NSF Project Summary: Max Engelstein

Overview: Free Boundary Problems (FBPs) are a class of problems in which a function, u, satisfies a partial differential equation on a domain, Ω, which itself may depend on u. Alt and Caffarelli, in [AC81], drew an analogy between the regularity of the free boundary of minimizers to the energy, $J(v) \equiv \int |\nabla v| + \chi_{\{v>0\}}\, dx$, and the regularity of minimal surfaces. This observation shaped subsequent study of free boundary problems, especially those which arise in a variational context, that is to say, as the minimizer of some energy functional (e.g. the obstacle problem, [Caf98]). Researchers working on FBPs adopted many tools and techniques originally developed to study minimal surfaces, such as monotonicity formulas and epiperimetric inequalities (see, e.g., [GP09], [We99a]).

On the other hand, there are many FBPs to which geometric techniques have not yet been sucessfully adapted, often because these problems cannot be stated in terms of a minimized energy (or an associated flow). The purpose of this project is to use harmonic analysis and geometric measure theory (GMT) to adapt ideas from geometric analysis to the study of free boundary problems which cannot be expressed variationally. The PI plans to study three broad categories of such problems; two-phase problems for harmonic measure, parabolic FBPs and almost-minimizers (in the sense of [DaT15]). The PI believes that a mix of ideas from geometric analysis, GMT and harmonic analysis should yield new insights into these problems.

Intellectual Merit: The three categories of FBPs above are all important areas of research; FBPs for harmonic measure are connected to questions in potential theory and the structure of rough sets amongst other areas (see [Tor10]). Parabolic problems are used to model a wide range of phenomena, both in the real world (e.g. tumor growth, see [QPV14]) and in other areas of mathematics (e.g. geometric flows, see [DH99]). Finally, almost-minimization is a widely studied concept (e.g., [Alm68]) which can model measurement errors and has several applications in pure mathematics (e.g. partition problems [Alm76]).

The PI's work on two-phase free boundary problems ([Eng16a], [BET15]) has shown how estimates from geometric measure theory and harmonic analysis can be used to adapt tools from geometric analysis (such as Łojasiewicz inequalities and monotonicity formulas) to free boundary problems for harmonic measure. Further work will focus on adapting other important tools such as epiperimetric inequalities and on applying these methods in different contexts (e.g. proving non-degeneracy for other non-variational FBPs).

For parabolic problems, the PI is inspired by a program of David Jerison and collaborators (see [DeJ09], [DeJ11], [JS15], [JKa14]) which relates global minimizers to the Alt-Caffarelli functional above to global minimal surfaces. The PI believes that this analogy can be extended to the parabolic case and that mean curvature flow will provide great insights into the study of parabolic FBPs. The PI's study of global solutions to parabolic problems ([Eng15], [Eng16b]), as well as recent important advances in mean curvature flow (e.g. [CM15]) indicate that the time is right to develop these connections.

Finally, the PI's work in progress with Guy David and Tatiana Toro on almost-minimizers to the Alt-Caffarelli functional ([DET16]), represents an early step in a larger program to understand almost-minimizers to singular energy functionals. This work proves the regularity of the one-phase free boundary, but future work should extend this to two-phase, variable coefficient, vectorial and higher co-dimension settings (some of which are still being understood for *minimizers*, see [CSY16]). These results are should have connections to many other FBPs, for example, optimal domains for eigenvalue functions (see e.g. [MTV16]).

Broader Impacts: The PI is engaged with the broader mathematical research community as evidenced by his active collaborations, participation in international conferences and invited seminar talks. The PI has also been exposed to several different mathematical communities (GMT, harmonic analysis and geometric analysis), and plans to use this wide range of exposure to better facilitate interdisciplinary research, e.g. by organizing conferences that bridge these three areas.

