Real Analysis

Wikipedia

Real analysis is a branch of mathematical analysis dealing with the set of real numbers.

Analysis has its beginnings in the rigorous formulation of calculus.

Calculus (Latin, *calculus*, a small stone used for counting) is a branch of mathematics that includes the study of limits, derivatives, integrals, and infinite series, and constitutes a major part of modern university education.

Math 8601/02 = Freshman Calculus redone abstractly and rigorously.

Real Analysis

Folland

The name “real analysis” is something of an anachronism. Originally applied to the theory of functions of a real variable, it has come to encompass several subjects of a more general and abstract nature that underlie much of modern analysis. … [These include] measure and integration theory, point set topology, and functional analysis … .

Math 8601/02 = measure and integration theory, point set topology, functional analysis …

Numbers

$\mathbb{N}$ = the set of positive integers $= \{1, 2, 3, \ldots\}$

$\mathbb{Z}$ = the set of integers

$\mathbb{Q}$ = the set of rational numbers

$\mathbb{R}$ = the set of real numbers

$\mathbb{C}$ = the set of complex numbers

Numbers

Why do we need integers?

To solve the equation

$$x + 1 = 0.$$

Why do we need rational numbers?

To solve the equation

$$2x - 1 = 0.$$

Why do we need algebraic numbers?

To solve the equation

$$x^2 - 2 = 0.$$

Numbers

Why do we need complex numbers?

To solve the equation

$$x^2 + 1 = 0.$$

Why do we need real numbers?

To take limits

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

and to sum series

$$\sum_{n=0}^{\infty} \frac{1}{n!}.$$

Fibonacci Sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

$$a_0 = a_1 = 1, \quad a_{n+1} = a_n + a_{n-1}$$

$$\frac{a_{n+1}}{a_n} = 1 + \frac{a_{n-1}}{a_n}$$

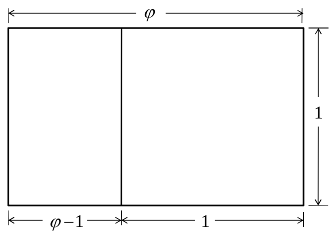
$$\text{Let } x_n = \frac{a_n}{a_{n-1}}, \quad n = 1, 2, \ldots .$$

$$\text{Then } x_{n+1} = 1 + \frac{1}{x_n}$$

$$\lim_{n\to\infty} x_n = \varphi, \quad \text{where} \quad \varphi = 1 + \frac{1}{\varphi},$$

$$\text{or} \quad \boxed{\varphi^2 - \varphi - 1 = 0}, \quad \text{the "golden ratio".}$$

Golden Ratio



$$\frac{\varphi}{1} = \frac{1}{\varphi - 1}, \quad \boxed{\varphi^2 - \varphi - 1 = 0}$$

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Golden Ratio

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}, \quad \varphi = 1 + \frac{1}{\varphi}, \quad \boxed{\varphi^2 - \varphi - 1} = 0.$$

Approximants

$$\varphi \approx \varphi_1 = 1, \quad \varphi \approx \varphi_2 = 1 + \frac{1}{1} = 2, \quad \varphi \approx \varphi_3 = 1 + \cfrac{1}{1 + \cfrac{1}{1}} = \frac{3}{2},$$

$$\varphi \approx \varphi_4 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{5}{3}, \quad \varphi \approx \varphi_5 = \frac{8}{5}, \quad \varphi \approx \varphi_6 = \frac{13}{8}, \quad \varphi \approx \varphi_7 = \frac{21}{13}, \quad \ldots$$

Note Fibonacci numbers.

