A classification of second-order raising operators for Hamiltonians in two variables

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We develop a group theoretic method based on results of Winternitz *et al.* to compute and classify all first- and second-order raising and lowering operators admitted by Hamiltonians of the form $\underline{H} = -(1/2)\Delta_2 + V(x, y)$. The key to our results, which generalize to higher dimensions, is a proof that $\underline{H}$ admits a second-order raising operator only if the Schrödinger equation separates in Cartesian, polar, or elliptic coordinates.

INTRODUCTION

We call an operator $\underline{R}$ a raising operator for a Hamiltonian $\underline{H}$ if $[\underline{H},\underline{R}] = \lambda\underline{R}$, where λ is a nonzero real constant. If ψ is an eigenvector of $\underline{H}$ with eigenvalue μ, $\underline{H}\psi = \mu\psi$, it follows easily that $\underline{H}(\underline{R}\psi) = (\mu+\lambda)\underline{R}\psi$. Thus, knowledge of $\underline{R}$ permits one to obtain new eigenvalues and eigenvectors of $\underline{H}$ from old ones.

In this paper we give a complete classification of all potentials occurring in the two-dimensional time independent Schrödinger equation $\underline{H}\psi = \mu\psi$ which admit first- and second-order raising operators. The classification of first-order operators is almost trivial, and it is only the second-order case which presents difficulties. Moreover, as one can see from the results of Secs. 2 and 3, there are very few potentials admitting second-order raising operators, and all such potentials are generalizations of the harmonic oscillator.

The principal interest in our results lies in the fact that they are exhaustive and in the method used to obtain them. Proceeding directly, one can show that a Hamiltonian admits a second-order raising operator if and only if the corresponding potential V satisfies the system (2.8)—(2.10) of second-order overdetermined partial differential equations. However, while one can easily find some solutions of these equations, it is extremely difficult to determine when one has found all solutions. We have not been able to solve these equations directly.

In order to solve (2.8)—(2.10) we have adopted an indirect method based on results of Winternitz *et al.*,[1] which relates this problem to the Euclidean group $E(2)$. In Ref. 1 the authors show that $\underline{H}$ admits a second-order symmetry operator if and only if the corresponding Schrödinger equation separates in Cartesian, polar, parabolic, or elliptic coordinates. In this paper we show in essence that if $\underline{H}$ admits a second-order raising operator, then it also admits a second-order symmetry operator, hence that the Schrödinger equation must separate in Cartesian, polar, or elliptic (but strangely, not in parabolic) coordinates. This means that we can restrict ourselves to a search for solutions of (2.8)—(2.10) which separate in one of these three coordinate systems. In this case (2.8)—(2.10) reduce to systems of ordinary differential equations which, though tedious to solve, are tractable. Thus we obtain a complete solution to our problem.

Our method can be generalized to the more interesting three-dimensional case[2] as well as to other types of differential equations, for example, wave equations or the time dependent Schrödinger equation.

The results of Refs. 1, 2, and this paper show the intimate connection between second-order raising and symmetry operators and the separation of the Schrödinger equation in some coordinate system. It appears that higher-order operators will not be of great interest unless and until one can find similar indirect means of characterizing them.

The paper is organized as follows: In Sec. 1 the problem of first-order raising operators is solved, while in Sec. 2 the problem for second-order operators is formulated as a system of overdetermined second-order partial differential equations. We then obtain some solutions, but not the most general class which must await the further development of the connection with separation of variables in Sec. 3, where we complete our classification of all solutions. Finally in Sec. 4, we give the action of the raising and lowering operators on a basis of eigenfunctions of the Schrödinger equation for each case.

1. FIRST-ORDER OPERATORS

Let $\underline{H}$ be the formal Hamiltonian

$$\underline{H} = -\tfrac{1}{2}(\partial_{xx} + \partial_{yy}) + V(x,y) \tag{1.1}$$

acting on the Hilbert space $L_2(R_2)$ of square integrable functions in the plane. Here $V(x,y)$ is a real-valued thrice-differentiable function of (x,y) to be determined. We search first for all Hamiltonians which admit a first-order raising operator $\underline{R}$, i.e., we look for all $\underline{H}$ which satisfy

$$[\underline{H},\underline{R}] = \lambda\underline{R}, \tag{1.2}$$

where λ is a nonzero real constant and $\underline{R}$ is a first-order partial differential operator

$$\underline{R} = \alpha_1(xy)\partial_x + \alpha_2(x,y)\partial_y + \alpha_3(x,y), \quad |\alpha_1|^2 + |\alpha_2|^2 \neq 0. \tag{1.3}$$

Without loss of generality, we can assume that $\underline{R}$ is real, i.e., that $\alpha_1, \alpha_2, \alpha_3$ are real-valued functions. Substituting (1.1) and (1.3) into (1.2) and equating coefficients of $\partial_{xx}, \partial_{xy}, \partial_{yy}, \partial_x, \partial_y, 1$ on both sides of the resulting expression, we obtain the conditions

$$\partial_x\alpha_1 = \partial_y\alpha_2 = 0, \quad \partial_x\alpha_2 + \partial_y\alpha_1 = 0, \tag{1.4}$$

$$\begin{aligned}&(\partial_{xx} + \partial_{yy})\alpha_1 + 2\partial_x\alpha_3 + 2\lambda\alpha_1 = 0,\\ &(\partial_{xx} + \partial_{yy})\alpha_2 + 2\partial_y\alpha_3 + 2\lambda\alpha_2 = 0,\end{aligned} \tag{1.5}$$

$$(\partial_{xx}+\partial_{yy})\alpha_3+2\alpha_1\partial_x V+2\alpha_2\partial_y V+2\lambda\alpha_3=0. \quad (1.6)$$

