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Math 4567. Homework Set # 5

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Chapter 3 (page 79, problems 1,2), (page 82, problems 1,2), (page 86, problems 2,3), Chapter 4 (page 93, problems 2,3), (page 98, problems 1,2), (page 102, problems 1,2,3).

Chapter 3, page 79, Problem 1 a. Let $z(\rho)$ be the static transverse displacements in a membrane, streched between circles $\rho = 1$ and $\rho = \rho_0 > 1$, the first circle in the plane $z = 0$ and the second in the plane $z = z_0$.

a. Show that the boundary problem can be written as

$$\frac{d}{d\rho}\left(\rho\frac{dz}{d\rho}\right) = 0, \quad 1 < \rho < \rho_0,$$

$$z(1) = 0, \quad z(\rho_0) = z_0.$$

b. Obtain the solution

$$z(\rho) = z_0\frac{\ln\rho}{\ln\rho_0}, \quad 1 \le \rho \le \rho_0.$$

Solution:

a. The wave equation for the vibrating membrane is $z_{tt} = a^2(z_{xx} + z_{yy})$. In polar coordinates $x = \rho\cos\phi, y = \sin\phi$ this equation reads

$$z_{tt} = a^2(\frac{d}{d\rho}\left(\rho\frac{dz}{d\rho}\right) + \frac{z_{\phi\phi}}{\rho^2}$$

. Steady state means that $z_t = 0$ and the rotational symmetry of the problem means that $z_\phi = 0$. Thus the equation for $z(\rho)$ reduces to $\frac{d}{d\rho}\left(\rho\frac{dz}{d\rho}\right) = 0$, $1 < \rho < \rho_0$, with boundary conditions $z(1) = 0$, $z(\rho_0) = z_0$.

b. Since $\frac{d}{d\rho}\left(\rho\frac{dz}{d\rho}\right) = 0$, we must have $\rho\frac{dz}{d\rho} = c_1$ where c_1 is a constant. Thus $\frac{dz}{d\rho} = c_1/\rho$. Integrating again we have $z(\rho) = c_1 \ln\rho + c_2$. Since $z(1) = 0$ we have $c_2 = 0$. Since $z(\rho_0) = z_0$ we have $z_0 = c_1 \ln\rho_0$. Thus $c_1 = z_0/\ln\rho_0$ and

$$z(\rho) = z_0\frac{\ln\rho}{\ln\rho_0}, \quad 1 \le \rho \le \rho_0.$$

Chapter 3, page 79, Problem 2 Show that the steady-state temperatures $u(\rho)$ in an infinitely long hollow cylinder $1 \le \rho \le \rho_0$, $-\infty < z < \infty$ also satisfy the boundary value Problem 1 if $u = 0$ on the inner cylindrical surface and $u = z_0$ on the outer one.

Solution: Here the heat equation is $u_t = k(u_{xx} + u_{yy})$. In polar coordinates $x = \rho\cos\phi, y = \sin\phi$ this is

$$u_t = k(\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) + \frac{u_{\phi\phi}}{\rho^2}.$$

Steady state means that $u_t = 0$, and axial symmetry means that $u_\phi = 0$. Thus the equation for $u(\rho)$ reduces to $\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) = 0$, $1 < \rho < \rho_0$, with boundary conditions $u(1) = 0$, $u(\rho_0) = z_0$.

Chapter 3, page 82, Problem 1 Use the general solution of the wave equation to solve the boundary value problem

$$y_{tt} = a^2 y_{xx}, \quad -\infty < x < \infty,\ t > 0,$$

$$y(x,0) = 0,\ y_t(x,0) = g(x), \quad -\infty < x < \infty.$$

Solution: The general solution of the wave equation is

$$y(x,t) = \phi(x + at) + \psi(x - at)$$

for arbitrary twice differentiable functions ϕ, ψ. We impose the boundary conditions on this general solution:

$$y(x,0) = 0 = \phi(x) + \psi(x),$$

$$y_t(x,0) = g(x) = a\left(\phi'(x) - \psi'(x)\right).$$

Thus $\psi(x) = -\phi(x)$ and $\phi'(x) = \frac{1}{2a}g(x)$. Integrating, we have

$$\phi(x) = C + \int_0^x \phi'(s)\ ds = C + \frac{1}{2a}\int_0^x g(s)\ ds,$$

$$\psi(x) = -\phi(x) = -C - \frac{1}{2a}\int_0^x g(s)\ ds = -C + \frac{1}{2a}\int_x^0 g(s)\ ds.$$

Thus

$$y(x,t) = \phi(x+at) + \psi(x-at) = \frac{1}{2a}\int_{x-at}^{x+at} g(s)\ ds.$$

Chapter 3, page 82, Problem 2 Let $Y(x,t)$ be d'Alembert's solution

$$Y(x,t) = \frac{1}{2}(f(x+at) + f(x-at))$$

of the boundary value problem solved in Section 27 and let $Z(x,t)$ denote the solution found in Problem 1. Verify that $y(x,t) = Y(x,t) + Z(x,t)$ solves the problem

$$y_{tt} = a^2 y_{xx}, \quad -\infty < x < \infty,\ t > 0,$$

$$y(x,0) = f(x),\ y_t(x,0) = g(x), \quad -\infty < x < \infty.$$

Solution: We have that $Z(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} g(s)\ ds$. solves the problem

$$y_{tt} = a^2 y_{xx}, \quad -\infty < x < \infty,\ t > 0,$$

$$y(x,0) = 0,\ y_t(x,0) = g(x), \quad -\infty < x < \infty.$$

whereas $Y(x,t)$ solves the problem

$$y_{tt} = a^2 y_{xx}, \quad -\infty < x < \infty,\ t > 0,$$

$$y(x,0) = f(x),\ y_t(x,0) = 0, \quad -\infty < x < \infty.$$

Thus by linearity, $y(x,t) = Y(x,t) + Z(x,t)$ solves the full initial value problem and yields the solution

$$y(x,t) = \frac{1}{2}(f(x+at) + f(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\ ds.$$

Chapter 3, page 86, Problem 2 Consider the equation

$$Ay_{xx} + By_{xt} + Cy_{tt} = 0,\ B^2 - 4AC > 0, \quad AC \neq 0,$$

where A, B, C are constants.

1. Use the transformation $u = x + \alpha t$, $v = x + \beta t$, $\alpha \neq \beta$, to derive the equation

$$(A+B\alpha+C\alpha^2)y_{uu}+[2A+B(\alpha+\beta)+2C\alpha\beta]y_{uv}+(A+B\beta+C\beta^2)y_{vv} = 0.$$

2. Show that $y_{uv} = 0$ if α, β have the values

$$\alpha_0 = \frac{-B + \sqrt{B^2 - 4AC}}{2C}, \quad \beta_0 = \frac{-B - \sqrt{B^2 - 4AC}}{2C}.$$

3. Conclude from the last result that the general solution of the original equation is $y = \phi(x + \alpha_0 t) + \psi(x + \beta_0 t)$ where ϕ, ψ are twice differentiable. Then verify that the solution of the wave equation $y_{tt} - a^2 y_{xx} = 0$ follows as a special case.

Solution:

1. We have

$$\partial_t = \alpha\partial_u + \beta\partial_v, \quad \partial_x = \partial_u + \partial_v.$$

thus

$$\begin{aligned} y_{xx} &= (\partial_u + \partial_v)(y_u + y_v) = y_{uu} + 2y_{uv} + y_{vv} \\ y_{xt} &= (\partial_u + \partial_v)(\alpha y_u + \beta y_v) = \alpha(y_{uu} + (\alpha + \beta)y_{uv} + \beta y_{vv}, \\ y_{tt} &= (\alpha\partial_u + \beta\partial_v)(\alpha y_u + \beta y_v) = \alpha^2 y_{uu} + 2\alpha\beta y_{uv} + \beta^2 y_{vv}. \end{aligned}$$