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Golden Ratio

$$\varphi_1 = \frac{1}{1}, \quad \varphi_2 = \frac{2}{1}, \quad \varphi_3 = \frac{3}{2} = \frac{1+2}{1+1}, \quad \varphi_4 = \frac{5}{3} = \frac{2+3}{1+2},$$

$$\varphi_5 = \frac{8}{5} = \frac{3+5}{2+3}, \quad \varphi_6 = \frac{13}{8} = \frac{5+8}{3+5}, \quad \varphi_7 = \frac{21}{13} = \frac{8+13}{5+8}, \quad \ldots$$

$$\text{If } \varphi_{k-1} = \frac{a}{b}, \text{ and if } \varphi_k = \frac{c}{d}, \text{ then } \varphi_{k+1} = \frac{a+c}{b+d}.$$

Exercise

If $\frac{a}{b} < \frac{c}{d}$ (in lowest terms), and if there is no rational number between $\frac{a}{b}$ and $\frac{c}{d}$ with denominator no greater than $\max(b, d)$, then $\frac{a+c}{b+d}$ is the rational number between $\frac{a}{b}$ and $\frac{c}{d}$ with the smallest denominator.

c.f. Farey Sequence

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Farey Sequences

Farey Sequence of order n: Arrange all the rational numbers with denominators less than or equal to n (in lowest terms) in order.

$$n = 2: \ \frac{0}{1}, \boxed{\frac{1}{2}}, \frac{1}{1}; \quad n = 3: \ \frac{0}{1}, \boxed{\frac{1}{3}}, \frac{1}{2}, \boxed{\frac{2}{3}}, \frac{1}{1}; \quad n = 4: \ \frac{0}{1}, \boxed{\frac{1}{4}}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \boxed{\frac{3}{4}}, \frac{1}{1};$$

$$n = 5: \ \frac{0}{1}, \boxed{\frac{1}{5}}, \frac{1}{4}, \frac{1}{3}, \boxed{\frac{2}{5}}, \frac{1}{2}, \boxed{\frac{3}{5}}, \frac{2}{3}, \frac{3}{4}, \boxed{\frac{4}{5}}, \frac{1}{1};$$

$$n = 6: \ \frac{0}{1}, \boxed{\frac{1}{6}}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \boxed{\frac{5}{6}}, \frac{1}{1};$$

$$n = 7: \ \frac{0}{1}, \boxed{\frac{1}{7}}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \boxed{\frac{2}{7}}, \frac{1}{3}, \frac{2}{5}, \boxed{\frac{3}{7}}, \frac{1}{2}, \boxed{\frac{4}{7}}, \frac{3}{5}, \frac{2}{3}, \boxed{\frac{5}{7}}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \boxed{\frac{6}{7}}, \frac{1}{1};$$

$$n = 8: \ \frac{0}{1}, \boxed{\frac{1}{8}}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \boxed{\frac{3}{8}}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \boxed{\frac{5}{8}}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \boxed{\frac{7}{8}}, \frac{1}{1}.$$

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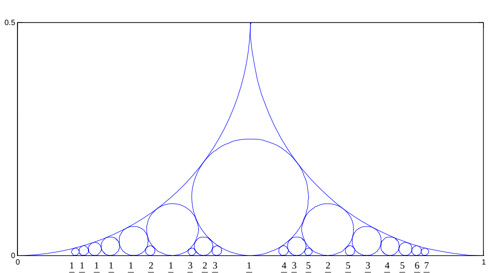
Farey Sequences



The Farey sequence of order 8

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Farey Sequences

Exercise

If $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent numbers in a Farey sequence, then

(1) $ad - bc = \pm 1$, and

(2) the next number to appear between $\frac{a}{b}$ and $\frac{c}{d}$ in a higher order Farey sequence is $\frac{a+c}{b+d}$.

Farey Sequences

Exercise

If $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent numbers in a Farey sequence, and if

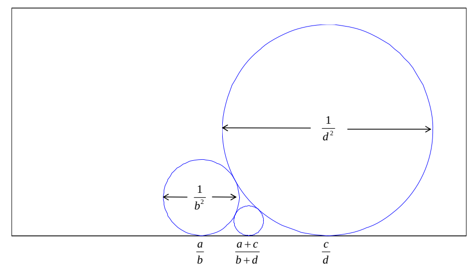
C_1 is a circle of diameter $\frac{1}{b^2}$ tangent to the x-axis at $\frac{a}{b}$, and if

C_2 is a circle of diameter $\frac{1}{d^2}$ tangent to the x-axis at $\frac{c}{d}$, then

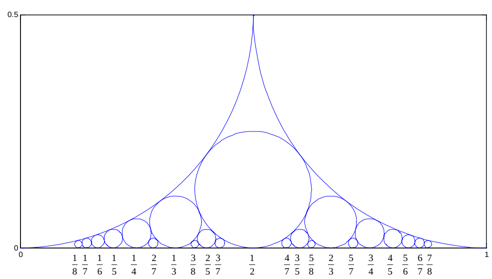
(1) C_1 and C_2 are tangent, and

(2) the circle of diameter $\frac{1}{(b+d)^2}$ tangent to the x-axis at $\frac{a+c}{b+d}$ is tangent to both C_1 and C_2.