Furthermore, the PI has been involved in the development of young mathematicians through his teaching at MathILy (a summer program for talented high schoolers), the University of Chicago's Research Experience for Undegraduates and University of Chicago's Directed Reading Program. The PI also has a track record of working with students from underrepresented groups at a variety of levels, from high school to beginning graduate school. At Cambridge he plans to continue activities in this vein, as well as supervise undergraduate research through MIT's Undergraduate Research Opportunities Program.

References

[Alm68] F. J. Almgren, Jr., Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. *Ann. Math. (2)* **87**, (1968) 321-391.

[Alm76] F. J. Almgren, Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Mem. Amer. Math. Soc.* **165** (1976).

[Alm79] F.J. Almgren Jr., Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents. in "Minimal Submanifolds and Geodesics" North-Holland, Amstrerdam, 1979, pp 1-6.

[AC81] H. Alt and L. Caffarelli, Existence and Regularity for a minimum problem with free boundary. *J. Reine Angew. Math.* **325** (1981), pp 105-144.

[AW09] J. Andersson and G. S. Weiss, A Parabolic Free Boundary Problem with Bernoulli Type Condition on the Free Boundary. *J. Reine Angew. Math.* **627** (2009), pp 213-235.

[AMT16] J. Azzam, M. Mourgoglou and X. Tolsa, Mutual Absolute Continuity of Interior and Exterior Harmonic Measure Implies Rectifiability. *C.P.A.M., to appear.* arXiv: 1602.01397v2. (2016).

[Bad11] M. Badger, Harmonic polynomials and tangent measures of harmonic measure, *Rev. Mat. Iberoamericana* **27** (2011), pp 841-870.

[Bad13] M. Badger, Flat points in zero sets of harmonic polynomials and harmonic measure from two sides. *J. London Math. Soc.* **87** (2013), pp 111-137.

[BET15] M. Badger, M. Engelstein and T. Toro, Structure of Sets which are Well Approximated by Zero Sets of Harmonic Polynomials. *Preprint*, arXiv:1509.03211. (2015).

[BL15] M. Badger and S. Lewis, Local set approximation: Mattila-Vuorinen type sets, Reifenberg type sets and tangent sets. *Forum Math. Sigma* **3** (2015) 63 pp.

[Caf87] L. Caffarelli, A Harnack inequality approach to the regularity of free boundaries. Part I, Lipschitz free boundaries are $C^{1,\alpha}$. *Rev. Math. Iberoamericana* **3** (1987), pp 139-162.

[Caf98] L. Caffarelli, The Obstacle Problem Revisited. *J. of Fourier Anal. and App.* **4** (1998), 383-402.

[CSY16] L. Caffarelli, H. Shahgholian and K. Yeressian, A minimization problem with free boundary related to a cooperative system. *Preprint* arXiv: 1608.07689 (2016).

[CM15] T. Colding and W.P. Minicozzi, Uniqueness of blowups and Łojasiewicz inequalities. *Ann. Math.* **182** (2015) pp 443-471.

[Dah77] B. Dahlberg, Estimates of harmonic measure. *Arch. Rational Mech. Anal.* **65** (1977), 275-288.

[Dah79] B. Dahlberg, On the Poisson integral for Lipschitz and C^1-domains. *Studia Math.* **66** (1979), 13-24.

[Das14] P. Daskalopoulos, The regularity of solutions in degenerate geometric problems. *Surveys in Diff. Geom.* **19** (2014), 83-110.

[DH99] P. Daskalopoulos and R. Hamilton, The Free Boundary on the Gauss Curvature Flow with Flat Sides. *J. Reine Angenw. Math.* **510** (1999), 187-227.

[DET16] G. David, M. Engelstein and T. Toro, Free boundary regularity for almost minimizers. *In preparation* (2016).

[DaS93] G. David and S. Semmes, "Analysis of and on Uniformly Rectifiable Sets." Math. Surveys Monog. **38** Amer. Math. Soc., Providence, 1993.

[DaT15] G. David and T. Toro, Regularity of almost minimizers with free boundary. *Calc. Var. P.D.E.* **54** (2015), 455-524.