It is easy to show that these equations have solutions if and only if

$$V(x,y)=f(bx-ay)+\tfrac{1}{2}\lambda^2(x^2+y^2)-\begin{cases}\lambda cx/a & \text{if } a\neq 0\\ \lambda cy/b & \text{if } a=0,\ b\neq 0.\end{cases} \quad (1.7)$$

Here, a, b, c are real constants with $a^2+b^2>0$ and f is an arbitrary real differentiable function. The raising operator is then

$$\underline{R}=a\partial_x+b\partial_y-\lambda(ax+by)+c. \quad (1.8)$$

By a simple translation and rotation of the (x,y) coordinates we can obtain new Cartesian coordinates X, Y in which

$$V(X,Y)=g(X)+\tfrac{1}{2}\lambda^2Y^2,\quad \underline{R}=\partial_Y-\lambda Y, \quad (1.9)$$

where $g(X)$ is arbitrary. In these coordinates the Schrödinger equation

$$\underline{H}\psi(X,Y)=\mu\psi(X,Y) \quad (1.10)$$

has solutions of the form

$$\psi_{\mu,n}=\exp(-|\lambda|Y^2/2)H_n(\sqrt{|\lambda|}Y)G(X),\quad n=0,1,2,\dots, \quad (1.11)$$

where H_n is a Hermite polynomial[3] and $G(X)$ is a square integrable solution of the equation

$$G''-2g(X)G=[-2\mu+|\lambda|(2n+1)]G.$$

It follows easily that

$$\underline{R}\psi_{\mu,n}=\begin{cases}-\sqrt{|\lambda|}\,\psi_{\mu+\lambda,n+1} & \text{if } \lambda>0\\ 2n\sqrt{|\lambda|}\,\psi_{\mu+\lambda,n-1} & \text{if } \lambda<0.\end{cases} \quad (1.12)$$

2. SECOND-ORDER OPERATORS

Next we consider the more interesting problem of computing those Hamiltonians $\underline{H}$ which admit a second-order raising operator $\underline{R}$:

$$\underline{R}=\alpha_1\partial_{xx}+\alpha_2\partial_{xy}+\alpha_3\partial_{yy}+\alpha_4\partial_x+\alpha_5\partial_y+\alpha_6. \quad (2.1)$$

Here $\alpha_j(x,y)$ is a real function of (x,y) and $\alpha_1^2+\alpha_2^2+\alpha_3^2>0$. Substituting (1.1) and (2.1) into (1.2) and equating coefficients of the third-order and second-order derivatives, we obtain equations for $\alpha_1,\dots,\alpha_5$ which can easily be solved to yield

$$\alpha_1=-A_1y+A_4,\quad \alpha_2=A_1x+A_2y+A_3,\quad \alpha_3=-A_2x+A_5,$$
$$\alpha_4=\lambda A_1xy-\lambda A_2y^2-\lambda A_3y-\lambda A_4x-A_6y+A_8, \quad (2.2)$$
$$\alpha_5=-\lambda A_1x^2+\lambda A_2xy-\lambda A_5y+A_6x+A_7,$$

where the A_i are real constants. The constraints on α_6 and V are obtained by equating coefficients of ∂_x, ∂_y, and 1 in (1.2):

$$\tfrac{1}{2}(\partial_{xx}+\partial_{yy})\alpha_4+\partial_x\alpha_6+2\alpha_1V_x+\alpha_2V_y=-\lambda\alpha_4, \quad (2.3)$$

$$\tfrac{1}{2}(\partial_{xx}+\partial_{yy})\alpha_5+\partial_y\alpha_6+\alpha_2V_x+2\alpha_3V_y=-\lambda\alpha_5, \quad (2.4)$$

$$\tfrac{1}{2}(\partial_{xx}+\partial_{yy})\alpha_6+\alpha_1V_{xx}+\alpha_2V_{xy}+\alpha_3V_{yy}+\alpha_4V_x+\alpha_5V_y=-\lambda\alpha_6. \quad (2.5)$$

Relations (2.3) and (2.4) yield

$$\partial_x\alpha_6=-2\alpha_1V_x-\alpha_2V_y-\lambda\alpha_4+A_2\lambda,$$
$$\partial_y\alpha_6=-\alpha_2V_x-2\alpha_3V_y-\lambda\alpha_5+A_1\lambda. \quad (2.6)$$

Substituting (2.6) into (2.5), we obtain an expression for the multiplier α_6 in terms of V:

$$2\lambda\alpha_6=(A_1-2\alpha_5)V_y+(A_2-2\alpha_4)V_x+\lambda^2(A_1y+A_2x-A_4-A_5). \quad (2.7)$$

Equations (2.3) and (2.4) may not be consistent with (2.7). To guarantee consistency, we differentiate (2.7) to compute $\partial_x\alpha_6$, $\partial_y\alpha_6$ and substitute into (2.3), (2.4). This yields the consistency conditions for the potential:

$$(A_2-2\alpha_4)V_{xx}+(A_1-2\alpha_5)V_{yx}+6\lambda\alpha_1V_x-2(\partial_x\alpha_5-\lambda\alpha_2)V_y=-2\lambda^2\alpha_4+\lambda^2A_2, \quad (2.8)$$

$$(A_2-2\alpha_4)V_{yx}+(A_1-2\alpha_5)V_{yy}-2(\partial_y\alpha_4-\lambda\alpha_2)V_x+6\lambda\alpha_3V_y=-2\lambda^2\alpha_5+\lambda^2A_1. \quad (2.9)$$

Thus, corresponding to any choice of the constants $A_1,\dots,A_8$, the Hamiltonian admits the raising operator $\underline{R}$, (2,1), (2,2), provided that V satisfies the partial differential equations (2.8) and (2.9). The multiplier α_6 for $\underline{R}$ is given by (2.7).

