Geometry and the Dirichlet Problem in Any Co-dimension

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(joint work with G. David (U. Paris Sud) and S. Mayboroda (U. Minn.))

Massachusetts Institute of Technology

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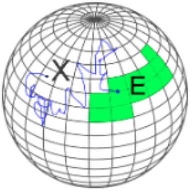

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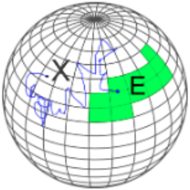

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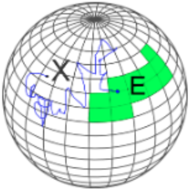

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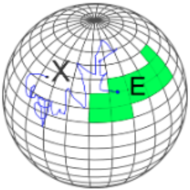

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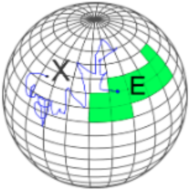

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- Very surprising (and recent): higher co-dimension analogues.

Walker goes to each neighbor with equal probability.

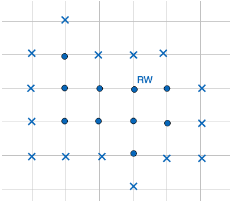

Figure: Each neighbor has probability 1/4

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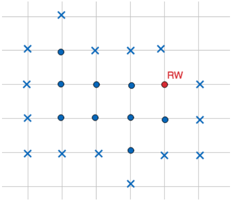

Figure: We keep going until we hit the boundary

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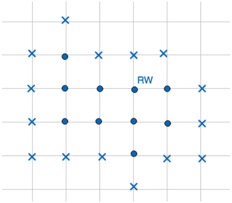

Figure: Backtracking is allowed

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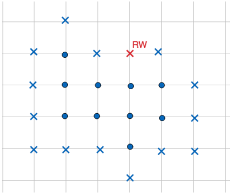

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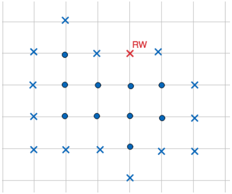

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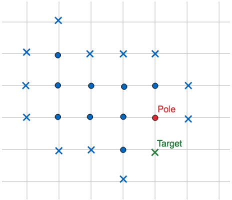

Figure: Hitting Measure of Green starting at Red $\equiv \omega^{\mathrm{Pole}}(\mathrm{Target})$

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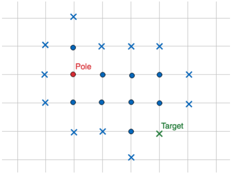

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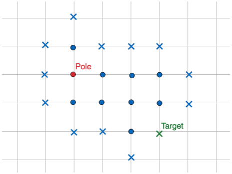

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Hard to compute!!! (Solve 10 eqns with 10 unknowns)
Average of neighbors!

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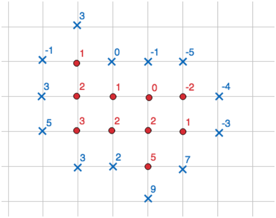

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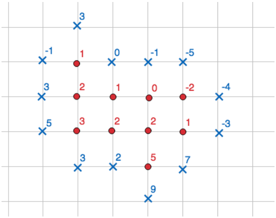

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Uniqueness: Maximum principle!

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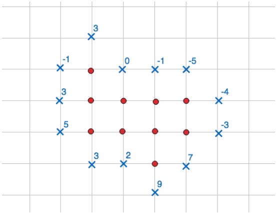

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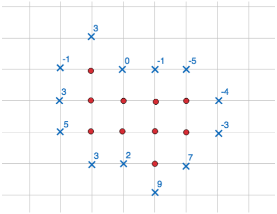

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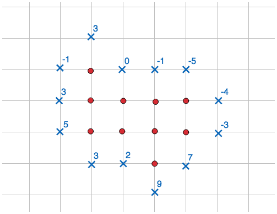

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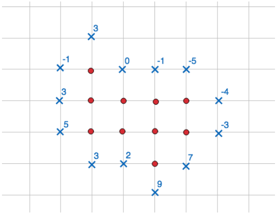

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Expected value!

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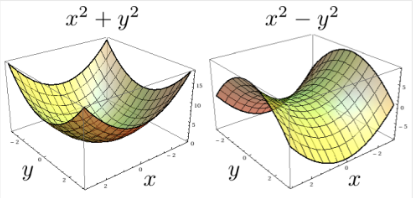

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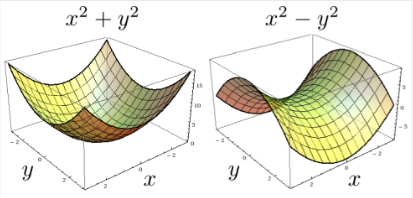

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Represents Equilibrium after diffusion.