[DSS15a] C. De Lellis, E. Spadaro and L. Spolaor, Regularity theory for 2-dimensional almost minimal currents I: Lipschitz approximation. *Preprint* arXiv: 1508.05507, (2015).

[DFS14] D. De Silva, F. Ferrari and S. Salsa, Two-Phase Problems with Distributed Source: Regularity of the Free Boundary. *Anal. & PDE* **7** (2014), 267-310.

[DeJ09] D. De Silva and D. Jerison, A Singular Energy Minimizing Free Boundary. *J. Reine Angew. Math.* **635** (2009), 1-22.

[DeJ11] D. De Silva and D. Jerison, Gradient bound for energy minimizing free boundary graphs. *Comm. Pure Appl. Math.* **64** (2011) 538-555.

[Eng15] M. Engelstein, A Free Boundary Problem for the Parabolic Poisson Kernel. *Preprint* arXiv: 1510.05704, 2015.

[Eng16a] M. Engelstein, A Two-Phase Free Boundary Problem for Harmonic Measure. *Ann. Sci. École Norm. Sup.* **49** (2016), 859-905

[Eng16b] M. Engelstein, Parabolic NTA domains $\mathbb{R}^2$. *Preprint* arXiv:1607.08650 (2016).

[FS16] M. Focardi and E. Spadaro, An epiperimetric inequality for the thin obstacle problem. *Adv. Differential Equations* **21** (2016), 153-200.

[GM05] J.B. Garnett and D.E. Marshall. "Harmonic Measure." Cambridge University Press, New York, 2005.

[GP09] N. Garofalo and A. Petrosyan, Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem. *Inv. Math.* **177** (2009), 415-461.

[GSP16] N. Garofalo, M. Smit Vega Garcia and A. Petrosyan, An epiperimetric inequality approach to the regularity of the free boundary in the Signorini problem with variable coefficients. *J. Math. Pures Appl.* **105** (2016) 745-787.

[GT16] D. Girela-Sarrión and X. Tolsa, The Riesz transform and quantitative rectifiability for general Radon measures. *Preprint*, arXiv: 1601.08079 (2016).

[HLN03] S. Hofmann, J.L. Lewis and K. Nyström, Existence of big pieces of graphs for parabolic problems. *Ann. Acad. Sci. Fenn. Math.* **28** (2003), 355-384.

[HLN04] S. Hofmann, J.L. Lewis and K. Nyström, Caloric measure in parabolic flat domains. *Duke Math. J.* **122** (2004), 281-346.

[HM14] S. Hofmann and J. M. Martell, Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in L^p. *Ann. Sci. École Norm. Sup.* **47** (2014) 577-654

[HMU14] S. Hofmann, J. M. Martell and I. Uriarte-Tuero, Uniform rectifiability and harmonic measure, II: Poisson kernels in L^p imply uniform rectifiability. *Duke Math. J.* **163** (2014) 1601-1654.

[Jer89] D. Jerison, Harmonic measure in convex domains. *Bull. Am. Math. Soc.* **21** (1989), 255-260.

[Jer90] D. Jerison, Regularity of the Poisson Kernel and Free Boundary Problems. *Coll. Math.* **60/61** (1990), 547-568.

[Jer92] D. Jerison, Prescribing harmonic measure on convex domains. *Invent. Math.* **105** (1991) 375-400.

[JKa14] D. Jerison and N. Kamburov, Structure of one-phase free boundaries in the plane. *I.M.R.N To Appear.* arXiv:1412.4106, 2014.

[JKe82] D. Jerison and C. Kenig, Boundary Behavior of Harmonic Functions in Non-Tangentially Accessible Domains. *Adv. Math.* **46** (1982), 80-147.

[JS15] D. Jerison and O. Savin, Some remarks on stability of cones for the one-phase free boundary problem. GAFA 25 (2015) 1240-1257

[KPT09] C. Kenig, D. Preiss, and T. Toro, Boundary structure and size in terms of interior and exterior harmonic measures in higher dimensions. *J.A.M.S.* **22** (2009), 771-796.

