\nu| = \nu^+ + \nu^-$, where $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν.

(4) **d.** Let ν be signed a measure and μ a positive measure on a measurable space $(X,\mathcal{M})$. Define $\nu \ll \mu$.

Solution. For every $E \in \mathcal{M}$, $\mu(E) = 0 \Rightarrow \nu(E) = 0.$

(4) **e.** Show that $|\nu| \ll \mu \Rightarrow \nu \ll \mu$.

Solution. Let $E \in \mathcal{M}$ satisfy $\mu(E) = 0$, and let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν. Since $|\nu| \ll \mu$, we know that $|\nu|(E) = 0$. Therefore $\nu^+(E) + \nu^-(E) = |\nu|(E) = 0$. Since ν^+ and ν^- are both positive measures, we have $\nu^+(E) = \nu^-(E) = 0$. Therefore, $\nu(E) = \nu^+(E) - \nu^-(E) = 0$, which implies that $\nu \ll \mu$.

(20) **2.**

(10) **a.** State the Lebesgue-Radon-Nikodym Theorem.

The Lebesgue-Radon-Nikodym Theorem. Let ν be a $\sigma-$finite signed measure and let μ be a $\sigma-$finite measure on a measurable space $(X,\mathcal{M})$. There exist unique $\sigma-$finite signed measures λ, ρ on $(X,\mathcal{M})$ such that $\lambda \perp \mu$, $\rho \ll \mu$, and $\nu = \lambda + \rho$. Furthermore, there is an extended $\mu-$integrable function $f : X \to \mathbb{R}$ such that $d\rho = f\, d\mu$, where f is unique up to sets of $\mu-$measure zero.

(10) **b.** Let μ be the Lebesgue-Stieltjes measure associated with $F(x) = x + \chi_{[0,\infty)}(x)$. Find the Lebesgue decomposition of μ with respect to Lebesgue measure on $\mathbb{R}$.

Solution. Let λ be the Dirac measure at $x = 0$, i.e.,

$$\lambda(E) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

Then the Lebesgue decomposition of μ is $\boxed{\mu = \lambda + m}$, where m is Lebesgue measure on $\mathbb{R}$.

Justification: Let $-\infty < a < b < \infty$. If $b < 0$, then

$$\lambda((a,b]) + m((a,b]) = 0 + b - a = b - a + \chi_{[0,\infty)}(b) - \chi_{[0,\infty)}(a) = F(b) - F(a),$$

while, if $b \geq 0$, then

$$\lambda((a,b]) + m((a,b]) = 1 + b - a = b - a + \chi_{[0,\infty)}(b) - \chi_{[0,\infty)}(a) = F(b) - F(a).$$

In either case, $(\lambda + m)((a,b]) = F(b) - F(a)$. Since μ is the unique measure satisfying $\mu((a,b]) = F(b) - F(a)$, we must have $\mu = \lambda + m$. Clearly, $m \ll m$. Since λ is null on $\mathbb{R} - \{0\}$ while m is null on $\{0\}$, we have $\lambda \perp m$, which implies that $\mu = \lambda + m$ is the Lebesgue decomposition of μ.

(20) **3.** Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of measurable functions, and let $f : [0,1] \to \mathbb{R}$ be a measurable function. Prove or disprove:

(10) **a.** If $f_n \to f$ a.e., then $f_n \to f$ in L^1.

Counterexample. Let $f_n = n\chi_{(0,1/n)}$. Then $f_n(x) \to 0$, for all $x \in [0,1]$, hence $f_n \to 0$ a.e. But $\int_0^1 |f_n(x) - 0| dx = \int_0^1 n\chi_{(0,1/n)}(x) dx = 1$, for all n, so f_n does not converge to 0 in L^1.

(10) **b.** If $f_n \to f$ in L^1, then $f_n \to f$ a.e.