Substituting into equation

$$Ay_{xx} + By_{xt} + Cy_{tt} = 0$$

we obtain the desired result

$$(A+B\alpha+C\alpha^2)y_{uu}+[2A+B(\alpha+\beta)+2C\alpha\beta]y_{uv}+(A+B\beta+C\beta^2)y_{vv} = 0.$$

2. The roots of the quadratic equation $A + B\alpha + C\alpha^2 = 0$ are $\alpha = \frac{-B\pm\sqrt{B^2-4AC}}{2C}$. Thus $\alpha = \alpha_0$ is a root. The roots of the quadratic equation $A + B\beta + C\beta^2 = 0$ are again $\beta = \frac{-B\pm\sqrt{B^2-4AC}}{2C}$. Thus $\beta = \beta_0$ is a root. With these substituions the equation becomes

$$[2A + B(\alpha_0 + \beta_0) + 2C\alpha_0\beta_0]y_{uv} = 0$$

or

$$(2A + \frac{-B^2}{C} + 2C\frac{B^2 - B^2 + 4AC}{4C^2})y_{uv} = \frac{4AC - B^2}{C}y_{uv} = 0,$$

so $y_{uv} = 0$.

3. Since $y_{uv} = 0$, the general solution of this eauation is $y = \phi(u) + \psi(v)$. Passing to the original variables x, t we have

$$y(x,t) = \phi(x + \alpha_0 t) + \psi(x + \beta_0 t)$$

as the general solution. In the special case of the equation $y_{tt} - a^2 y_{xx} = 0$ we have $A = -a^2$, $B = 0$ and $C = 1$, so $B^2 - 4AC > 0$, $AC \neq 0$ and $\alpha_0 = a$, $\beta_0 = -a$. Thus we recover the solution

$$y(x,t) = \phi(x + at) + \psi(x - at).$$

Chapter 3, page 86, problem 3 Show that with the transformation $u = x$, $v = \alpha x + \beta t$ for $\beta \neq 0$, the equation of Problem 2 becomes

$$Ay_{uu} + (2A\alpha + B\beta)y_{uv} + (A\alpha^2 + B\alpha\beta + C\beta^2)y_{vv} = 0.$$

Then show that the new equation reduces to (a) $y_{uu} + y_{vv} = 0$ when $B^2 - 4AC < 0$ and

$$\alpha = \frac{-B}{\sqrt{4AC - B^2}}, \quad \beta = \frac{2A}{\sqrt{4AC - B^2}};$$

(b) $y_{uu} = 0$ when $B^2 - 4AC = 0$ and $\alpha = -B$, $\beta = 2A$.

Solution:

1. We have
$$\partial_t = \beta\partial_v, \quad \partial_x = \partial_u + \alpha\partial_v,$$
so
$$y_{xx} = (\partial_u + \alpha\partial_v)(y_u + \alpha y_v) = y_{uu} + 2\alpha y_{uv} + \alpha^2 y_{vv},$$
$$y_{xt} = (\partial_u + \alpha\partial_v)(\beta y_v) = \beta y_{uv} + \alpha\beta y_{vv},$$
$$y_{tt} = (\beta\partial_v)\beta y_v = \beta^2 y_{vv}.$$
Thus the original equation transforms to
$$Ay_{uu} + (2A\alpha + B\beta)y_{uv} + (A\alpha^2 + B\alpha\beta + C\beta^2)y_{vv} = 0.$$

2. Suppose $B^2 - 4AC < 0$ and
$$\alpha = \frac{-B}{\sqrt{4AC - B^2}}, \quad \beta = \frac{2A}{\sqrt{4AC - B^2}}.$$
Then $2A\alpha + B\beta = \frac{-2AB+2AB}{\sqrt{4AC-B^2}} = 0$ and
$$A\alpha^2 + B\alpha\beta + C\beta^2 = \frac{AB^2 - 2AB^2 + 4A^2C}{4AC - B^2}$$
$$= \frac{-AB^2 + 4A^2C}{4AC - B^2} = A \neq 0,$$
because $4AC > B^2 \geq 0$. Thus we can divide by A to get $y_{uu} + y_{vv} = 0$.