We can obtain another consistency relation for V by differentiating (2.8) with respect to x, differentiating (2.9) with respect to y, and subtracting the second equation from the first:

$$(A_1x+A_2y+A_3)V_{xx}+2(A_1y-A_2x-A_4+A_5)V_{xy}-(A_1x+A_2y+A_3)V_{yy}+3A_1V_x-3A_2V_y=-\lambda(-3A_1\lambda x+3A_2\lambda y+\lambda A_3+2A_6). \quad (2.10)$$

Although (2.10) is a consequence of (2.8) and (2.9), it is useful in its own right.

In conclusion, to find the potentials V admiting raising operators, we must solve the system (2.8)—(2.10) of overdetermined second order partial differential equations.

To simplify the solution of these equations, let us consider the action of the Euclidean group $E(2)$. Under the action of a Euclidean transformation the coordinates (x,y) go into new coordinates (x',y'), where

$$x'=x\cos\phi+y\sin\phi+a,\quad \phi,a,b\in R,$$
$$y'=-x\sin\phi+y\cos\phi+b. \quad (2.11)$$

Since Euclidean transformations preserve the Laplace operator, we have

$$-\tfrac{1}{2}(\partial_{x'x'}+\partial_{y'y'})+V(x',y')=-\tfrac{1}{2}(\partial_{xx}+\partial_{yy})+V'(x,y),$$

where $V'(x,y)=V(x',y')$. Thus the Hamiltonian $\underline{H}$ is transformed into a new Hamiltonian $\underline{H}'=-\frac{1}{2}(\partial_{xx}+\partial_{yy})+V'(x,y)$. Similarly the raising operator $\underline{R}$ is transformed into a new raising operator $\underline{R}'$ satisfying $[\underline{H}',\underline{R}']=\lambda\underline{R}'$. Considering the set M of all pairs $\{V,\underline{R}\}$ which satisfy (1.2), we see that $E(2)$ acts on M as a transformation group. We will consider two solutions of (1.2) as *equivalent* if one solution can be obtained from the other by a transformation (2.11), i.e., if both solutions lie on the same $E(2)$ orbit. Clearly, it will be enough for us to find a solution, if one exists, corresponding to a single point on each orbit.

For the orbit analysis we make use of (2.1) and (2.2) to write a general raising operator as

$$\underline{R}=\sum_{j=1}^{8} A_j\underline{Q}_j+\alpha_6(x,y), \tag{2.12}$$

where

$$\underline{Q}_1=MP_1-\lambda xM,\quad \underline{Q}_2=-MP_2+\lambda yM,\quad \underline{Q}_3=P_1P_2-\lambda yP_1,$$
$$\underline{Q}_4=P_1^2-\lambda xP_1,\quad \underline{Q}_5=P_2^2-\lambda yP_2,\quad \underline{Q}_6=M,\quad \underline{Q}_7=P_2,\quad \underline{Q}_8=P_1. \tag{2.13}$$

Here,

$$P_1=\partial_x,\quad P_2=\partial_y,\quad M=x\partial_y-y\partial_x \tag{2.14}$$

are the basis operators for the Lie algebra action of $E(2)$. We see that the pure differential operator component of $\underline{R}$ is described by the vector $(A_1,\ldots,A_8)$ and that the action of $E(2)$ induces an orbit structure on the set of all such vectors. A direct computation shows that a rotation through the angle θ [Eqs. (2.11) with $a=b=0$] transforms (A_j) into (A_j') with

$$\begin{aligned}
&A_1'=\cos\theta A_1+\sin\theta A_2,\quad A_2'=-\sin\theta A_1+\cos\theta A_2,\\
&A_3'=\cos2\theta A_3+\sin2\theta\,(A_4-A_5),\\
&A_4'=-\sin\theta\cos\theta A_3+\cos^2\theta A_4+\sin^2\theta A_5,\\
&A_5'=\sin\theta\cos\theta A_3+\sin^2\theta A_4+\cos^2\theta A_5,\\
&A_6'=\lambda\sin^2\theta A_3-\lambda\sin\theta\cos\theta A_4+\lambda\sin\theta\cos\theta A_5+A_6,\\
&A_7'=\cos\theta A_7+\sin\theta A_8,\quad A_8'=-\sin\theta A_7+\cos\theta A_8.
\end{aligned} \tag{2.15}$$

Similarly, the translation $x\to x+a$ yields

$$\begin{aligned}
&A_1'=A_1,\quad A_2'=A_2,\quad A_3'=aA_1+A_3,\\
&A_4'=A_4,\quad A_5'=-aA_2+A_5,\\
&A_6'=-2a\lambda A_1+A_6,\quad A_7'=-a^2\lambda A_1+aA_6+A_7,\\
&A_8'=-a\lambda A_4+A_8,
\end{aligned} \tag{2.16}$$

and the translation $y\to y+b$ yields

$$\begin{aligned}
&A_1'=A_1,\quad A_2'=A_2,\quad A_3'=bA_2+A_3,\quad A_4'=-bA_1+A_4,\\
&A_5'=A_5,\quad A_6'=b\lambda A_2+A_6,\\
&A_7'=-b\lambda A_5+A_7,\quad A_8'=-b^2\lambda A_2-b\lambda A_3-bA_6+A_8.
\end{aligned} \tag{2.17}$$

Using these results, we will choose a point on each $E(2)$-orbit. We start with a general operator $\sum A_j\underline{Q}_j$. Noticing that $A_1^2+A_2^2$ is an $E(2)$-invariant, we see that there are three cases:

Case 1: $A_1^2+A_2^2>0$.