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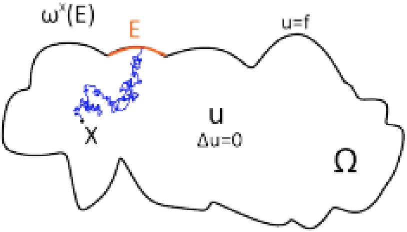

FIGURE: Brownian Motion exiting a domain (figure credit Matthew Badger)

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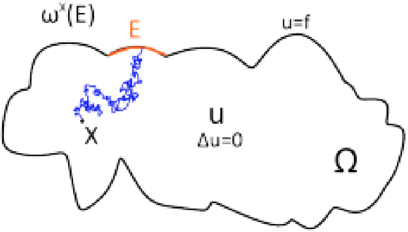

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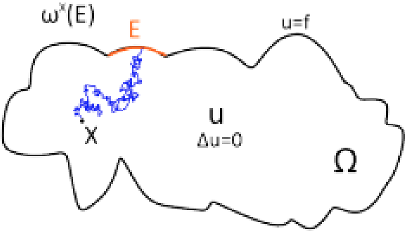

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What is the temperature in the interior, given the temperature on the edge?

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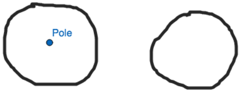

Figure: ω^{Pole} cannot see the other component

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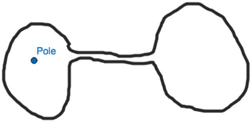

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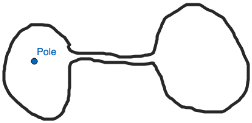

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- Cusps

Figure: Brownian motion cannot get to the cusp

So what if ω^X doesn't see large sets?

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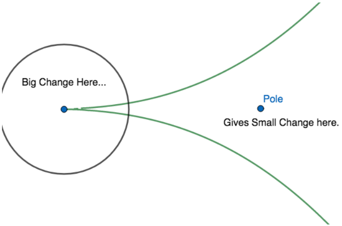

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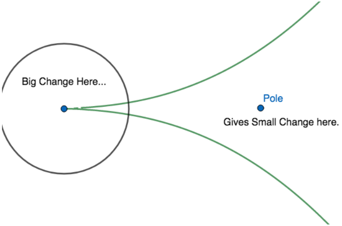

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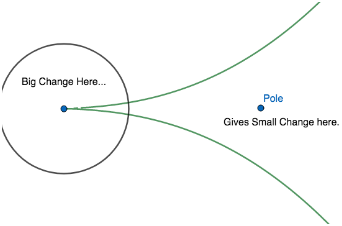

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Three Examples: ω vs Length

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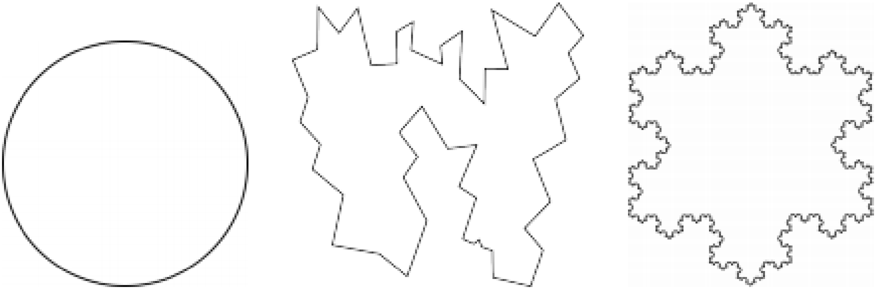

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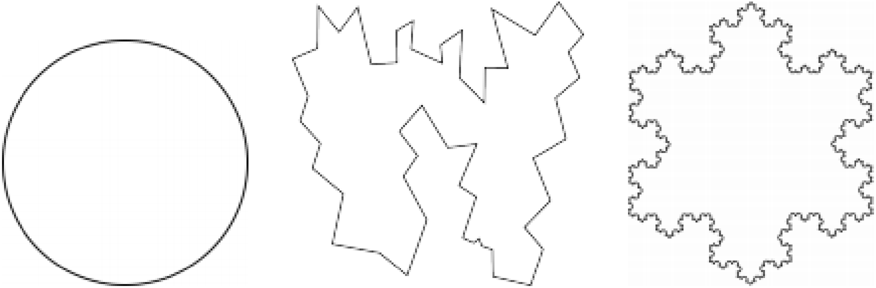

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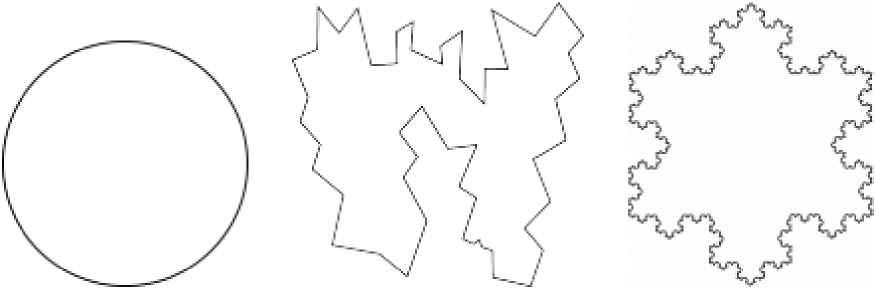