3. Suppose $B^2 - 4AC = 0$ and $\alpha = -B$, $\beta = 2A$. Then
$$2A\alpha + B\beta = -2AB + 2AB = 0,$$
$$A\alpha^2 + B\alpha\beta + C\beta^2 = AB^2 - 2AB^2 + 4A^2C = A(4AC - B^2) = 0.$$
Thus the equation reduces to $Ay_{uu} = 0$ or $y_{uu} = 0$ unless the equation is vacuous.

Chapter 4, page 93, Problem 2 Use the operators $L = x$ and $M = \partial_x$ to illustrate that LM and ML are not always the same.

Solution: Let $u(x)$ be a continuously differentiable function. Then $Lu = xu(x)$ and

$$M(Lu) = M(xu(x) = \partial_x(xu(x)) = u(x) + xu'(x).$$

But

$$LMu = L(Mu) = L(u'(x)) = xu'(x).$$

so $ML \neq LM$.

Chapter 4, page 93, Problem 3 Verify that each of the functions

$$u_0 = y, \; u_n = \sinh ny \cos nx, \quad n = 1, 2, \cdots$$

satisfies Laplace's equation

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad 0 < x < \pi, \; 0 < y < 2,$$

and the three boundary conditions

$$u_x(0,y) = u_x(\pi, y) = 0, \; u(x,0) = 0.$$

Then use the superposition proncipel to show, formally, that the series

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh ny \cos nx$$

satisfies the differential equation and boundary conditions.

Solution:

1.
$$(\partial_{xx} + \partial_{yy})u_0 = (\partial_{xx} + \partial_{yy})y = 0,$$
$$\partial_x u_0(0,y) = \partial_x y = 0, \; \partial_x(u_0(\pi, y) = \partial_x y = 0, \; u_0(x,0) = 0,$$
$$(\partial_{xx} + \partial_{yy})u_n = -n^2 \sinh ny \cos nx + n^2 \sinh ny \cos nx = 0,$$
$$\partial_x u_n(0,y) = -n \sinh ny \sin 0 = 0,$$
$$\partial_x u_n(\pi, y) = -n \sinh ny \sin n\pi = 0, \; u_n(x,0) = \sinh 0 \cos nx = 0.$$

2. Since the equation is linear and the boundary conditions are homgeneous, an arbitray linear combination of these special solutions also satisfies the equation and boundary conditions, formally, Thus

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh ny \cos nx$$

satisfies the differential equation and boundary conditions.

Chapter 4, page 98, Problem 1 Consider the boundary value problem

$$u_{xx}(x, y) + u_{yy}(x, y) = 0, \quad 0 < x < \pi,\ 0 < y < 2,$$

with homogeneous boundary conditions

$$u_x(0, y) = u_x(\pi, y) = 0,\ u(x, 0) = 0.$$

Use separation of variables $u = X(x)Y(y)$ and the results of Section 31 to show how the functions

$$u_0 = y,\ u_n = \sinh ny \cos nx, \quad n = 1, 2, \cdots$$

can be discovered. Proceed formally to derive the solution of the problem with nonhomogenous condition $u(x, 2) = f(x)$ as

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh ny \cos nx,$$

where

$$A_0 = \frac{1}{2\pi} \int_0^{\pi} f(x)dx,\ A_n = \frac{2}{\pi \sinh 2n} \int_0^{\pi} f(x) \cos nx\ dx, \quad n = 1, 2, \cdots.$$

Solution:

1. Set $u = X(x)Y(y)$, Substituting into the differential equation and separating variables, we have

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda.$$

Thus the Sturm-Liouville problems are

$$(a) \quad X'' + \lambda X = 0, \ X'(0) = X'(\pi) = 0,$$

$$(b) \quad Y'' - \lambda Y = 0, Y(0) = 0.$$

Working on (a), we see that if $\lambda = -a^2 < 0$ then $X(x) = Ae^{ax} + Be^{-ax}$, so $X'(x) = a(Ae^{ax} - Be^{-ax})$. Thus $X'(0) = a(A - B) = 0$ implies $A = B$, so $X'(\pi) = aB(e^{a\pi} + e^{-a\pi})$ which implies $B = 0$. Thus we can't satisfy the boundary conditions if $\lambda < 0$.

If $\lambda_0 = 0$ then $X(x) = Ax + b$. $X'(0) = X'(\pi) = 0$ implies $A = 0$. Thus $\lambda_0 = 0$ is an eigenvalue and we can take the eigenfunction as $X_0(x) = 1$.

If $\lambda = a^2 > 0$ with $a > 0$ then $X(x) = A\cos ax + B\sin ax$. Since $X'(x) = -Aa\sin ax + Ba\cos ax$ we have the requirement $X'(0) = Ba = 0$ so $B = 0$. The requirement $X'(\pi) = -Aa\sin a\pi = 0$ means that $a = n$. Thus the eigenvalues are $\lambda_n = n^2$, $n = 1, 2, \cdots$ with eigenfunctions $X_n(x) = \cos nx$.

For (b) we need consider only $\lambda \geq 0$. For $\lambda_0 = 0$ we have $Y(t) = Ay + B$ and the boundary condition $Y(0) = 0$ implies $B = 0$. Thus we have $Y_0(y) = y$.

For $\lambda_n = n^2$ we have $Y(y) = A\sinh ny + B\cosh ny$. The boundary condition $Y(0) = B = 0$ implies that the eigenfunctions are $Y_n(y) = \sinh ny$.

We conclude that the special solutions are

$$u_0 = y, \quad u_n = \cos nx \sinh ny, \quad n = 1, 2, \cdots.$$

2. Taking, formally, a linear combination of the special solutions u_0, u_n we get

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh ny \cos nx.$$

The inhomogeneous condtion $u(x, 2) = f(x)$ imposes the requirement

$$f(x) = 2A_0 + \sum_{n=1}^{\infty} A_n \sinh 2n \cos nx.$$

This is a Fourier Cosine series on the interval $[0, \pi]$, so we must have

$$4A_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx,\ A_n \sinh 2n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx,\ n = 1, 2, \cdots$$

from which we can obtain A_0, A_n.

Chapter 4, page 98, Problem 2 Show that if in Section 31 we had written

$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)} = -\lambda$$

to separate variables, we would still have obtained the same results.

Solution: Here $u(x,t) = X(x)T(t)$ and the boundary conditions are

$$u_x(0,t) = 0,\ u_x(c,c) = 0, \quad t > 0.$$

Thus the Sturm-Liouville problem is

$$X'' + \frac{\lambda}{k}X = 0,\ X'(0) = X'(c) = 0,$$

and there is the additional equation

$$T' + \lambda T = 0.$$

If $\lambda/k = 0$ then $X(x) = Ax + B$, and the conditions $X'(0) = X'(c) = 0 = A$ imply $A = 0$. Thus $\lambda_0 = 0$ is an eigenvalue with eigenfunction $X_0(x) = 1$. The corresponding solution for T is $T_0(t) = 1$.