Case 2: $A_1=A_2=0,\quad A_3^2+A_4^2+A_5^2>0$.

Case 3: $A_1=A_2=\cdots=A_5=0,\quad A_6^2+A_7^2+A_8^2>0$.

In Case 3, $\underline{R}$ is first order and has already been treated in Sec. 1. In Case 1 we can perform a rotation so that $A_1'>0$, $A_2'=0$ and then translate so that $A_3''=A_4''=0$. Thus the vectors (A_j) of the form

$$(A_1,0,0,0,A_5,A_6,A_7,A_8),\quad A_1>0, \tag{2.18}$$

cut each Case 1 orbit exactly once. In Case 2 we can perform a rotation such that $A_3'=0$. Thus, vectors of the form

$$(0,0,0,A_4,A_5,A_6,A_7,A_8),\quad A_4^2+A_5^2>0, \tag{2.19}$$

cut each Case 2 orbit at least once.

These Case 2 solutions of (2.8)–(2.10) are easy to find. Indeed, assuming that $\underline{R}=\sum A_j\underline{Q}_j+\alpha_6$ and V are Case 2 solutions, we can use (2.19) to require $A_1=A_2=A_3=0$. Then (2.10) becomes

$$2(A_5-A_4)V_{xy}=-2\lambda A_6. \tag{2.20}$$

Suppose first that $A_5-A_4\neq0$. Then (2.20) has the general solution

$$V=-[\lambda A_6xy/(A_5-A_4)]+f(x)+g(y),$$

where f and g are arbitrary. Substituting this solution into (2.8) and (2.9), we find $A_6=0$ for consistency, and so $V=f(x)+g(y)$. If both A_4 and A_5 are nonzero, we can perform translations (2.16), (2.17) to achieve $A_7=A_8=0$. Thus Eqs. (2.8), (2.9) reduce to

$$xf''+3f'=\lambda^2x,\quad \lambda g''+3g'=\lambda^2y$$

with general solution

$$V(x,y)=\frac{\lambda^2}{8}(x^2+y^2)+\frac{a}{x^2}+\frac{b}{y^2}+c,\quad a,b,c\in R,$$
$$(A_j)=(0,0,0,A_4,A_5,0,0,0),\quad A_4,A_5\neq0. \tag{2.21}$$

If $A_4\neq0$, $A_5=0$, we can perform a translation (2.16) to achieve $A_8=0$. Then Eqs. (2.8), (2.9) reduce to

$$xf''+3f'=\lambda^2x,\quad (g''-\lambda^2)A_7=0$$

with solutions

$$V(x,y)=\frac{\lambda^2}{8}(x^2+4y^2)+\frac{a}{x^2}+by+c,$$
$$(A_j)=(0,0,0,A_4,0,0,A_7,0),\quad A_4,A_7\neq0 \tag{2.22}$$

and

$$V(x,y)=\frac{\lambda^2x^2}{8}+\frac{a}{x^2}+g(y),\quad g(y)\ \text{arbitrary},$$
$$(A_j)=(0,0,0,A_4,0,0,0,0),\quad A_4\neq0. \tag{2.23}$$

The cases $A_5\neq0$, $A_4=0$ are identical to (2.22), (2.23) with x and y interchanged.

Finally, suppose $A_4=A_5\neq0$. Then (2.20) yields $A_6=0$, and by applying translations (2.16), (2.17) we can achieve $A_7=A_8=0$. Thus Eqs. (2.8), (2.9) reduce to

$$\frac{d}{dx}(xV_x+yV_y+2V)=\lambda^2x,\quad \frac{d}{dy}(xV_x+yV_y+2V)=\lambda^2y$$

with general solution

$$V(x,y)=\frac{\lambda^2}{8}(x^2+y^2)+\frac{g(x/y)}{y^2}+a=\frac{\lambda^2}{8}r^2+\frac{f(\theta)}{r^2}+a,$$
$$(A_j)=(0,0,0,A_4,A_4,0,0,0),\quad A_4\neq0. \tag{2.24}$$

Here g is arbitrary, $x=r\cos\theta$, $y=r\sin\theta$, $f(\theta)=g(\tan\theta)/\sin^2\theta$.

This completes the analysis of Case 2. However, Case 1, Eqs. (2.18), is much more difficult. We have not been able to discover a direct practical means of computing all solutions of (2.8)–(2.10) corresponding to this case. In the next section we develop an indirect group-theoretic procedure which not only enables us to solve these equations but also provides clear insight into the structure of second-order raising operators.

3. SEPARATION OF VARIABLES

Let us note that raising and lowering operators occur in pairs: If $\underline{R}$ is a raising operator for $\underline{H}$,

$$[\underline{H},\underline{R}]=\lambda\underline{R},\quad \lambda\neq 0,\quad \lambda\in R \tag{3.1}$$

then, taking the formal adjoint, we have

$$[\underline{H},\underline{R}^+]=-\lambda\underline{R}^+ \tag{3.2}$$

so that $\underline{R}^+$ is a lowering operator (raising operator by $-\lambda$). In particular, if $\underline{R}$ takes the form (2.12), then

$$\underline{R}^+=\sum_{j=1}^{8}A_j\underline{Q}_j^+ +\alpha_6=\sum_{j=1}^{8}\tilde{A}_j\underline{Q}_j(-\lambda)+\tilde{\alpha}_6, \tag{3.3}$$

where

$$\begin{aligned}&\tilde{A}_j=A_j,\quad 1\leq j\leq 5,\\ &\tilde{A}_6=-A_6,\quad \tilde{A}_7=A_1-A_7,\quad \tilde{A}_8=A_2-A_8,\\ &\tilde{\alpha}_6=\alpha_6-\lambda(A_1y+A_2x-A_4-A_5).\end{aligned} \tag{3.4}$$