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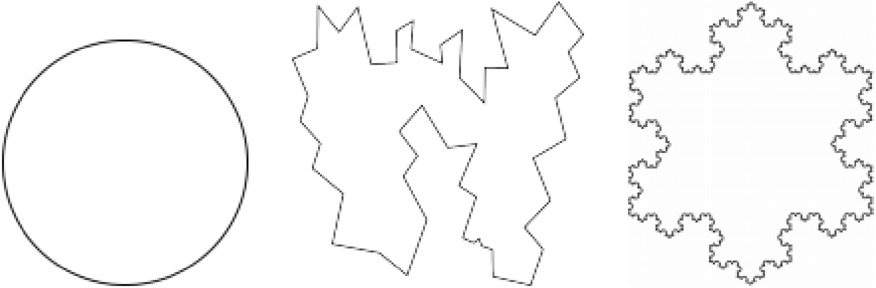

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For the snowflake $\omega^0 = k\sigma$ and $k = +\infty$ or 0 at every point.

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When $n = 2$: connections to the complex analysis.

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FIGURE: George Green (figure credit wikipedia)

$$\begin{cases} G(X,Y) > 0 & X \neq Y \in \Omega \\ G(X,Q) = 0 & Q \in \partial\Omega \\ \Delta_Y G(X,Y) = \delta_X(Y) & Y \in \Omega. \end{cases}$$

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Really hard to compute! Known only for a few domains (half plane, disc, polygons...)

Poisson Kernel

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Takeaway: Geometry of a set is characterized by solutions of Laplacian in complement of the set!

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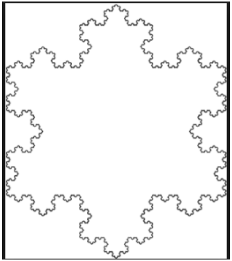

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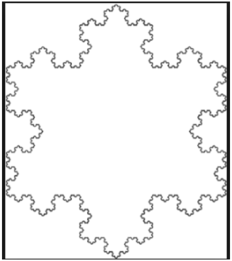

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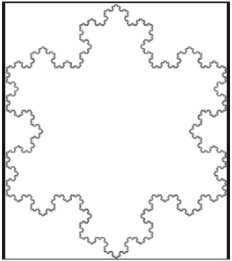

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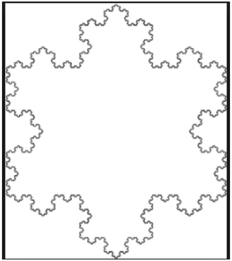

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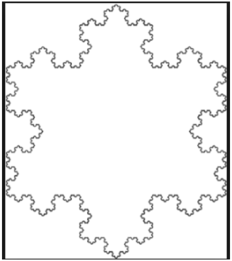

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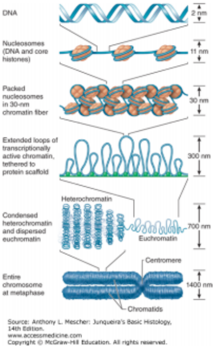

Figure: DNA Straightens and Curls up to Attract/Avoid Enzymes

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Question: Geometry of E characterized by ω_L vs σ?

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Regularized Distance I: A Better Scent

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Let E be the graph of a Lipschitz $\phi : \mathbb{R}^d \to \mathbb{R}^{n-d}$ with small Lip constant. Then $\omega^X_{L_\alpha} = k^X d\sigma$ and k^X is not too small or too big too often (A_∞ weight).

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Note: applies to much more general scents (i.e. any suitably smooth replacement for $D_\alpha(x)^{-(n-d-1)} I$ works).

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D_α is too nice a scent!

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Can compute: see that for $\alpha = n - d - 2$ we have

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When α is magic $D_\alpha(x) = \left(\int_E \frac{1}{|x-y|^{n-2}} d\sigma\right)^{-1/\alpha}$. Note: $\frac{1}{|x|^{n-2}}$ is harmonic!

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3. What does $\alpha \mapsto D_\alpha$ look like?
    - The power $-\frac{1}{\alpha}$ makes this question harder.

Open Questions about the Magic α

1. Why is magic α magic?
   - D_α satisfies an equation but what is really going on?
   - Physical/geometric/probabilistic interpretation?
2. Is this emblematic or pathological?
   - Is any other β magic?
   - Can we prove the converse for ω_β with β not magic?
3. What does $\alpha \mapsto D_\alpha$ look like?
   - The power $-\frac{1}{\alpha}$ makes this question harder.
4. Can we do this in co-dimension one? Two?

Thank You For Listening!

The way of Laplace!