If $\lambda/k = \alpha^2 > 0$ where $\alpha > 0$ then $X(x) = A\sin\alpha x + B\cos\alpha x$. The condition $X'(0) = 0 = A\alpha$ implies $A = 0$. The condition $X'(c) = 0 = -B\alpha\sin\alpha c$ implies $\alpha c = n\pi$, $n = 1, 2, \cdots$. Thus there are eigenvalues $\lambda_n = kn^2\pi^2/c^2$ with corresponding eigenfunctions

$$X_n(x) = \cos\frac{n\pi x}{c}, \quad T_n(t) = \exp(-\frac{kn^2\pi^2 t}{c^2}).$$

If $\lambda/k = -\alpha^2 < 0$ where $\alpha > 0$ then $X(x) = Ae^{\alpha x} + Be^{-\alpha x}$. The condition $X'(0) = 0 = \alpha(A - B)$ implies $B = A$. The condition

$X'(c) = 0 = A(e^{\alpha c} - e^{-\alpha c})$ implies $A = 0$ Thus there are no eigenvalues for this case.

We conclude that the separated solutions are

$$u_0 = 1,\ u_n = \cos(\frac{n\pi x}{c}) \exp{(-\frac{kn^2\pi^2 t}{c^2})}, \quad n = 1, 2, \cdots,$$

just as before.

Chapter 4, page 102, Problem 1 By assuming a product solution obtain conditions

$$X'' + \lambda X = 0,\ X(0) = X(c) = 0,$$
$$T'' + \lambda a^2 T = 0,\ T'(0) = 0,$$

from the homogeneous conditions

$$y_{tt} = a^2 y_{xx}, \quad 0 < x < c,\ t > 0,$$

$$y_t(0, t) = 0,\ y(c, t) = 0, y_t(x, 0) = 0.$$

Solution: Assume $y(x, t) = X(x)T(t)$ satisfies the wave equation. Then $XT'' = a^2 X'' T$ so we have

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda.$$

Thus

$$X'' + \lambda X = 0,\ T'' + \lambda a^2 T = 0.$$

The boundary condition $y_t(0.t) = 0 = T'(t)X(0)$ implies $X(0) = 0$ since we never have $T'(t) \equiv 0$ even for $\lambda = 0$.. The boundary condition $y(c, t) = 0 = X(c)T(t)$ implies $X(c) = 0$. The initial condition $y_1(x, 0) = 0 = T'(0)X(x)$ implies $T'(0) = 0$.

Chapter 4, page 102, Problem 2 Derive the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$X'' + \lambda X = 0,\ X(0) = X(c) = 0.$$

Solution: If $\lambda = 0$ then $X(x) = Ax + B$. Since $X(0) = 0 = B$ we have $B = 0$. Since $X(c) = 0 = Ac$ we have $A = 0$, so $\lambda = 0$ is not an eigenvalue.

If $\lambda = -a^2$ with $a > 0$ we have $X(x) = Ae^{ax} + Be^{-ax}$. The condition $X(0) = 0 = A + B$ implies $B = -A$. The condition $X(c) = 0 = A(e^{ac} - e^{-ac})$ implies $A = 0$. Thus no such $\lambda < 0$ is an eigenvalue.

If $\lambda = a^2$ with $a > 0$ we have $X(x) = A \sin ax + B \cos ax$. The condition $X(0) = 0 = B$ implies $B = 0$. The condition $X(c) = 0 = A \sin ac$ implies $a = n\pi/c$, $n = 1, 2, \cdots$. Thus the possible eigenvalues are $\lambda_n = n^2\pi^2/c^2$ with eigenfunctions $X_n(x) = \sin(\frac{n\pi x}{c})$, $n = 1, 2, \cdots$.

Chapter 4, page 102, Problem 3 Point out how it follows from expression

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{c} \cos \frac{n\pi a t}{c},$$

that for each fixed x, the displacement function $y(x,t)$ is periodic in t with period $T_0 = \frac{2c}{a}$.

Solution: From the expansion above, if you replace t by $t + \frac{2c}{a}$ then

$$\cos(\frac{n\pi a(t + \frac{2c}{a})}{c}) = \cos(\frac{n\pi a t}{c} + 2\pi n) = \cos \frac{n\pi a t}{c},$$

so $y(x, t + T_0) = y(x,t)$. Thus y is periodic in t with period T_0.