Here $\underline{Q}_j(-\lambda)$ is obtained from $\underline{Q}_j$, (2.13), by replacing λ with $-\lambda$. These results follow from (2.13) and the following facts:

$$\begin{aligned}&\underline{Q}_1^+=MP_1+\lambda xM+P_2-\lambda y,\quad \underline{Q}_2^+=-MP_2-\lambda yM+P_1-\lambda x,\\ &\underline{Q}_3^+=P_1P_2+\lambda yP_1,\quad \underline{Q}_4^+=P_1^2+\lambda xP_1+\lambda,\\ &\underline{Q}_5^+=P_2^2+\lambda yP_2+\lambda,\quad \underline{Q}_6^+=-M,\quad \underline{Q}_7^+=-P_2,\quad \underline{Q}_8^+=-P_1.\end{aligned} \tag{3.5}$$

Moreover, it follows from (3.1) and (3.2) that $[\underline{H},\underline{S}]=0$, where $\underline{S}=[\underline{R},\underline{R}^+]$, i.e., $\underline{S}$ is a symmetry of $\underline{H}$. We are concerned with the case where $\underline{R}$ and $\underline{R}^+$ are both second-order differential operators, so that we would expect that $\underline{S}$ was in general a third-order operator. However, we see from (2.12), (2.13), and (3.3) that the purely second-order terms in $\underline{R}$ and $\underline{R}^+$ are identical. This means that $\underline{S}$ is at most a second-order operator. Indeed for $\underline{S}=\mathcal{S}+\beta$, where $\mathcal{S}$ is a pure differential operator and β is a multiplier function, a straightforward computation yields

$$\begin{aligned}\mathcal{S}=&P_1^2(4\lambda A_4^2+2\lambda A_3^2+A_1^2-2A_1A_7+2A_3A_6)+P_2^2(4\lambda A_5^2+A_2^2\\ &-2A_2A_8-2A_3A_6)+P_1P_2(2A_1A_8-2A_1A_2+2A_2A_7+6\lambda A_3A_5\\ &+2\lambda A_3A_4-4A_4A_6+4A_5A_6)+M^2(4\lambda A_1^2+4\lambda A_2^2)+(MP_1\\ &+P_1M)(-\lambda A_1A_5+4\lambda A_1A_4-A_2A_6-3\lambda A_2A_3)+(MP_2\\ &+P_2M)(-A_1A_6+2\lambda A_1A_3-4\lambda A_2A_5+\lambda A_2A_4).\end{aligned} \tag{3.6}$$

At this point we can make use of the results of Ref. 1. There one studies differential operators

$$\begin{aligned}\underline{L}=&AP_1^2+BP_1P_2+CP_2^2+DM^2+E(P_1M+MP_1)\\ &+F(P_2M+MP_2)+\gamma(x,y)\end{aligned} \tag{3.7}$$

such that $[\underline{H},\underline{L}]=0$, where $\underline{H}$ is given by (1.1). A principal result of Ref 1 is essentially that if $\underline{H}$ commutes with a nontrivial $\underline{L}$, then the Schrödinger equation $\underline{H}\psi=\mu\psi$ separates in one of four orthogonal coordinate systems. More specifically the authors study the action of $E(2)$ on the set of all operators $\underline{L}$ via the coordinate transformations (2.11). They show that the $E(2)$-orbits are of five types.

I. $P_1^2-P_2^2+a(P_1^2+P_2^2)+\beta$,

II. $P_1M+MP_1+a(P_1^2+P_2^2)+\beta$,

III. $M^2+a(P_1^2+P_2^2)+\beta$, (3.8)

IV. $M^2+\frac{1}{2}l^2(P_1^2-P_2^2)+a(P_1^2+P_2^2)+\beta$,

V. $a(P_1^2+P_2^2)+\beta,\quad a,l\in R,\quad l>0.$

Every $\underline{L}$ lies on the same orbit as a constant multiple of exactly one of the elements I–V. (The term $P_1^2+P_2^2$ occurs with an arbitrary constant because the Hamiltonian always commutes with itself.) Thus by applying an appropriate $E(2)$ transformation we can always assume that $\underline{L}$ is equal to one of these five forms.

If $\underline{L}$ takes the form I, then, according to Ref. 1,

$$V(x,y)=f(x)+g(y), \tag{3.9}$$

and the Schrödinger equation separates in rectangular coordinates. If $\underline{L}$ takes the form II, then

$$V=\frac{f(\xi_1)+g(\xi_2)}{\xi_1^2+\xi_2^2},\quad x=\tfrac{1}{2}(\xi_1^2-\xi_2^2),\quad y=\xi_1\xi_2, \tag{3.10}$$

and the Schrödinger equation separates in parabolic coordinates, while if $\underline{L}$ takes form III,

$$V=f(r)+g(\theta)/r^2,\quad x=r\cos\theta,\quad y=r\sin\theta \tag{3.11}$$

and the equation separates in polar coordinates. If $\underline{L}$ takes form IV, then

$$V=\frac{f(\sigma)+g(\rho)}{\cos^2\sigma-\cosh^2\rho},\quad x=l\cosh\rho\cos\sigma,\quad y=l\sinh\rho\sin\sigma, \tag{3.12}$$

and the equation separates in elliptic coordinates. Finally, if $\underline{L}$ takes form V, then $\underline{L}$ is a multiple of $\underline{H}$ and there is no information about V.