[KT97] C. Kenig and T. Toro, Harmonic Measure on Locally Flat Domains. *Duke Math. J.* **87** (1997), 509-550.

[KT03] C. Kenig and T. Toro, Poisson Kernel characterization of Reifenberg flat chord arc domains. *Ann. Sci. École Norm. Sup.* **36** (2003), 323-401.

[KT04] C. Kenig and T. Toro, On the Free Boundary Regularity Theorem of Alt and Caffarelli. *Disc. and Cont. Dyn. Systems* **10** (2004), 397-422.

[KT06] C. Kenig and T. Toro, Free boundary regularity below the continuous threshold: 2-phase problems. *J. Reine Angew. Math.* **596** (2006), 1-44.

[KNS78] D. Kinderlehrer, L. Nirenberg and J. Spruck, Regularity in Elliptic Free Boundary Problems I. *J. D' Analyse Math.* **34** (1978), 86-119.

[LM95] J.L. Lewis and M.A. Murray, The Method of Layer Potentials for the Heat Equation in Time-Varying Domains. *Mem. Am. Math. Soc.* **114** (1995), no 545.

[MTV16] D.Mazzoleni, S. Teracini and B. Velichkov, Regularity of the optimal sets for some spectral functionals. *Preprint* arXiv: 1609.01231 (2016).

[Nys06a] K. Nyström, On Blow-ups and the Classification of Global Solutions to a Parabolic Free Boundary Problems. *Ind. Univ. Math. J.* **55** (2006), 1233-1290.

[Nys06b] K. Nyström, Caloric Measure and Reifenberg Flatness. *Ann. Acad. Sci. Fenn. Math.* **31** (2006), 405-436.

[Nys12] K. Nyström, On an inverse type problem for the heat equation in parabolic regular graph domains. *Math. Zeitschrift* **270** (2012), 197-222.

[Pre87] D. Preiss, Geometry of measures in $\mathbb{R}^n$: distributions, rectifiability and densities. *Ann. of Math.* **125** (1987), 537-643.

[QPV14] F. Quiros, B. Perthame and J.-L. Vazquez, The Hele-Shaw Asymptotics for mechanical models of tumor growth. *ARMA* **212** (2014), 93-127.

[SP07] H. Shahgholian and A. Petrosyan (appendix by T. Arnarson), Parabolic obstacle problems applied to finance: free-boundary regularity approach in **D. Danielli ed.** Recent Developments in Nonlinear Partial Differential Equations. Contemporary Matematics **439** (2007), 117?133.

[Sim83] L. Simon, Asymptotics for a class of evolution equations, with applications to geometric problems, *Annals of Math.* **118** (1983), 525-571.

[Sim93] L. Simon, Cylindrical tangent cones and the singular set of minimal submanifolds. *J. Diff. Geom.* **38** (1993), 585-652.

[Sim95] L. Simon, Rectifiability of the singular sets of multiplicity 1 minimal surfaces and energy minimizing maps. In *Surveys in differential geometry*, **Vol. II** (Cambridge, MA, 1993), 246?305, Int. Press, Cambridge, MA, 1995.

[Tay76] J. Taylor, The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces. *Ann. Math.* **103** (1976) 489-539.

[Tor10] T. Toro, Potential Analysis Meets Geometric Measure Theory. *Proc. of the ICM* Hyderabad, India, (2010), 1485-1497.

[We99a] G. Weiss, A homogeneity improvement approach to the obstacle problem. *Invent. Math.* **138** (1999) 23-50.

[We99b] G. Weiss, Partial Regularity for a Minimum Problem with Free Boundary. *Journal of Geom. Anal.* **9** (1999), 317-326.

[Wei03] G. Weiss, A singular limit arising in combustion theory: Fine properties of the free boundary. *Calc. Var. PDE* **17** (2003), 311-340.