Farey Sequences



Farey Sequences



The Farey sequence of order 8

Diophantine Approximation

For each rational $\frac{p}{q} \in [0,1]$ and for $C > 0$ define the interval

$$I_{p/q}(C) = \left(\frac{p}{q} - \frac{C}{q^2}, \frac{p}{q} - \frac{C}{q^2} \right), \text{ and let}$$

$$V_C = \bigcup_{p/q \in [0,1]} I_{p/q}(C).$$

Note that V_C is an open subset of $[0,1]$ containing all the rationals.

If $C \geq \frac{1}{2}$, then V_C contains all numbers in $[0,1]$.

What about smaller values of C ?

Does V_C contain the golden ratio φ ?

Diophantine Approximation

Let $f(x) = x^2 - x - 1$. Then $f(\varphi) = 0$, and

$$f\left(\frac{p}{q}\right) = f\left(\frac{p}{q}\right) - f(\varphi) = \left(\frac{p}{q}\right)^2 - \frac{p}{q} - 1 - \left(\varphi^2 - \varphi - 1\right) = \left(\frac{p}{q} - \varphi\right)\left(\frac{p}{q} + \varphi - 1\right).$$

$$\left| f\left(\frac{p}{q}\right) \right| \leq M \left| \frac{p}{q} - \varphi \right|, \text{ for } \frac{p}{q} \in [1,2].$$

$$\text{But } \left| q^2 f\left(\frac{p}{q}\right) \right| = \left| p^2 - pq - q^2 \right| \geq 1, \text{ so}$$

$$Mq^2 \left| \frac{p}{q} - \varphi \right| \geq 1, \text{ so } \left| \frac{p}{q} - \varphi \right| \geq \frac{1}{Mq^2},$$

$$\boxed{\varphi \notin V_{1/M}}$$

Diophantine Approximation

Exercise

If y is an algebraic number of degree n, i.e., y is the zero of an irreducible polynomial of degree n, then there is a $C > 0$ such that

$$\left| y - \frac{p}{q} \right| \geq \frac{C}{q^n}, \text{ for every rational } \frac{p}{q}.$$

Diophantine Approximation

As before, for each rational $\frac{p}{q} \in [0,1]$ and for $C > 0$ define the interval

$$I_{p/q}(C) = \left(\frac{p}{q} - \frac{C}{q^n}, \frac{p}{q} - \frac{C}{q^n} \right),$$ and let

$$V_C = \bigcup_{p/q \in [0,1]} I_{p/q}(C).$$

Note that V_C is an open subset of $[0,1]$ containing all the rationals.
For small C, V_C misses a lot of algebraic numbers.
What else does it miss?

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Diophantine Approximation

A Little Measure Theory

$$\mu\left(I_{p/q}(C)\right) = \frac{2C}{q^n}.$$

For each $q > 1$, there are at most q rationals in the interval $[0,1]$ with denominator equal to q .

Therefore $\mu(V_C) \le \sum_{q=1}^{\infty} q \frac{2C}{q^n} = 2C \sum_{q=1}^{\infty} \frac{1}{q^{n-1}} < \infty$ for $n \ge 3$.

Thus we have constructed an open set of arbitrarily small measure containing all the rationals in an interval.

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Liouville Numbers

Definition

A Liouville number is a real number x such that, for every positive integer n , there exists a rational number $\frac{p}{q}$, with $q > 1$, such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

Exercise

Liouville numbers exist and are transcendental (not algebraic).

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Morals

The real numbers have a lot of structure.

They are much more interesting than a homogeneous line.

Real Analysis doesn't have to be boring.