The above results apply immediately to our study of the operator $\underline{S}$. First of all, by putting $\underline{R}$ in one of the forms (2.18), (2.19), we see from (3.6) that if $\underline{R}$ is truly second-order, then $\underline{S}$ is truly second-order (never first-order).

Note that the coefficient of M^2 in (3.6) is proportional to $A_1^2+A_2^2$. If this coefficient is nonzero, then $\underline{S}$ lies on a type III or IV orbit, i.e., the Schrödinger equation separates in either polar or elliptic coordinates. If $A_1=A_2=0$, then $\underline{S}$ lies on a type I, II, or V orbit.

We consider Case 2$(A_1=A_2=0)$ first. Then from (2.19) we can also require $A_3=0$, $A_4^2+A_5^2>0$. Substituting into (3.6), we find

$$\underline{S}=4\lambda A_4^2P_1^2+4\lambda A_5^2P_2^2+4A_6(A_5-A_4)P_1P_2+\beta. \tag{3.13}$$

It follows that type II orbits never appear, only type I and V orbits are possible. Moreover, our analysis of (2.20) has shown that we can find a potential V only if $A_6=0$. Thus $\underline{S}$ corresponds to a type I orbit if $A_4^2\neq A_5^2$ and to a type V orbit if $A_4^2=A_5^2$. The method of Ref. 1 yields no information for type V orbits but our direct approach in Sec. 2 has yielded the solutions (2.24), separation in polar coordinates, and the special case $A_4=-A_5$ of (2.21), separation in rectangular coordinates. For $A_4^2\neq A_5^2$ the results of Ref. 1 show that $\underline{H}$ lies on the same orbit as a Hamiltonian whose potential takes the form $V=f(x)+g(y)$. This agrees with the results (2.21)–(2.23).

So far we have merely verified previous results. However, the method of Ref. 1 now allows us to find all solutions of (2.8)—(2.10) corresponding to Case 1. Indeed, if $A_1^2+A_2^2>0$, we know that $\underline{H}$ lies on the same orbit as a Hamiltonian with potential of the form (3.11) or of the form (3.12). Thus, we can find all Case 1 solutions of (2.8)—(2.10) by requiring that V take either the form (3.11) or (3.12). That is, every solution V lies on the same orbit as a V which separates in either polar or elliptic coordinates. This fact is of great importance for it allows us to separate variables in (2.8)—(2.10) and reduce these coupled partial differential equations to uncoupled ordinary differential equations for f and g.

At this point we have proved the following fact: If a Hamiltonian $\underline{H}$ admits a second-order raising operator then the Schrödinger equation $\underline{H}\psi=\mu\psi$ separates in either rectangular, polar or elliptic coordinates. Of course the converse is false.

To find all cases when $\underline{S}$ is type III we substitute the polar coordinate expression (3.11) into (2.8)—(2.10) and find all solutions which correspond to type III orbits. A tedious computation yields the single solution

$$V=\frac{\lambda^2r^2}{2}+\frac{a\sin\theta+b}{r^2\cos^2\theta}+c=\frac{\lambda^2}{2}(x^2+y^2)+\frac{ay}{x^2\sqrt{x^2+y^2}}+\frac{b}{x^2}+c, \tag{3.14}$$

$$(A_j)=(A_1,0,0,0,0,0,\tfrac{1}{2}A_1,0),\quad A_1\neq 0.$$

Every type III solution lies on the same orbit as (3.14).

To find all cases when $\underline{S}$ is type IV we substitute the elliptic coordinate expression (3.12) into (2.8)—(2.10) and find all solutions which correspond to type IV orbits. We obtain

$$V=\frac{\tfrac{1}{2}\lambda^2(\cosh^2\rho+\cos^4\sigma-\cosh^4\rho-\cos^2\sigma)+b(1/\cosh^2\rho-1/\cos^2\sigma)}{\cos^2\sigma-\cosh^2\rho}$$
$$+c=\tfrac{1}{2}\lambda^2(x^2+y^2)+b/x^2+c, \tag{3.15}$$
$$(A_j)=(A_1,0,0,0,0,0,A_7,0),\quad A_1\neq 0,\quad 2A_7\neq A_1,$$
$$x=\cosh\rho\cos\sigma,\quad y=\sinh\rho\sin\sigma,$$
$$V=\frac{\lambda^2}{2}\frac{(\cosh^2\rho+\cos^4\sigma-\cosh^4\rho-\cos^2\sigma)}{\cos^2\sigma-\cosh^2\rho}+c=\frac{\lambda^2}{2}(x^2+y^2)+c, \tag{3.16}$$
$$(A_j)=(A_1,0,0,0,0,0,A_7,A_8),\quad A_1,A_8\neq 0.$$

The determination of all solutions of Eqs. (2.8)—(2.10) for elliptic coordinates is extremely tedious due to the complicated nature of the coefficients in the resultant coupled ordinary differential equations. Our method is to examine these equations in the vicinity of some convenient point which may or may not be a singularity of the potential. [This singularity cannot be essential since from (2.10) in elliptic coordinates one can see that if there is a singular point it is regular.] For example, examination of (2.8)—(2.10) about the points $\sin\sigma=0$ yields six differential equations involving only $g(\rho)$ which must be compatible. In this way one can proceed until all possibilities for the parameters A_j and potentials V are exhausted.

4. EXAMPLES

In this section we explicitly solve the Schrödinger equations corresponding to the above potentials and examine the action of our second-order raising operators. Without loss of generality we can assume $\lambda>0$ and set the additive constant c equal to zero for each potential. In each case we solve the equation $\underline{H}\psi=\mu\psi$ corresponding to appropriate choices of the potential parameters.