Counterexample. Let $g_{jk} = \chi_{[j/2^k,(j+1)/2^k)}$, $j = 0,\ldots,2^k - 1$, $k = 0,1,\ldots$, and let $f_n = g_{jk}$, where $n = 2^k + j$. Then $\int_0^1 g_{jk}(x) dx = 2^{-k}$, so $\int_0^1 |f_n(x) - 0| dx \to 0$ as $n \to \infty$, which implies that $f_n \to 0$ in L^1. However, for all $x \in [0,1)$, $f_n(x) = 1$ for infinitely many n, and $f_n(x) = 0$ for infinitely many n. Therefore f_n does not converge to 0 a.e.

(20) **4.**

(5) **a.** State Tonelli's Theorem for functions in L^+.

Tonelli's Theorem. Let $(X,\mathcal{M})$ and $(X,\mathcal{M})$ be $\sigma-$finite measure spaces, let $f\in L^+(X\times Y)$, let $g(x)=\int f_x d\nu$, and let $h(y)=\int f^y d\mu$. Then $g\in L^+(X)$, $h\in L^+(Y)$, and

$$\int f\,d(\mu\times\nu)=\int g\,d\mu=\int h\,d\upsilon.$$

For parts (b), (c), and (d), let $X=Y=[0,1]$ with the σ-algebra of Borel measurable sets. Let m be Lebesgue measure on X, and let ν be counting measure on Y. Let $D=\{(x,x):x\in[0,1]\}$.

(5) **b.** Show that $\iint \chi_D dm d\nu=0$.

Solution.

$$\iint \chi_D dm d\nu=\int\left(\int(\chi_D)^y\,dm\right)d\nu(y)=\int\left(\int\chi_{\{y\}}dm\right)d\nu(y)=\int m(\{y\})d\nu(y)=\int 0 d\nu=0$$

(5) **c.** Show that $\iint \chi_D d\nu dm=1$.

Solution.

$$\iint \chi_D d\nu dm=\int\left(\int(\chi_D)_x\,d\nu\right)dm(x)=\int\left(\int\chi_{\{x\}}d\nu\right)dm(x)=\int \nu(\{x\})dm(x)=\int 1dm=m(X)=1$$

(5) **d.** Explain why parts (b) and (c) do not contradict Tonelli's Theorem.

Solution. Counting measure is not $\sigma-$finite on $[0,1]$.

(20) 5. Let $F:[a,b]\to\mathbb{R}$, where $-\infty<a<b<\infty$.

(5) a. Define "F is of bounded variation on $[a,b]$."

Solution. F is of bounded variation on $[a,b]$ if

$$\sup\left\{\sum_{k=1}^{n}\left|F(x_k)-F(x_{k-1})\right|:n\in\mathbb{N},a=x_0<\cdots<x_n=b\right\}<\infty.$$

(5) b. Define "F is absolutely continuous on $[a,b]$."

Solution. F is absolutely continuous on $[a,b]$ if, for every $\varepsilon>0$, there exists $\delta>0$ such that

$$\sum_{k=1}^{N}(b_k-a_k)<\partial \quad\Rightarrow\quad \sum_{k=1}^{N}\left|F(b_k)-F(a_k)\right|<\varepsilon,$$

for every finite set $(a_1,b_1),\ldots,(a_N,b_N)$ of subintervals of $[a,b]$.

(5) c. Prove or disprove: If F is absolutely continuous on $[a,b]$, then F is uniformly continuous on $[a,b]$.

Proof. To establish uniform continuity, we must show that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in [a,b]$

$$|x - y| < \partial \quad \Rightarrow \quad |F(x) - F(y)| < \varepsilon,$$

If we write the interval spanned by x and y as (a_1, b_1), this condition is implied by the condition for absolute continuity, with $N = 1$.

(5) d. Prove or disprove: If F is continuous and of bounded variation on $[a,b]$, then F is absolutely continuous on $[a,b]$.

Counterexample. Let F be the middle third Cantor function on $[0,1]$ Then F is nondecreasing and bounded, so F is of bounded variation on $[0,1]$. However, F' is zero almost everywhere, which implies that

$$F(1) - F(0) = 1 \neq 0 = \int_0^1 F'(t)\,dt,$$

which means that F does not satisfy the fundamental theorem of calculus, which implies that F is not absolutely continuous.