Consider first the potential (2.21),

$$V(x_1,x_2)=\frac{\lambda^2}{8}(x_1^2+x_2^2)+\frac{a_1}{x_1^2}+\frac{a_2}{x_2^2}.$$

Bound states exist for $a_i>-\frac{1}{8}$, and the normalized eigenfunctions are

$$\psi^{\pm\pm}_{k_1k_2}(x_1,x_2)=\frac{\lambda}{2}\prod_{i=1}^{2}\left(\frac{k_i!\,(\lambda^2/4)^{\pm\nu_i}}{\Gamma(k_i\pm\nu_i+1)}\right)^{1/2}\exp(-\lambda x_i^2/2)\times x_i^{1/2\pm\nu_i}L_{k_i}^{\pm\nu_i}(\lambda x_i^2/2), \tag{4.1}$$
$$\nu_i=\tfrac{1}{2}(1+8a_i)^{1/2},\quad \mu^{\pm\pm}=\lambda(k_1+k_2+1)+\tfrac{1}{2}\lambda(\pm\nu_1\pm\nu_2),$$
$$k_i=0,1,2,\ldots .$$

For details on the degeneracies see Ref. 1.

Here $L_n^\nu(x)$ is a generalized Laguerre polynomial.[3] The raising operator in x_1 takes the form

$$\underline{R}=-\tfrac{1}{4}(2/\lambda\ \partial_{x_1x_1}-2x_1\partial_{x_1}+\tfrac{1}{2}\lambda x_1^2-4a_1/\lambda x_1^2-1) \tag{4.2}$$

with action

$$\underline{R}\psi_{k_1k_2}=\sqrt{(k_1+1)(k_1+\nu_1+1)}\,\psi_{k_1+1,k_2},$$
$$\underline{R}^+\psi_{k_1k_2}=\sqrt{k_1(k_1+\nu_1)}\,\psi_{k_1-1,k_2}. \tag{4.3}$$

There is a similar operator in x_2 which raises the k_2 index.

The potential (2.22),

$$V(x_1,x_2)=\tfrac{1}{8}\lambda^2(x_1^2+4x_2^2)+(a_1/x_1^2)+a_2x_2,$$

has bound states for $a_1>-\frac{1}{8}$, with eigenfunctions

$$\psi^{\pm}_{k_1k_2}(x_1,x_2)=\left(\frac{\lambda}{\pi}\right)^{1/4}\left(\frac{\lambda}{2}\right)^{(1\pm\nu)/2}\left(\frac{k_1!\,1}{k_2!\,2^{k_2}\Gamma(k_1\pm\nu+1)}\right)^{1/2}$$
$$\times\exp\left[-\frac{\lambda}{2}\left(x_2+\frac{a_2}{\lambda^2}\right)^2\right]H_{k_2}\left[\sqrt{\lambda}\left(x_2+\frac{a_2}{\lambda^2}\right)\right]$$
$$\times\exp\left(-\frac{\lambda}{4}x_1^2\right)x_1^{1/2\pm\nu}L_{k_1}^{\pm\nu}\left(\frac{\lambda}{2}x_1^2\right),$$
$$\nu=\tfrac{1}{2}(1+a_1)^{1/2},$$
$$\mu^{\pm}=\lambda\left[k_1+k_2+1\pm\frac{\nu}{2}-\frac{a_2^2}{2}\left(\frac{\lambda^2}{2}\right)^{-3/2}\right],\quad k_1,k_2=0,1,\ldots . \tag{4.4}$$

Details on the degeneracies can again be found in Ref. 1. This potential takes the form (1.7) with $x_1=x_1=x$, x_1, $x_2=y$, $a=0$ so that it admits a first-order raising operator (1.12) in x_2. It also admits the second-order operator (4.2) with action (4.3). Similarly the potential (2.23) admits a second-order raising operator in $x=x_1$ with the form (4.2).

The potential (2.24)

$$V=\tfrac{1}{8}\lambda^2r^2+f(\theta)/r^2$$

corresponds to eigenfunctions

$$\psi_{n,s}(r,\theta)=\left(\frac{\lambda}{2}\right)^{(s+1)/2}\left(\frac{2(n!)}{\Gamma(n+s+1)}\right)^{1/2}$$

$$\times\exp\left(-\frac{\lambda r^2}{4}\right)r^s L_n^s\left(\frac{\lambda r^2}{2}\right)\Theta_s(\theta),\qquad n=0,1,\cdots. \tag{4.5}$$

where Θ_s is a solution of

$$\Theta_s'' + [s^2 - 2f(\theta)]\Theta_s = 0,$$

and $\mu=\lambda(n+\frac{1}{2}s+\frac{1}{2})$, $s>-1$. The raising operator takes the form

$$\underline{R} = -\frac{1}{4}\left[\partial_{rr}+\left(\frac{1}{r}-\lambda r\right)\partial_r - \frac{s^2}{r^2}+\frac{\lambda^2 r^2}{4}-\lambda\right]$$

with action

$$\underline{R}\psi_{n,s}=\tfrac{1}{2}\lambda[(n+s+1)(n+1)]^{1/2}\psi_{n+1,s},$$
$$R^*\psi_{n,s}=\tfrac{1}{2}\lambda[(n+s)n]^{1/2}\psi_{n-1,s}. \tag{4.7}$$

For the above potentials it was always possible to choose coordinates such that $\underline{R}$ could be expressed as a differential operator in a single variable. In the remaining three cases this is no longer possible and the action of the raising operator is more complicated.

The potential (3.14),

$$V=\frac{\lambda^2 r^2}{2}+\frac{a\sin\theta+b}{r^2\cos^2\theta},$$

has, for example, in the case $a=(\alpha^2-\beta^2)/4$, $b=-\frac{1}{8}+(\alpha^2+\beta^2)/4$, $\alpha>\frac{1}{2}$, $\beta>\frac{1}{2}$, normalized eigenfunctions

$$\psi_{n,k}(r,\theta)=\left(\frac{\lambda(n!)(k!)\Gamma(\alpha+\beta+k+1)(\alpha+\beta+2k+1)}{\Gamma(n+\frac{1}{2}(\alpha+\beta)+k+\frac{3}{2})\Gamma(\alpha+k+1)\Gamma(\beta+k+1)2^{\alpha+\beta}}\right)^{1/2}$$
$$\times\exp(-\lambda r^2/2)(\sqrt{\lambda}r)^{(\alpha+\beta)/2+k+1/2}L_n^{(\alpha+\beta+1+k)/2}(\lambda r^2)$$
$$\times(1+\sin\theta)^{\beta/2+1/4}(1-\sin\theta)^{\alpha/2+1/4}P_k^{\alpha,\beta}(\sin\theta),\qquad n,k=0,1,2,\cdots, \tag{4.8}$$

and energy eigenvalues $\mu=\lambda[2n+k+\frac{1}{2}(\alpha+\beta)+1]$. Here, $P_k^{\alpha,\beta}(x)$ is Jacobi polynomial.[3] The raising operator is

$$\underline{R}=-\frac{\sin\theta}{r}\partial_{\theta\theta}+\cos\theta\,\partial_{r\theta}-\tfrac{1}{2}\sin\theta\partial_r-\left(\frac{\cos\theta}{2r}+\lambda r\cos\theta\right)\partial_\theta$$
$$+\frac{(\alpha^2-\beta^2)}{4r}\frac{(\sin^2\theta+1)}{\cos^2\theta}+\left(\frac{\alpha^2+\beta^2}{2}-\frac{1}{4}\right)\frac{\sin\theta}{r\cos^2\theta}+\frac{\lambda r}{2}\sin\theta \tag{4.9}$$

and its action takes the form

$$\underline{R}\psi_{n,k}=\gamma_{n,k}\psi_{n,k+1}+\xi_{n,k}\psi_{n+1,k-1},$$
$$\underline{R}^*\psi_{n,k}=\gamma_{n,k-1}\psi_{n,k-1}+\xi_{n-1,k+1}\psi_{n-1,k+1} \tag{4.10}$$

where γ, ξ are rather complicated real constants, nonzero in general. Thus, $\underline{R}$ no longer raises a single index n or k.

The potential (3.15),

$$V=\tfrac{1}{2}\lambda^2(x^2+y^2)+b/x^2,$$

is of the form (1.9) in y. Thus, it admits a first-order raising operator with action (1.11), (1.12). Furthermore, this potential corresponds to a special case of (2.21) so that its eigenfunctions are given by (4.1) with $a_1=b$, $a_2=0$, and it admits two second-order raising operators of the form (4.2). The potential is also a special case of (2.24) and (3.14) so that it admits the raising operators (4.6) and (4.9). However, the potential admits the further raising operator

$$\underline{R}=y(\partial_{xx}-\lambda^2x^2-2b/x^2-\lambda)-x(\partial_x-\lambda x)(\partial_x+\lambda y), \tag{4.11}$$

which is not admitted by the earlier mentioned potentials in their generality. In Cartesian coordinates the Hamiltonian has (unnormalized) eigenvectors

$$\psi^{\pm\nu}_{k_1,k_2}(x,y)=\exp(-\lambda r^2/2)(x)^{\pm\nu+1/2}L^{\pm\nu}_{k_1}(\lambda x^2)H_{k_2}(\sqrt{\lambda}y), \tag{4.12}$$

$$\nu=(2b+\tfrac{1}{4})^{1/4},\quad b>-\tfrac{1}{8},$$

and eigenvalues

$$\mu=\lambda(2k_1+k_2\pm\nu+1),\quad k_1,k_2=0,1,\cdots.$$

The action of $\underline{R}$ is given by

$$\underline{R}\psi^{\pm\nu}_{k_1,k_2}=-\lambda^{1/2}(2k_1\pm\nu+\tfrac{1}{2})\psi_{k_1,k_2+1}-4\lambda^{1/2}(k_1+1)k_2\psi_{k_1+1,k_2-1}, \tag{4.13}$$

$$\underline{R}^*\psi^{\pm\nu}_{k_1,k_2}=-2\lambda^{1/2}(k_1\pm\nu)\psi_{k_1-1,k_2+1}-2\lambda^{1/2}k_2(2k_1\pm\nu+\tfrac{3}{2})\psi_{k_1,k_2-1}.$$

The potential (3.16), isotropic harmonic oscillator, is a special case of all previous potentials except (2.22) and it admits all of the raising operators allowed by these potentials.

The raising operator $Q_1+\alpha_6$ admitted by potentials (3.15) and (3.16) implies via our procedure that the corresponding Schrödinger equations separate in elliptic coordinates. Thus one might expect that the action of these raising operators would be simplest in elliptic coordinates. This is not the case. The elliptic coordinate solutions of the harmonic oscillator Hamiltonian are Ince polynomials,[4] but the corresponding polynomial solutions for (3.15) in elliptic coordinates appear not to have been studied in any detail. In any event, the action of the raising operator on an elliptic basis is not transparent.

In conclusion, we remark that Refs. 5 and 6 contain results related to our work.

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